Elliptic fibrations on the rational elliptic surface

MSc Thesis

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Contents

1 Introduction

The investigation of elliptic fibrations is an interesting topic of its own right in differential topology. Since the classification of complex surfaces is quite a new area in differential geometry, the main result in the study of elliptic fibrations on complex surfaces came from K. Kodaira [9] in 1963. He gave the complete list of singular fibers, which can occur in elliptic fibrations. The question, which combination of these singular fibers are possible on the rational elliptic surface, have been answered by U. Persson [14] and R. Miranda [10]. An interesting feature of this topic is, that elliptic fibrations on the rational elliptic surface can be considered from algebraic geomertic point of view as well. One can construct elliptic fibrations with certain types of singular fibers via blowing up pencils on the rational elliptic surface. Showing the existence of singular fiber combinations in elliptic fibrations this way is possible, because birational maps between surfaces are composites of blow ups. However in higher dimension the structure of birational maps are more complex, which shows the difficulty of this area of differential geometry.

The classification of complex surfaces, also with fibrations on rational elliptic surfaces, has connection with many further areas in geometry. In [4] the authors, R. Gompf and A. Stipsicz, use the basic concepts of this topic in Kirby calculus. It turns out, that fibrations with different kind of degenerated fibers are extremally useful in knot theory or in the construction of exotic smooth structures on closed 4-manifolds [4], [12], [13], [15]. Moreover the investigation of fibrations on the rational elliptic surface provides interest in Hodge theory. Hitchin fibrations on a 2-dimensional moduli space of irregular Higgs bundles over $\mathbb{C}P^1$ has been shown to be biregular to the complement of a singular fiber in an elliptic fibration on the rational elliptic surface (which is diffeomorphic to the 9-fold blow up $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ of the complex projective plane), see [6], [7], [8]. The investigation of these spaces has further applications in Painlevé equations or in mathematical physics. In these questions it is very important to show the existence of singular fiber configurations via explicitly constructing them. All fibrations studied in this topic arise from blow up of pencils (see [17]), and this is the process we will use to construct such fibrations. These constructions of elliptic fibrations with certain types of singular fibers have been described in the above cited papers, provided by A. Stipsicz, Sz. Szabó and P. Ivanics, except the case of fibration with singular fiber I_1^*. The main result of our investigation is the explicit, algebraic geometric construction of the 13 possible configurations of elliptic fibrations with I_1^* fiber. These constructions on the Hirzebruch surface, and the pencils from which they arise, have not been described yet. We will show them in the second part of this thesis.

In the first part we describe the most important definitions and theorems related to elliptic fibrations on the rational surface. In Section 2 we recall intersection number of analytic subvarieties on complex manifolds, linear systems (pencils) on manifolds and the blow up process. We also discuss here the rational, and elliptic surfaces, and pay special attention to the Hirzebruch surfaces. The main literature used here is the book *Principles of algebraic geometry* [5] by P. Griffiths and J. Harris, but also several others are cited in this Section. Section 3 is about the list and description of types of singular fibers, and their properties. Here we also introduce, which combinations can occure with an I_1^* type fiber, based on the paper of A. Stipsicz, Z. Szabó, and Á. Szilárd [16]. Finally in Section 4, we present our results, i.e. the constructions of rational elliptic surfaces, containing the 13 possible singular fiber configurations. First we define the method of constructions of elliptic fibrations through blowing up pencils of elliptic curves on the second Hirzebruch surface $\mathbb{F}_2$, and on $\mathbb{C}P^2$. We detail this in the above mentioned 13 cases, by choosing the right curves, and supporting our calculations with figures, illustrating the process.

2 Preparatory material

In this section we will review those definitions, theorems, examples, which are necessary to understand the structure of the elliptic fibrations we want to describe. Consider this, as a beginning of a setup, which will lead us to the construction of the fibrations in the final section. The literature dealing with the topics in this section is very wide, we will use several books and notes: [1],[2],[3],[4],[5],[11].

2.1 Intersection number and Poincaré duality

For this topological introduction we will use reference [5, Section 0.4], and [2, Section 1.6] Let M be a compact, connected, oriented real 4-manifold, and $S_1, S_2 \subset M$ two piecewise smooth real 2-cycles.

Definition 2.1. i) $p \in S_1 \cap S_2$ is a transverse intersection point, if $T_pS_1 \oplus T_pS_2 \to T_pM$ is a bijective correspondence.
ii) S_1 is transverse with respect to S_2, if every $p \in S_1 \cap S_2$ is a transverse intersection point. Notation: $S_1 \pitchfork S_2$

If $\{e_1, e_2\}$ is a basis of T_pS_1, and $\{e_3, e_4\}$ is a basis of T_pS_2, then transverse intersection at p means, that $\{e_1, e_2, e_3, e_4\}$ is a basis of T_pM.

Definition 2.2. i) If $S_1 \pitchfork S_2$, the intersection index of S_1 with S_2 at $p \in S_1 \cap S_2$, denoted by $\iota_p(S_1, S_2)$, is $+1$, if $\{e_1, e_2, e_3, e_4\}$ is an oriented basis for $T_pS_1 \oplus T_pS_2 = T_pM$, and -1 if not.
ii) If $S_1 \pitchfork S_2$, the intersection number of the cycles is defined to be

$$S_1 \cap S_2 = \sum_{p \in S_1 \cap S_2} \iota_p(S_1, S_2) \in \mathbb{Z}.$$

For the completness of this definition, we need to remark, that this sum is finite, since S_1, S_2 are compact, and because of $S_1 \pitchfork S_2$, the set $S_1 \cap S_2$ is discrete. The main observation is, that the intersection number $S_1 \cap S_2$ depends only on the homology class of S_1 and S_2. That is, if we consider the two homology classes $[S_1] = \alpha, [S_2] = \beta \in H_2(M, \mathbb{Z})$, we may find C^∞ piecewise smooth cycles S_1', S_2' represeting α, β, and intersecting transversely. Thus the intersection number defines a bilinear pairing:

$$H_2(M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \to \mathbb{Z},$$

called the intersection pairing, denoted by $\alpha \cap \beta$. With this, the intersection number $S_1 \cap S_2$ makes sense, even if S_1 and S_2 fail to meet transversely. It is also true, that $\alpha \cap \beta$ is independent of the choice of the representing cycles.

Theorem 2.3 (Poincaré duality)**.** *If M is a compact oriented 4-manifold, the intersection pairing $H_2(M,\mathbb{Z}) \times H_2(M,\mathbb{Z}) \to \mathbb{Z}$ is unimodular, i.e. any group homomorphism $H_2(M,\mathbb{Z}) \to \mathbb{Z}$ is expressible as intersection with some $\alpha \in H_2(M,\mathbb{Z})$, and if $\alpha \in H_2(M,\mathbb{Z})$ has intersection number 0 with all classes in $H_2(M,\mathbb{Z})$, then α is the zero class.* (*We can also write $H_2(M,\mathbb{Z}) \cong H_2(M,\mathbb{Z})^*$*).

Suppose now, that M is a compact, oriented, complex 2-manifold, and $S \subset M$ is an arbitrary complex 1-cycle on M. Then the correspondence in the theorem can be realized, with group homomorphsim $\int_S \in H^2(M,\mathbb{Z})^*$:

$$\int_S : H^2(M,\mathbb{Z}) \to \mathbb{R}, \qquad [\omega] \mapsto \int_S \omega\big|_S.$$

And the theorem states, that there exists an η_S closed 2-form, such that

$$\int_S \omega\big|_S = \int_M \omega \wedge \eta_S,$$

where the cohomology class of η_S is the Poincaré dual of S, and also called the fundamental class of S. In fact the definition of intersection number for (possibly singular) analytic subvarieties is the same as above. (Some difference is, that analytic subvarieties may intersect with higher multiplicity $m \geq 1$ at point p. In such case, the intersection multiplicity is $\iota_p(S_1,S_2) = \pm m$.) Since on analytic subvarieties there is a natural orientation coming from the orientation of M, we may observe the following.

Remark 2.4. *The intersection index of analytic subvarieties meeting transversely is always positive.*

The $S \cap S \in \mathbb{Z}$ is called the self-intersection number of S, and can be computed via $\int_M \eta_S \wedge \eta_S$.

2.2 Line bundles and linear systems

In this Section we will use the setup of [5, Section 1.1]. Some of the basic concepts of this topic are the projective algebraic varieties, which are defined to be the set of complex zeros of homogenous polynomials in projective space. In general, if M is an n-dimensional complex manifold, with $V \subseteq M$ $(n-1)$-dimensional analytic subvariety, V is called analytic hypersurface, if there is a single f holomorphic function, such that for all $p \in V$, in some neighborhood of p, V can be given as the zero locus of f local defining function (and all holomorphic g vanishing on V is divisible by f in some negighborhood of p). Moreover V is said to be irreducible, if cannot be written as the union of $V_1, V_2 \neq V$ analytic hypersurfaces.

Definition 2.5. A divisor D on M is the formal linear combination $D = \sum a_i V_i$ of V_i irreducible analytic hypersurfaces, where for all $p \in M$, there is a neighborhood of p, meeting only finitely many V_i.

Remark 2.6. *i) There is a natural additive group structure on the set of divisors in M.*
ii) D divisor is called effective, if $a_i \geq 0$ for all i.

Two divisors D and D' are said to be linearly equivalent, if $D' = D + (f)$, where $(f) = (p) - (z)$ for the divisor of zeros (z), and poles (p) of some meromorphic function f on M. Denote by $|D|$ the set of effective divisors on M, which are all equivalent to D. So if $D' \in |D|$, then there is an f, such that $D' = D + (f)$, and f is defined up to multiplication with nonzero scalar, since $\lambda(f) = (\lambda f)$. Hence $|D|$ is the projective space associated to the vector space of f meromorphic funtcions: $\mathcal{L}(f, D) := \{f \mid D + (f) \text{ is effective}\}$. With this notation, $|D|$ can be identified with $\mathcal{L}(f, D)$ space.

Definition 2.7. A linear system is a projective subspace of $|D|$, for some effective divisor D.

Now let $L \xrightarrow{\pi} M$ be a line bundle on M, i.e. vector bundle, with 1-dimensional fibers (in complex sense). To this belongs $\{U_\alpha\}$ open cover of M, and $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ local trivializations. Relative to this, we can define transition functions $g_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathrm{Aut}(\mathbb{C})$, by

$$g_{\alpha,\beta}(z) = \varphi_\alpha \circ \varphi_\beta^{-1}\big|_{L_z} \in \mathrm{Aut}(\mathbb{C}).$$

The functions $g_{\alpha,\beta}$ are clearly holomorphic, nonvanishing, and satisfy for all α, β, γ the cocycle conditions

$$g_{\alpha,\alpha} = g_{\alpha,\beta} \cdot g_{\beta,\alpha} = g_{\alpha,\beta} \cdot g_{\beta,\gamma} \cdot g_{\gamma,\alpha} = \text{id}.$$

Conversely, if we have some collection of $\{g_{\alpha,\beta}\}$ functions satisfying the above conditions, we can construct a line bundle L with transition functions $\{g_{\alpha,\beta}\}$, by taking the union of all $U_\alpha \times \mathbb{C}$, and identifying the points over $U_\alpha \cap U_\beta$ via the $g_{\alpha,\beta}$ functions.

If D is a divisor with local defining function $\{f_\alpha\}$ over $\{U_\alpha\}$ open cover, then $g_{\alpha,\beta} = f_\alpha \cdot f_\beta^{-1}$ functions clearly satisfy the above conditions, that is we can associate a $\mathcal{D}$ line bundle to D with transition functions $\{f_\alpha \cdot f_\beta^{-1}\}$.

Remark 2.8. *There is an identification between the $\mathcal{L}(f, D)$ vectorspace and the $\Gamma(M, \mathcal{D})$ vectorspace of sections of line bundle $\mathcal{D}$. Let s_0 be a global meromorphic section of $\mathcal{D}$, such that $(s_0) = D$, then for $f \in \mathcal{L}(f, D)$: $f \leftrightarrow f \cdot s_0 \in \Gamma(M, \mathcal{D})$ gives the identification.*

The dimension of a linear system is the dimension of the projective subspace parametrizing it. Thus the dimension of $|D|$ is $\dim(\mathcal{L}(f, D)) - 1 = \dim(\Gamma(M, \mathcal{D})) - 1$. The linear system of dimension 1 is called pencil.

If $\{D_\lambda\}_{\lambda \in \mathbb{C}P^n}$ is a linear system, then for any $\lambda_0, ..., \lambda_n \in \mathbb{C}P^n$ linearly independent

$$D_{\lambda_0} \cap ... \cap D_{\lambda_n} = \bigcap_{\lambda \in \mathbb{C}P^n} D_\lambda.$$

Definition 2.9. The set of common intersection points of divisors in a linear system is called the base locus of the system (base points if they are finitely many).

The following example shows the concrete case, we will use in Section 4.

Example 2.10. Let $D_1 = (f_1), D_2 = (f_2)$ be two linearly equivalent divisors on $\mathbb{C}P^2$ complex 2-manifold. The set

$$S = \{(t_1 f_1 + t_2 f_2) \mid t = [t_1 : t_2] \in \mathbb{C}P^1\}$$

is a complete family of linearly equivalent divisors. The linear system S is family of divisors of two parameters, thus it is a pencil indeed, and the base points are the intersection points $D_1 \cap D_2$.

Theorem 2.11 (Bertini)**.** *The generic element of a linear system is smooth away from the base locus of the system.*

The consequence of this theorem, that we will use in Section 4., is that if $\{D_\lambda\}_{\lambda\in\mathbb{C}P^1}$ is a pencil with base locus Z, then it gives a holomorphic mapping $M\setminus Z \to \mathbb{C}P^1$, because every point in $M\setminus Z$ lies on a unique D_λ.

2.3 The blow up process

The blow up process, as an algebraic geometric construction, is a very useful tool in our hands, to resolve singularities, and in some sense, the blow up is an 'elementary' map between complex surfaces. We devote this section to the definition, some properties, and examples of the blow up according to [5, Section 1.4 and 4.1], and [4, Section 2.2].

Denote $D \subseteq \mathbb{C}^n$ the disc containing the origin, with euclidean coordinates $(z_1, ..., z_n)$. Now consider in $D\times\mathbb{C}P^{n-1}$ the submanifold

$$\widetilde{D} = \{(z,l) \in D \times \mathbb{C}P^{n-1} | z \in \mathbb{C}^n \text{ incident to } l \in \mathbb{C}P^{n-1}\}.$$

For each $z \in \mathbb{C}^n$, there is a unique $l \in \mathbb{C}P^{n-1}$, such that the line determined by l in $\mathbb{C}^n$, goes through z, except the $0 \in D$. Hence the projection map

$$\pi : \widetilde{D} \to D \qquad (z,l) \mapsto z$$

is an isomorphism away from the origin.

Definition 2.12. The $(\widetilde{D},\pi)$ pair is called the blow up of D in 0.

The definition extends to n-dimensional complex manifold M, since some U neighborhood of $x \in M$ is diffeomorphic to D via the φ_U map. Now $\pi' = \varphi_U \circ \pi : \widetilde{D} \to U$, and the restriction of π' on $\widetilde{D} - (\pi')^{-1}x$ yields an isomorphism to $U - \{x\}$.

Definition 2.13. (i) The connected sum $\widetilde{M_x} = (M - \{x\})\#_{\pi'}\widetilde{D}$, together with the natural extension $\pi' : \widetilde{M_x} \to M$ of the projection map, is called the blow up of M at x.
(ii) The $E = (\pi')^{-1}x$ in $\widetilde{M_x}$ is called the exceptional curve (or

exceptional divisor) of the blow up.
(iii) The proper transform of $V \subseteq M$ submanifold (or subvariety) is the closure of the inverse image: $\widetilde{V} = \overline{(\pi')^{-1}(V - \{x\})}$.

Remark 2.14. *One can easily check, that the blow up $\widetilde{M}_x \to M$ is independent of the coordinates chosen in D.*

Again the $\pi' : \widetilde{M}_x - \{(\pi')^{-1}x\} \to M - \{x\}$ restriction of the blow up is an isomorphism, and the point x is where the blow up is really interesting. That is, the lines passing through x in D are blown up into disjoint lines, which have distinct intersection points with E, diffeomorphic to $\mathbb{C}P^{n-1}$. More generally, by blowing up, we can reduce the number of intersection points of transversally intersecting complex curves. Now consider how intersection, and self-intersection numbers change under the blow up process.

Lemma 2.15. *Let $M = \mathbb{C}P^2$, with one-point blow up $\widetilde{M}$. Then the exceptional divisor E in $\widetilde{M}$ has self-intersection number -1.*

Proof. Let $p = 0 \in \mathbb{C}P^2$ the point, which we blow up. Now the image of $p \neq 0$ is itself, while the image of $p = 0$ is $E \cong \mathbb{C}P^1$ under the blow up. Let $B_1^4 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 < 1\} \subseteq \mathbb{C}P^2$ open ball around $p = 0$, and $S_1^3 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1\}$ its boundary. Now take two copy of $\mathbb{C}P^2 \setminus B_1^4$, the second one under the map

$$t : \mathbb{C}^2 \to \mathbb{C}^2, \qquad (z_1, z_2) \mapsto \frac{1}{|z_1|^2 + |z_2|^2}(\overline{z_1}, \overline{z_2}).$$

Thus we can obtain $\widetilde{M}$, as $(\mathbb{C}P^2 \setminus B_1^4) \cup_{S_3^1} \overline{(\mathbb{C}P^2 \setminus B_1^4)}$. That is, one inverted copy of $\mathbb{C}P^2 \setminus B_1^4$ (with conjugation, for the sake of orientation) can be sticked to the other copy along the S_1^3. Thus the infinite circle $\mathbb{C}P^1$ of the second copy will be placed, where 0 was in the first copy, this will be the exceptional divisor E. Since E is a line ($\mathbb{C}P^1$ curve), it would have intesection number 1 with any other line (also with itself), but because of the change of orientation E will have self-intersection number -1. ⊠

Remark 2.16. *In the proof of this lemma we also saw, that the blow up of $\mathbb{C}P^2$ at one point, is the same, as the connected sum $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.*

It is in general also true, that the exceptional divisor of blow up has self-intersection number -1. Now if D is a divisor on M, with proper transform $\widetilde{D}$ in $\widetilde{M}$, then obviously $\pi^*D \cap E = 0$, because the preimage of E is a single point (here π^*D is the pullback of the divisor D). Thus the intersection number:

$$\widetilde{D} \cap \widetilde{D} = (\pi^*D - E) \cap (\pi^*D - E) = \pi^*D \cap \pi^*D + E \cap E = D \cap D - 1.$$

It is important to notice, that if $\{D_\lambda\}_\lambda$ is a linear system on M, then the $\{\widetilde{D}_\lambda\}_\lambda$ proper transforms of the D_λ curves on $\widetilde{M}$ do not necessarily form a linear system. It is a linear system, if and only if the D_λ curves have the same multiplicity in the p base point, where the blow up happens. If this is not the case, under the proper transform of linear system, we mean the linear system $\{D_\lambda - mE\}_\lambda$, where m is the minimum of multiplicities of D_λ curves at p.

The decreasing self-intersection number shows, how blow ups resolve singularities of curves.

Proposition 2.17. *If M is a nonsingular complex surface, and $S \subset M$ a (possibly singular) complex curve, then there is a complex surface $\widetilde{M}$, and $\pi : \widetilde{M} \to M$ composition of blow ups, such that the proper transform $\widetilde{S} \subset \widetilde{M}$ is a smooth complex curve.*

The following examples of desingularisation, will be very important in the constructions in Section 4.

Example 2.18. $S = \{[x : y : z] \in \mathbb{C}P^2 | zy^2 = x^3 + zx^2\} \subset \mathbb{C}P^2$ is smooth, except at $P = [0 : 0 : 1]$. This curve is topologically a sphere, with self-intersection 0, and with one positive double point. If we blow up at P, the proper transform $\widetilde{S}$ will be a nonsingular sphere ($\mathbb{C}P^1$), with self-intersection -2, because it decreases by 2, after blowing up a double point. Similarly the exceptional curve E will be a $\mathbb{C}P^1$, with self-intersection -2, while E and $\widetilde{S}$ intersect each other transversally in two points (see Fig.1). For more details see [4, Section 2.3].

Example 2.19. $S = \{[x : y : z] \in \mathbb{C}P^2 | zy^2 = x^3\} \subset \mathbb{C}P^2$ is smooth, except at $P = [0 : 0 : 1]$. This curve is topologically a sphere with a cusp singularity at P. Now under the blow up, the proper transform $\widetilde{S}$ will be a smooth line ($\mathbb{C}P^1$), tangent to

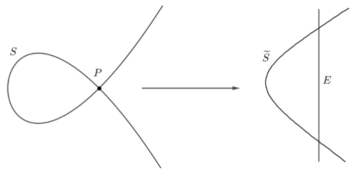

Figure 1: The blow up of the so called fishtail curve (denoted by I_1) at its singular point.

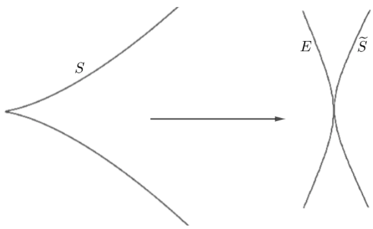

Figure 2: The blow up of the so called cusp curve (denoted by II) at its singular point.

the exceptional curve E at P (see Fig.2). For more details see [4, Section 2.3].

Example 2.20. $S = \{[x : y : z] \in \mathbb{C}P^2 | zy^2 = x^4\} \subset \mathbb{C}P^2$ is smooth, except at $P = [0 : 0 : 1]$. This curve is topologically two spheres (S_1, S_2) of self-intersection -2, tangent to each other at P. After blow up at P, the proper transform of the two branches will meet transversally at some point of the exceptional curve (which corresponds to their common tangent line at P). After one more blow up at P, the second proper transforms will be smooth disjoint lines (see Fig.3). For more details see [5, Section 4.2].

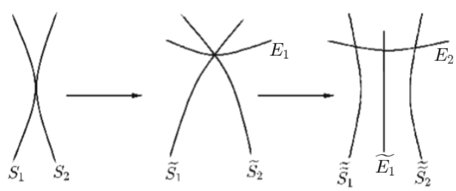

Figure 3: Evolution of curves under the blow up process.

2.4 Rational and elliptic surfaces

The literature used in this section can be found at [5, Section 4.2], [4, Section 3.1], and [1].

Definition 2.21. A rational (or meromorphic) map of a complex manifold M to $\mathbb{C}P^n$ is a map $f : z \mapsto [1 : f_1(z) : ... : f_n(z)]$, where f_i-s are global meromorphic functions on M.

Equvivalently, a rational map $f : M \to \mathbb{C}P^n$ is given by a holomorphic map $f : M \setminus V \to \mathbb{C}P^n$ defined on the complement of a subvariety V of codimension 2 or more in M. A rational map $f : M \to V$ to algebraic subvariety $V \subseteq \mathbb{C}P^n$, is a rational map $f : M \to \mathbb{C}P^n$, whose image is on V. In fact, if $\{D_\lambda\}_{\lambda \in \mathbb{C}P^n}$ is a linear system on M with base locus of codimension at least 2, then the map

$$f : M \to \mathbb{C}P^n, \qquad p \mapsto \{\lambda | p \in D_\lambda\}$$

is rational, and well defined away from the base locus of the system. That is exactly the case in Example 2.10., if D_1, D_2 divisors are determined by polynomials p_1, p_2 on $\mathbb{C}P^2$, with $(p_1, p_2) = 1$, what we will use in Section 4.

Definition 2.22. i) $f : M \to V$ rational map is birational, if there is a rational map $g : V \to M$, such that $f \circ g$ is the identity as a rational map, i.e. defined away from a subvariety of codimension at least 2.
ii) Two algebraic varieties are said to be birational, if there is a birational map between them.
iii) A variety is said to be rational, if it is birational to $\mathbb{C}P^n$.

Remark 2.23. *i) Any holomorphic map is clearly rational.*
ii) If $\pi : \widetilde{M} \to M$ is the blow up of M in some $\{p_i\}$ points, then the inverse map $\pi^{-1} : M \setminus \{p_i\} \to \widetilde{M}$ is trivially rational, so π is a birational isomorphism.

Moreover, if M has complex dimension 2, i.e. M is a complex surface, then a partial converse of the second point of the remark is also true.

Theorem 2.24. *If M and N are complex algebraic surfaces, and $f : M \to N$ a birational map, then there exists a surface $\widetilde{M}$, and $\pi_1 : \widetilde{M} \to M$, $\pi_2 : \widetilde{M} \to N$ blow ups, such that $f = \pi_2 \circ \pi_1^{-1}$.*

$$
\begin{tikzcd}
 & \widetilde{M} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
M \arrow[rr, "f"] & & N
\end{tikzcd}
$$

This theorem means, that every birational map is a sequence of blow ups and blow downs, and on the other hand, for any $f : M \to N$ a birational map, there exists $\pi_1 : \widetilde{M} \to M$ blow up, such that $f \circ \pi_1$ is holomorphic. Consequently, if $\{D_\lambda\}_{\lambda \in \mathbb{C}P^n}$ is a linear system on M with isolated base points, then there exists $\pi_1 : \widetilde{M} \to M$ blow up, such that the proper transform of the system in $\widetilde{M}$ has no base points. This resolution of base points with blow ups, is what we will use in Section 4.

Another important question is, whether a curve $S \subset M$ can be considered, as an exceptional curve of some blow up, and can be contracted to a point by blowing down.

Theorem 2.25 (Enriques-Castelnuovo)**.** *Let M be an algebraic surface, $S \subset M$ a smooth rational curve, with self-intersection number -1. Then there exists a smooth algebraic surface N, and $\pi : M \to N$ map, which is the blow up of N at $p \in N$, and $S = \pi^{-1}(p)$.*

A smooth rational curve of self-intersection -1 on a surface is called an exceptional divisor of the first kind. Also a complex surface is called minimal, if it does not contain rational -1-curve, so it is not the blow up of any other surface. A minimal model of complex surface M, is a complex surface M', if they are birational, and M' is minimal.

Definition 2.26. M complex surface is called elliptic surface, if there is a $\pi : M \to C$ holomorphic map to C complex curve, such that for generic $x \in C$ $\pi^{-1}(x)$ is a smooth elliptic curve.

The $\pi^{-1}(x)$ preimage is topologically a real 2-torus. The π map is called elliptic fibration. However a map $\pi : M \to C$ is called elliptic fibration as well, if each (possibly singular) $\pi^{-1}(x)$ fiber "looks like" a fiber in an elliptic surface. Precisely, each $\pi^{-1}(x)$ has an U neighborhood, and an orientation preserving diffeomorphism

$\varphi : U \to M$ into M elliptic surface, with $\varphi \circ \pi = \pi \circ \varphi$. A map $\sigma : C \to M$ is called section of the fibration, if $\sigma \circ \pi = \mathrm{id}_C$.

Our main interest is in the 9-fold blow up of $\mathbb{C}P^2$, and the 8-fold blow up of second the Hirzebruch surface (see Section 2.5.), which provide elliptic fibrations, as we will see at the beginning of Section 4.

2.5 On the Hirzebruch surface

For more details of the setup of this section, see [5, Section 4.3], [4, Section 3.4], [11] and [3, Section V.4].

Definition 2.27. S rational surface, with $\{C_\lambda\}$ pencil of disjoint irreducible (smooth) rational curves on S is called rational ruled surface.

If $\{C_\lambda\}$ is such a pencil of disjoint smooth rational curves on S rational ruled surface, then the pencil determines the natural map $p : S \to \mathbb{C}P^1$, called the ruling. In fact, this $p : S \to \mathbb{C}P^1$ is a holomorphic fibre bundle over over $\mathbb{C}P^1$, with $\mathbb{C}P^1$ fibers. In general, if $E \to M$ is a holomorphic vector bundle over M complex manifold, then the associated projective bundle $\mathbb{P}(E) \to M$ is the fibre bundle over M, whose fiber over $x \in M$ is the projective space $\mathbb{P}(E_x)$, associated to the vector space E_x.

Now if $L \to M$ line bundle, we can see from the transition functions, that $\mathbb{P}(E) = \mathbb{P}(E \otimes L)$, and conversely if $\mathbb{P}(E) = \mathbb{P}(E')$, then $E' = E \otimes L$ for some L line bundle over M. Also true, that any rational ruled surface is of the form $\mathbb{P}(E)$ for some $E \to \mathbb{C}P^1$ holomorphic 2-rank vector bundle. The following proposition gives a complete description of such vector bundles.

Proposition 2.28. *Any holomorphic vector bundle on $\mathbb{C}P^1$ is a direct sum of lines bundles.*

Let $\underline{\mathbb{C}} = \mathbb{C}P^1 \times \mathbb{C}$ be the trivial line bundle, and let

$$\mathcal{O}(-1) = \{(p, z) \in \mathbb{C}P^1 \times \mathbb{C}^2 | p \text{ is incident to } z\}.$$

be the tautological bundle. Its dual bundle (taking the dual space fiberwise) is the hyperplane bundle, denoted by $\mathcal{O}(1)$. The n-th tensor power of $\mathcal{O}(1)$ is denoted by $\mathcal{O}(n)$, thus because of the proposition, for any rational ruled surface, and for some n:

$$\mathbb{P}(E) = \mathbb{P}(L_1 \oplus L_2) = \mathbb{P}((L_1 \otimes L_2^*) \oplus \underline{\mathbb{C}}) = \mathbb{P}(\mathcal{O}(n) \oplus \underline{\mathbb{C}}).$$

Definition 2.29. $\mathbb{F}_n = \mathbb{P}(\mathcal{O}(n) \oplus \underline{\mathbb{C}})$ is the n-th Hirzebruch surface.

Let σ be any holomorphic section of $\mathcal{O}(n)$, then $E_\sigma \subset \mathbb{F}_n$ is the image of the $(\sigma, 1)$ section of $\mathcal{O}(n) \oplus \underline{\mathbb{C}}$. Particularly E_0 is called the zero section, and obviously E_σ is homologous to E_0 for any σ. The $(\sigma, 0)$ section of $\mathcal{O}(n) \oplus \underline{\mathbb{C}}$ for any σ, also determines a curve in $\mathbb{F}_n$ away from the zeros of σ, the closure of this curve is the E_∞ infinite section of $\mathbb{F}_n$. The E_∞ infinite section is independent of the choice of σ. If F is any fiber of the ruling $p : \mathbb{F}_n \to \mathbb{C}P^1$, we may obtain the intersection numbers: $E_0 \cdot E_0 = n$, $E_0 \cdot E_\infty = 0$, $E_0 \cdot E_\sigma$ =number of zeros of σ, $E_\sigma \cdot E_\infty$ =number of poles of σ, $E_0 \cdot F = E_\infty \cdot F = E_\sigma \cdot F = 1$.

One can easily see, that $\mathbb{F}_n - F - E_0$ is a $\mathbb{C}$-bundle over $\mathbb{C}P^1$ minus a point, which is isomorphic to $\mathbb{C}$, and therefore is contractible. Thus the second homology group $H_2(\mathbb{F}_n, \mathbb{Z})$ is generated by the homology class of F and E_0. Since two curves on $\mathbb{F}_n$ are linearly equivalent if and only if they are homologous, we can write: $E_\infty \sim m_1 E_0 + m_2 F$. From the above intersection numbers: $E_\infty \sim E_0 - nF$, and also we can compute $E_\infty \cdot E_\infty = -n$. Similarly $E_\infty \sim E_0 - mF$, where m is the number of poles of σ. From this we can conclude, that if $D \neq E_\infty$ is an irreducible curve on $\mathbb{F}_n$, then $D \cdot D \geq 0$, thus E_∞ is the only irreducible curve on $\mathbb{F}_n$ with negative self-intersection. This shows, that the $\mathbb{F}_n$ surfaces are distinct, and $\mathbb{F}_n$ is the unique $\mathbb{C}P^1$ bundle over $\mathbb{C}P^1$ having an irreducible curve of self-intersection $-n$. Also true, that the $\mathbb{F}_n$ Hirzebruch surfaces are all obtained from one another by blowing up and down, thus they are all birational, and since $\mathbb{F}_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ is rational, they are all rational.

Theorem 2.30. *Rational surfaces are the blow ups of $\mathbb{C}P^2$ or $\mathbb{F}_n$.*

This theorem might complete our picture of rational surfaces, and shows that the minimal model of each rational surface is $\mathbb{C}P^2$ or $\mathbb{F}_n$. This also proves our claim, that the rational elliptic surface can be obtained as the blow up of $\mathbb{C}P^2$ or $\mathbb{F}_n$.

A different point of view on the Hirzebruch surfaces shows, how we will construct curves and pencils on them. Let $\mathbb{C}[u]$ and $\mathbb{C}[v^{-1}]$ be polynomial rings with generators u and v^{-1}. Their spectra (set of all their prime ideals) $S_1 = \mathrm{Spec}(\mathbb{C}[u])$ and $S_2 = \mathrm{Spec}(\mathbb{C}[v^{-1}])$ can be obtained as subsets in $\mathbb{C}P^1$. Hence if we consider $S_1 \times \mathbb{C}P^1$ and $S_2 \times \mathbb{C}P^1$ with gluing identification $(u, 1) \leftrightarrow (v^{-1}, v^{-1})$, we get exactly the compactification of the hyperplane line bundle $\mathcal{O}(1)$. This is indeed a $\mathbb{C}$ bundle over $\mathbb{C}P^1$, i.e. this is a line bundle over $\mathbb{C}P^1$. The n-th tensor power of $\mathcal{O}(1)$, denoted by $\mathcal{O}(n)$, can be obtained with gluing map $(u, 1) \leftrightarrow (v^{-1}, v^{-n})$. Now

$$A_0 = \bigoplus_{n \geq 0} \mathcal{O}(n)$$

is a graded algebra over the polynomial ring. We can also glue its projectivizations over $\mathbb{C}[u]$ and $\mathbb{C}[v^{-1}]$ rings, by identifying the 0-th element of the first direct sum with the k-th element of the second direct sum. Hence we obtain the k-th Hirzebruch surface.

The canonical bundle of $\mathbb{C}P^1$ is the line bundle $K = \mathcal{O}(-2)$. If D is a divisor of total length 4, for example $D = t_1 + 2t_2 + t_3$, then by adding D to the canonical bundle: $K(D) \cong \mathcal{O}(2)$. Furthermore $\mathcal{O}(4) \cong \mathcal{O}(2)^{\otimes 2} \cong K^{\otimes 2}(2D)$, where the sections of $K^{\otimes 2}(2D)$ (or double sections of $K(D)$) are called meromorphic quadratic differentials, and can be given by homogenous degree-4 polynomials over $\mathbb{C}P^1$. For example:

- The polynomial u^4, as a section of $\mathcal{O}(4)$ provides two sections in $\mathcal{O}(2)$, which are tangent to each other over $[0:1] \in \mathbb{C}P^1$. See Example 2.20.

- The polynomial u^2v^2, as a section of $\mathcal{O}(4)$ provides two sections in $\mathcal{O}(2)$, which intersect each other transversally over $[0:1], [1:0] \in \mathbb{C}P^1$.

- The polynomial $u^2v(u+v)$, as a section of $\mathcal{O}(4)$ provides a double section of $\mathcal{O}(2)$, which has a node singularity over $[0:1] \in \mathbb{C}P^1$. See Example 2.18.

- The polynomial u^3v, as a section of $\mathcal{O}(4)$ provides a double section of $\mathcal{O}(2)$, which has a cusp singularity over $[0:1] \in \mathbb{C}P^1$. See Example 2.19.

3 Singular fibers in elliptic fibrations

Consider $f : M \to \mathbb{C}P^1$ elliptic fibration, with $F_t = f^{-1}(t)$ singular fiber for $t \in \mathbb{C}P^1$. If we restrixt the fibration to a small circle (which contains no more singular fiber) around t, then by traversing along this circle we get a diffeomorphism of the typical fiber T^2 torus.

Definition 3.1. The Γ_1 group of orientation preserving homeomorphisms of the torus, up to isotopy and conjugation, is called the mapping class group of the torus.

Here under isotopy we mean $h_t : T^2 \to T^2$ homotopy, such that $h_t(T^2)$ is diffeomorphic to T^2 for all t. Thus each singular fiber in the fibration determines an element in the mapping class group, which is called the mondoromy of the singular fiber. The Γ_1 mapping class group has presentation

$$\Gamma_1 = \langle a, b | aba = bab, (ab)^6 = 1 \rangle.$$

The symbols a and b can be interpreted as Dehn twists around the T_1, T_2 cycles of the torus, thus providing self-homeomorphisms. There is also a map sending a to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and b to $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, which provides an isomorphism $\varphi : \Gamma_1 \to SL_2(\mathbb{Z})$.

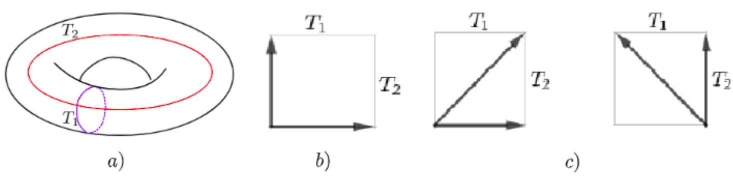

Figure 4: a,b) T_1, T_2 cycles on the torus c) The Dehn twists of the torus shows the integer matrix correspondence.

This way, the monodromy of singular fibers can be given by 2×2 integer matrices. We assume, that the fibration we investigate are relatively minimal, that is, no fiber contains an embedded sphere with self-intersection -1, because such sphere could be blown down, without changing the structure of the fibration. We also assume, that the fibration contains no multiple fibers, equivalently the fibration admits section (see [16, Prop. 2.1]). Accordingly the fibrations we consider, all originate from pencil by blow ups, thus all admit sections.

There is a method for construct fibration, based on the mondromies of singular fibers. This construction differs from the ones, we will apply in Section 4, however it shows, what kind of configurations of singular fibers can occur. Suppose, that ω is a word in Γ_1, which is a composition of 12 right-handed Dehn twists $\omega_1, ..., \omega_{12}$ around the T_1, T_2 cycles ($\omega_i \in \{a, b\}$). Now if

$$\omega = \omega_1 \cdots \omega_{12} = (\omega_1 \cdots \omega_{i_1})(\omega_{i_1+1} \cdots \omega_{i_2}) \cdots (\omega_{i_k+1} \cdots \omega_{12}) = 1$$

satisfies, and $(\omega_{i_j+1} \cdots \omega_{i_{j+1}})$ are conjugate to the monodromies of F_j $(j = 1, ..., k)$ singular fibers, then there is a fibration on the rational elliptic surface with singular fibers $F_1, ..., F_k$. For more details see [16, Section 2.2].

3.1 Classification of elliptic singular fibers

The singular fibers in elliptic fibrations have been classified by Kodaira in [9], providing the following theorem.

Theorem 3.2 (Kodaira). *A singular fiber of a locally holomorphic elliptic fibration without multiple fibers is one of the following types:* I_n $(n \geq 1)$, II, III, IV, I_n^* $(n \geq 0)$, $\widetilde{E_6}, \widetilde{E_7}, \widetilde{E_8}$.

In the following we will review the most important topological datas of the singular fibers of this list. See [16, Section 2.1].

***I_n*-fibers ($n \geq 1$).** The I_1 fiber (also called fishtail corresponds to the complex curve described in Example 2.18. For $n \geq 2$, the fiber I_n is the chain of n spheres of self-intersection -2 along a circle, where each one intersects transversally in a unique point the neighbouring spheres (which are connected by and edge on the chain graph). The monodromy of I_n is conjugate to a^n, or in matrix form to $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. The monodromy of I_1 is also conjugate to b, which can be easily seen from the matrix form of the monodromy. The Euler characteristics $\chi(I_n) = n$, and one can conclude, that the generic elliptic fibration on $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ contains 12 fishtail fibers.

Type *II* fiber. This fiber (also called cusp) corresponds exactly to Example 2.19, and has monodromy conjugate to ba, or to

$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. It has Euler characteristic 2, so one can obtain at most six of them in elliptic fibration on $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$.

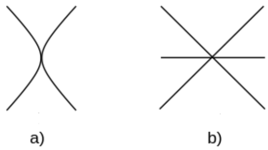

Figure 5: Singular fibers types III (a) and IV (b).

Type *III* fiber. This fiber is topologically the union of two spheres of self-intersection -2, tangent to each other in a unique point. The monodromy of this fiber is $aba = bab$ or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, while $\chi(III) = 3$.

Type *IV* fiber. Topologically this fiber is the union of three spheres of self-intersection -2, intersecting each other transversally in a unique point. Thus its Euler characteristic equals to 4, and its mondoromy is conjugate to $baba$, or in matrix terms to $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

I_n^*-fibers ($n \geq 0$). The structure of these fibers can be described with chain graphs on Figure 6.

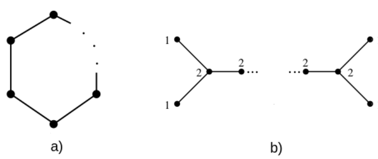

Figure 6: Chain diagram of fibers I_n (a) and I_n^* (b).

Here the vertices denote (-2)-spheres, while two vertices are connected with an edge, if the spheres intersect each other. The numbers on the vertices denote homological multiplicity of the spheres. The monodromy of the I_n^* fibers are $(ab)^3a^n$, or $\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$. Since the fiber I_n^* contains $n+5$ spheres, we can see $\chi(I_n^*) = n+6$.

The $\widetilde{E_6}, \widetilde{E_7}, \widetilde{E_8}$ fibers. These singular fibers are the least interesting for us, so we will omit their description. We focus on fibrations with singular fiber I_1^*, which has Euler characteristics 7, while $\widetilde{E_6}, \widetilde{E_7}, \widetilde{E_8}$ fibers has Euler characteristics $8, 9, 10$. This shows, that I_1^* can not appear with any of these fibers in fibration on $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$, because $\chi(\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}) = 12$.

3.2 Possible singular fiber combinations with I_1^*

The monodromy of I_1^* is $(ab)^3b$, hence from the expression

$$1 = (ab)^6 = (ab)^3b(ababa)$$

we can see, that the other singular fibers next to I_1^* have to provide the word of monodromies $ababa = aabaa$ (since $bab = aba$), according to the construction described above. Now we can obtain the following possibilities: $a + baba$ a fishtail and a type IV fiber, $a + bab + a$ two fishtails and a type III fiber, $aba + ba$ a type III fiber and a cusp, $a + ba + ba$ a fishtail and two cusps, $aa + ba + a$ an I_2 fiber, a cusp and a fishtail, $aa + b + aa$ two I_2 fibers and a fishtail, $a + a + ba + a$ a cusp with three fishtails, $aa + b + a + a$ an I_2 fiber with three fishtails, and $a + a + b + a + a$ five fishtails.

Not trivially, but we can also notice, that $aabaa = a^3(a^{-1}baa) = a^3(ba)^{a^{-1}}$ shows that an I_3 fiber and a cusp, $ababa = (ababa^{-1})aa^2 = (bab)^a a^2$ shows that a type III fiber and an I_2 fiber, $aabaa = aaaaa^{-2}ba^2 = a^4(b)^{a^{-2}}$ shows that an I_4 fiber and a fishtail, while $aabaa = aaaa^{-1}baa = a^3(b)^{a^{-1}}a$ shows that an I_3 fiber and two fishtails can be also obtained in the fibration containing I_1^* fiber.

Finally I_5 and $I_2 + I_3$ can be still set out from the letters next to I_1^* fiber, but the existence of these fibrations are excluded (see [16, Section 3]). And one can see, that there is no more possibility.

$I_4 + I_1$	$IV + I_1$	$I_3 + 2I_1$	$I_3 + II$
$III + I_2$	$III + II$	$III + 2I_1$	
$2I_2 + I_1$	$I_2 + II + I_1$	$I_2 + 3I_1$	
$2II + I_1$	$II + 3I_1$	$5I_1$	

Table 1: Possible singular fibers next to I_1^*

4 Pencils and fibrations with fiber I_1^*

In this Section we will give explicit constructions of elliptic fibrations with fiber-type I_1^* on the Hirzebruch surface $\mathbb{F}_2$, and on $\mathbb{C}P^2$ in some cases. The method will be similar as in the previous works in this topic [16], [6], [7], [8]. It will go as follows.

First we describe the $\mathbb{C}P^2$-case. Let p_1 and p_2 be two homogenous degree-3 polynomials in three variables, and their zero-sets two cubic curves C_1 and C_2 on the surface. Consider the pencil (see Section 2.2) generated by p_1, p_2:

$$\{p_t = t_1p_1 + t_2p_2 \mid t = [t_1 : t_2] \in \mathbb{C}P^1\}$$

and complex curves C_t defined by its elements. The intersection points of these curves are the base points of the pencil, as we defined earlier: $Z(p_1, p_2) = \{P \in \mathbb{C}P^2 | p_1(P) = p_2(P) = 0\}$. We suppose that $(p_1, p_2) = 1$, so that we have fintely many base points. According to Bézout-theorem, C_1 and C_2 have 9 intersection points counted with multiplicity, since p_1, p_2 are cubics. Counting them without multiplicity, we get the number of base points: $k \leq 9$. The map

$$\pi : \mathbb{C}P^2 \setminus Z(p_1, p_2) \to \mathbb{C}P^1, \qquad P \mapsto [p_1(P) : p_2(P)]$$

is well-defined, and holomorphic. Now consider the resolution of the indeterminacy of π, which is a 9-fold blow up of $\mathbb{C}P^2$ in the k base points: in each point a number of infinitely close blow ups, equal to the multiplicity of the intersection of C_1 and C_2 is in that base point. Thus the map π extends on the exceptional curves of the blow ups, and we get a $\pi : \mathbb{C}P^2\#9\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ holomorphic fibration, whose generic fibers are biregular to cubic curves, hence the generic fiber is a smooth elliptic curve (i.e. a torus). One can easily see, that if all fibers were smooth tori, then $\chi(\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}) = 0$, but it equals to 12 (see Section 2.4). This shows the existence of singular fibers in this construction, now our job is to choose the polynomials p_1, p_2 properly, in order to get the desired combination of singular fibers.

The construction of the elliptic fibrations on the second Hirzebruch surface $\mathbb{F}_2$ (see Section 2.5) will be similar, yet a little bit different. Similarly we pick two curves on the surface denoted by C_0, C_∞, and consider the pencil, that they generate. Precisely $C_0 = C(q)$, and $C_\infty = 2\sigma_\infty + F_1 + 2F_2 + F_3$ (denoted as divisors). Here F_1, F_2, F_3 are fibers of the ruling $p : \mathbb{F}_2 \to \mathbb{C}P^1$, and $C(q), \sigma_\infty$ are sections of the ruling. The σ_∞ is infinity section, that is we choose the infinity point in each $\mathbb{C}P^1$ fiber of the ruling: $\sigma_\infty = \{[1_x : 0_x] | x \in \mathbb{C}P^1\}$, where x parametrizes the underlying $\mathbb{C}P^1$. The $C(q)$ is disjoint from σ_∞, and will be provided by the solutions of the equation $C(q) = \zeta^2 - p^*q$, for the pullback of q degree-4 homogenous polynomial in the homogenous coordinates $[u : v]$ on $\mathbb{C}P^1$, as a section of $\mathcal{O}(4) \to \mathbb{C}P^1$ holomorphic bundle (see Section 2.5). Here we use the identification $\mathcal{O}(2) \otimes \mathcal{O}(2) \cong \mathcal{O}(4)$, to get a section of the bundle $\mathcal{O}(2) \to \mathbb{C}P^1$. This is necessary to get the wanted I_1^* fiber after the blow ups, as we will see later. Thus we take those $\zeta \in \mathcal{O}(2)$, which satisfy $\zeta_P \otimes \zeta_P = C(q)_P$ over each $P \in \mathbb{C}P^1$. This way we get σ_1, σ_2 components of the section $C(q)$, which are sections in $\mathcal{O}(2)$, if q is a complete square, or one double section in $\mathcal{O}(2)$, if q is not a complete square.

We need to check, that the two curves C_0, C_∞ are homologous, and hence there is a pencil of curves containing both. All the components $C(q), F_i$ are diffeomorphic to $\mathbb{C}P^1$, hence they can be represented with a single element in $H_2(\mathbb{F}_2, \mathbb{Z})$, which is generated by $[C(q)], [F]$. Obviously $[F_1] = [F_2] = [F_3] =: [F]$, and there is a linear connenction between σ_0 section of the ruling, F and σ_∞, (and thus between their homology classes), as we saw in Section 2.5. This looks like: $[\sigma_\infty] = [\sigma_0] - 2[F]$. By computing the intersection number of both sides with $[\sigma_0]$, we can conclude that $[\sigma_0]^2 = 2$. Hence $[C_0] = [C(q)] = 4[F] + 2[\sigma_\infty] = [C_\infty]$, shows that the two curves are homologous.

In some cases we have to construct more than two singular fibers in the fibration, so we need more curves than C_0, C_∞ to describe these cases. To this end, we need the following lemma (see [6]).

Lemma 4.1. *Let C' and C'' two homologous curves in $\mathbb{F}_2$, intersecting each other in points $P_1, ..., P_k$, with multiplicities $n_1, ..., n_k$, satisfying: $\sum_{i=1}^k n_i = 8$. Let C be a complex curve in $\mathbb{F}_2$ homologous to C' and C'', passing through $P_1, ..., P_k$, intersecting C' in these points with multiplicities $n_1, ..., n_k$. Then C lies in the pencil generated by C' and C''.*

Proof. Pick any point P on C different from $P_1, ..., P_k$. There exists a unique C_t element of the pencil

$$\{C_t = t_1 C' + t_2 C'' \mid t = [t_1 : t_2] \in \mathbb{C}P^1\}$$

going through P. Also $\sum_{i=1}^k n_i = 8$ is the self intersection number of C. Let C_t intersect C with multiplicity N in P. But then the intersection number of C and C_t is: $N + \sum_{i=1}^k n_i > 8$. This can only occur, if $C = C_t$, and C lies in the pencil. ⊠

We will apply this lemma in some cases with $C' = C_0 = C(q)$, $C'' = C_\infty$ and in other cases with $C' = C(q_1), C'' = C(q_2)$, where q_1, q_2 are two different homogenous degree-4 polynomials. The points $P_0, ..., P_k$ will be the base points of the pencil generated by the two curves. That means, we have to apply 8 blow ups in the base points (perhaps some of them infinitely close, if $n_i > 1$), in order to resolve indeterminacy. Similarly to the $\mathbb{C}P^2$-case, to invoke the combination of singular fibers we want, our job is to choose homogenous degree-4 polynomials q_1, q_2, and arrange the base points $P_0, ..., P_k$ properly. Let us see the details.

4.1 The construction of the fibrations

We will go through the 13 cases listed in Table 1, the configurations containing I_1^* as one of the singular fibers in elliptic fibration on the rational elliptic surface, and give the promised examples of the pencils, providing these singular fiber configurations.

Case 1. (fibration with an I_4 fiber) Such a fibration can be obtained simply by a proper geometric arrangement of two curves, generating the pencil, thus we will describe the $\mathbb{C}P^2$-case first. Let $C_1 = l_1 + 2l_2$ (denoted red on the pictures), two lines in $\mathbb{C}P^2$, one of them with multiplicity 2, intersecting each other in P. Let $C_2 = L_1 + L_2 + L_3$ (denoted blue on the pictures), three lines, L_1 passing through P, and L_2, L_3 intersecting each other on l_1, but not in P. Consider the pencil generated by these cubic curves, and blow up the projective plane at the base points. To see what happens, we describe this in details.

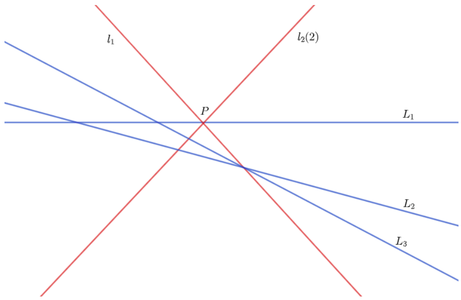

Figure 7: Step 1 of Case 1

Since $H_2(\mathbb{C}P^2, \mathbb{Z}) \cong \mathbb{Z}$, denote its generator with h. This means, $h = [H]$ is the class of the generating hyperplane H in $\mathbb{C}P^2$. From Bézout-theorem the intersection numbers: $C_1 \cap H = C_2 \cap H = 3$, hence $[C_1] = [C_2] = 3h$ in the homology group. First apply the blow up at point P.

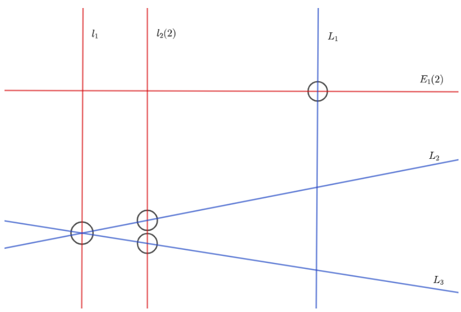

Figure 8: Step 2 of Case 1

The proper transforms of the curves C_1, C_2 will be $C_1 + aE_1$, $C_2 + bE_1$ for some a, b. The intersection number of the exceptional curve E_1 with $\widetilde{C_1}$ is 3 (because $[C_1] = 3h$), and with $\widetilde{C_2}$ is 1 (because $[L_1] = h$, and L_2, L_3 don't go through P).

The self-intersection number of the exceptional curve is always -1 (see Section 2.3). Thus by taking the intersection number of both sides of $[\widetilde{C_1}] = [C_1] + a[E_1]$ with $[E_1]$, we get

$$3 = [\widetilde{C_1}] \cdot [E_1] = [C_1] \cdot [E_1] + a[E_1][E_1] = 0 - a,$$
$$1 = [\widetilde{C_2}] \cdot [E_1] = [C_2] \cdot [E_1] + a[E_1][E_1] = 0 - b,$$

so $a = -3, b = -1$, and the homology classes of the proper transforms $\widetilde{C_1}, \widetilde{C_2}$ will be $3h - 3e, 3h - e$, where e represents the homology class of the exceptional divisor of the blow up. Thus in the proper transform of the pencil, the exceptional divisor will appear in the fiber of C_1 with multiplicity 2, as we saw in Section 2.3. Now apply 4 blow ups at the 4 denoted points in Figure 8.

Similarly to the previous step, after the blow up at $l_1 \cap L_2 \cap L_3$, the proper transforms will be $l_1 - E_2$ and $L_2 + L_3 - 2E_2$ (because

$l_1 \cap E_2 = 1$ and $(L_2 + L_3) \cap E_2 = 2)$, representing homology classes $h - e$ and $2h - 2e$, so E_2 will appear in the fiber of L_2, L_3 (and hence the fiber of C_2), with multiplicity 1.

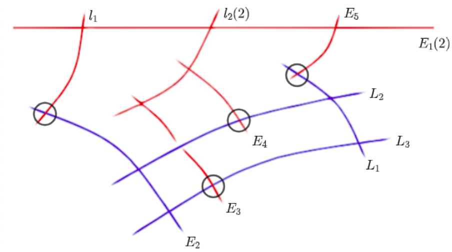

Figure 9: Step 3 of Case 1

At the point $l_2 \cap L_3$, the proper transforms will be $2l_2 - 2E_3$ (because l_2 has multiplicity 2), and $L_3 - E_3$, with homology classes $2h - 2e$ and $h - e$, so E_2 joins to the fiber of l_2 (which is the fiber of C_1), with multiplicity 1. And the same with $l_2 \cap L_2$ and E_4.

At the point $L_1 \cap E_1$ the line E_1 has multiplicity 2, while L_1 has multiplicity 1, so the curve E_5 joins to the fiber of C_1, with multiplicity 1. Finally apply 4 more blow ups, at the 4 denoted points. One can see, that the new exceptional curves will be sections:

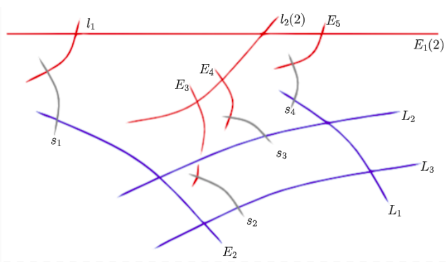

Figure 10: Step 4 of Case 1

At each of the 4 base points the intersecting curves have multiplicity 1, so they represent $h \in H_2(\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}, \mathbb{Z})$, and their proper transforms represent $h - e$, thus the exceptional curves s_i, $(i = 1, 2, 3, 4)$ (denoted grey), won't appear in any of the fibers, they will be sections.

Consider the fibers of C_1 (red), and C_2 (blue) in Figure 10. We can see, that the first one is of type I_1^*, the second one is of type I_4. We applied 9 blow ups on $\mathbb{C}P^2$, so we got the fibration $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ with fiber-types I_1^*, I_4, and necessarily a further I_1 fishtail fiber by the classification of possible combinations.

This looks very similar on the Hirzebruch surface: consider C_∞, C_0 as the generators of the pencil, where C_∞ as above, and C_0 has two components σ_1, σ_2, which are sections of the ruling, and intersect each other once on F_1 and once on F_3 (see Fig. 11). Such

C_0 can be obtained e.g. with $q(u, v) = u^2v^2$ as section of $\mathcal{O}(4)$, and thus F_1 and F_3 are over the points $[1 : 0], [0 : 1] \in \mathbb{C}P^1$, where the two components of $q(u, v) = u^2v^2$ intersect each other. In order to get the desired geometric arrangement, it is importatnt, that multiplicity-2 fiber F_2 is not over these points.

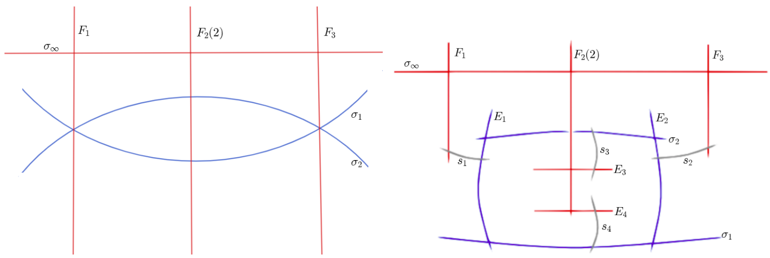

Figure 11: The two diagrams show the pencils before and after the 8 blow ups on $\mathbb{F}_2$.

With blow ups at $F_1 \cap \sigma_1 \cap \sigma_2$ and $F_3 \cap \sigma_1 \cap \sigma_2$ we get the exceptional curves E_1, E_2 in the fiber of C_0, and after two more blow ups the sections s_1, s_2. At each of $F_2 \cap \sigma_1$ and $F_2 \cap \sigma_2$ we need two blow ups as well, to see the exceptional divisors E_3, E_4, and the sections s_3, s_4 also. These steps can be checked by very similar homological computations, as above. See the final arrangement in Figure 11 (right), and notice, that the red fiber is of type I_1^*, while the blue one is of type I_4. Thus we got an elliptic fibration on the 8-fold blow up of $\mathbb{F}_2$ (which is a rational elliptic surface), with singular fibers I_1^*, I_4, and necessarily with an I_1 (because of the list of possible combinations).

Case 2. (fibration with a type IV fiber) This case is very similar to the previous one: the right arrangement of the generators of the pencil will guarantee the existence of the corresponding fiber types (first on $\mathbb{C}P^2$, and on $\mathbb{F}_2$ as well). Let $C_1 = l_1 + 2l_2$ as in Case 1, and let $C_2 = L_1 + L_2 + L_3$, three lines in $\mathbb{C}P^2$ going through a same point, while L_1 goes through $P = l_1 \cap l_2$ (see Fig. 12 left). We will give just a rough sketch of the steps of the blow up processes at the base points, because the guiding homological computations are very similar to Case 1.

First blow up at P, this concerns the same lines as in Case 1, so the result will be the same. The exceptional divisor E_1 will appear in the fiber of C_1 with multiplicity 2. Now apply 5 blow ups at the 5 denoted points in Figure 12 (right).

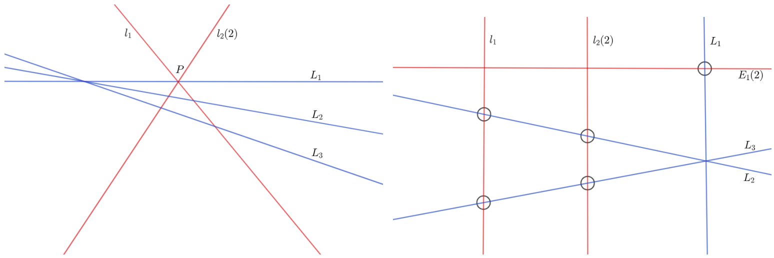

Figure 12: The first two steps of the blow ups on $\mathbb{C}P^2$ in Case 2.

After blow up at $l_2 \cap L_2$, the proper transforms will be $l_2 + aE_2$, and $L_2 + bE_2$, for exceptional curve E_2, and for some a, b. The intersection numbers of E_2 with the proper transform of l_2 is 2, because l_2 has multiplicity 2, and thus the intersection number of E_2 with the proper transform of L_2 is 1. The intersection number of E_2 with l_2, L_2 is 0, and with itself is -1. So $a = -2, b = -1$, and hence the homology classes represented by the two proper transforms, will be $2h - 2e_2, h - e_2 \in H_2(\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}, \mathbb{Z})$. The comparison with the homology classes before the blow up shows, that E_2 will appear in the fiber of C_1, with multiplicity 1.

The same calculation holds for $l_2 \cap L_3$ and $L_1 \cap E_1$, while at $l_1 \cap L_2$ and $l_1 \cap L_3$ the multiplicity of the intersecting lines coincide, so the exceptional curves will be s_1, s_2 sections (see Fig. 13 left).

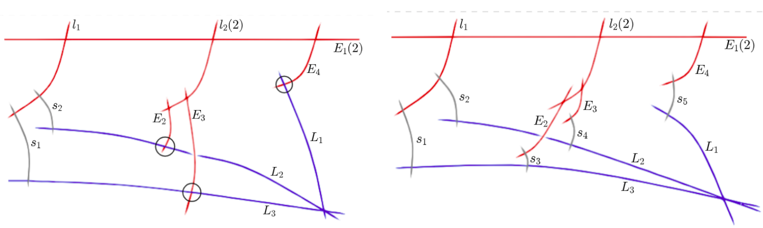

Figure 13: The remainig steps of the blow ups on $\mathbb{C}P^2$ in Case 2.

Finally if we apply the 3 blow ups denoted on the left diagram in Figure 13, then we get the s_3, s_4, s_5 sections, in the same way, as in the previous step. We can see the result after the 9 blow ups on the right diagram in Figure 13. The red fiber is of type

I_1^*, while the blue one is of type IV, exactly what we wanted. By the classification, there must be one more fishtail, i.e. an I_1 type fiber in this configuration, so we described the $I_1^* + IV + I_1$ case on $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$.

On the Hirzebruch surface set C_∞ as earlier, while C_0 a section of $\mathcal{O}(4)$. The two components σ_1, σ_2 are sections of the ruling, and they are tangent to each other at a point of F_1. For example let $q(u,v) = u^4$, it has two component, tangent to each other over $[0:1] \in \mathbb{C}P^1$, set the fiber F_1 over this point.

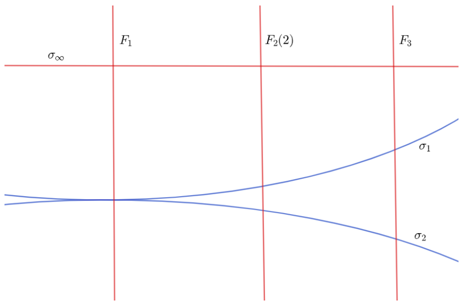

Figure 14: The pencil on $\mathbb{F}_2$ is generated by the red C_∞, and blue C_0 curves.

It is also important that the tangential point is not on the F_2 multiplicity 2 fiber. After blowing up in this base point, because of the tangency, the curve C_0 will not fall apart, but with the exceptional curve the σ_1, σ_2 will provide three lines passing through one point, this is how the fiber type IV appears (see Example 2.20). One can easily check, that after the 8 blow ups, we will get the same arrangement, as in the $\mathbb{C}P^2$-case.

In the remaining cases we need a slighltly different point of view, than in the previous ones. We have to specify more than one singular fiber-type next to the I_1^*, so we attack the problem from two sides. First we do, what we did in Case 1,2: determine a geometric arrangement of two curves (C_0, C_1 in $\mathbb{C}P^2$, and $C_0 = C(q_0), C_\infty$ in $\mathbb{F}_2$), the generators of a pencil, which contains I_1^*, and a certain singular fiber coming from the arrangement. We will not give details, just give the beginning position of the curves, because the homological computations go the same as earlier. On the other hand, we consider two curves $C_0 = C(q_0), C_1 = C(q_1)$ in $\mathbb{F}_2$, and investigate the pencil, which they generate. We settle the polynomilas q_0, q_1, so that the pencil contains the wanted singular fibers, and satisfy the conditions of Lemma 4.1. So the pencils generated by C_0, C_∞ and C_0, C_1 are the same, and we see the existence of two different types of singular fiber next to I_1^*. This also shows, that we will be more focus on the Hirzebruch surface case in the latters.

Case 3. (fibration with an I_3 fiber) In $\mathbb{C}P^2$, C_1 is again the union of two lines l_1, l_2, the second one with multiplicity two, while C_2 is the union of three lines: L_1 passing through $l_1 \cap l_2$, and L_2, L_3 intersecting each other in a generic point (which is not on L_1 and on any l_i). On $\mathbb{F}_2$ consider two sections σ_1, σ_2 of the ruling, intersecting each other in F_1 (which has multiplicity 1), and in a further point, which is not on the chosen F_2, F_3 fibers. One can easily check, that after the blow ups, these pencils will have an I_3 fiber next to the I_1^*.

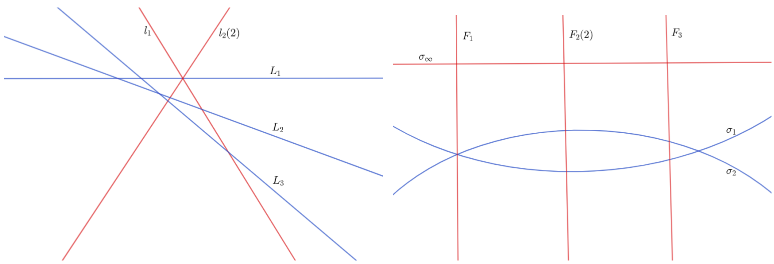

Figure 15: The diagram depicts two pencils (one in $\mathbb{C}P^2$ and the other one in $\mathbb{F}_2$) giving fibrations with an I_1^* fiber and an I_3 fiber.

Besides these two singular fibers, the fibration on the rational elliptic surface admits further singular fiber: either a cusp (II), or two fishtails ($2I_1$). Distinguish these two cases in the following.

a) ($I_1^* + I_3 + 2I_1$) Consider the pencil generated by $C(q_0), C(q_1)$, where $q_0(u,v) = u^2v^2$, and $q_1(u,v) = v(u+v)(u+\alpha v)^2$, two sections of $\mathcal{O}(4)$, for some $\alpha \in \mathbb{C}^\times$. The first one is a complete square, so the equation $\zeta \otimes \zeta = q_0$ over each point of $\mathbb{C}P^1$ defines two components σ_1, σ_2, which are sections of $\mathcal{O}(2)$. The two components intersect each other transversally over $[0 : 1], [1 : 0]$, this corresponds to the picture above. The second polynomial isn't a complete square, so the equation $\zeta \otimes \zeta = q_1$ defines a double section, with nodal point over $[u : v] = [-\alpha : 1]$. The base points of the pencil are determined by the equation

$$u^2v^2 = v(u+v)(u+\alpha v)^2.$$

The roots of this are $[u : v] = [1 : 0], [u_1 : 1], [u_2 : 1], [u_3 : 1]$, where u_1, u_2, u_3 are the roots of the cubic equation

$$u^2 = (u+1)(u+\alpha)^2.$$

First of all, consider the two intersection points of σ_1, σ_2: $[1 : 0]$ is a base point (the fiber F_1 lies above it in the above picture), while $[0 : 1]$ is not ($u_i = 0$ is not a root of the equation), so we will blow up the I_2 type fiber at one of the intersection points, hence it evolves to an I_3 type fiber. Second, q_1 provides an I_1 fiber with node over $[-\alpha : 1]$, and we do not blow up over this point, since it is not a base point ($\alpha \neq 0$), so the I_1 type fiber remains in the pencil. And third, the pencil must have its base points on the Hirzebruch surface over three different points. That is, because the intersection points of $C(q_0), C(q_1)$ must coincide with the intesection points of $C(q_0), C_\infty$, in order to satisfy Lemma 4.1. It means, that the equation $u^2 = (u+1)(u+\alpha)^2$ must have only two different roots (e.g. $u_2 = u_3$). We can provide it, by choosing an α, such that the discriminant of the polynomial $(u+1)(u+\alpha)^2 - u^2$ is 0.

$$\Delta_u(\alpha) = -4\alpha^5 + 13a^4 - 32a^3,$$

and the equation $\Delta_u(\alpha) = 0$, has roots $\alpha_\pm = \frac{13}{8} \pm \frac{7\sqrt{7}i}{8}$ besides $\alpha = 0$. One can easily check, that $(u+1)(u+\alpha_\pm)^2 - u^2 = 0$ has two different u_1, u_2 roots, so with $\alpha = \alpha_+$ or $\alpha = \alpha_-$, we can guarantee the conditions of Lemma 4.1, and we have the freedom, to place F_2, F_3 fibers over $[u_1 : 1], [u_2 : 1]$. Because of Lemma 4.1, the curves $C(q_0), C(q_1), C_\infty$ are in the same pencil, which contains fiber types I_1^*, I_3, I_1, and thus necassarily one more I_1.

(b) ($I_1^* + I_3 + II$) Similarly to case a), let $q_0(u,v) = u^2v^2$, and $q_1(u,v) = v(u+\alpha v)^3$. As above, over $[u:v] = [1:0]$ will be a base point, becasue it is a solution of the equation $u^2v^2 = v(u+\alpha v)^3$, but $[u:v] = [0:1]$ is not, so the singular fiber determined by q_0 will be an I_3 after the blow ups. The singular fiber determined by q_1 is a cuspidal curve, with singularity over $[-\alpha:1]$, and it isn't a base point, so the II type fiber remains in the pencil. Similarly to case a), the equation $u^2 = (u+\alpha)^3$ must have two different u_1, u_2 roots, which we can be ensured by setting the discriminant of $(u+\alpha)^3 - u^2$ to 0.

$$\Delta_u(\alpha) = 4\alpha^3 - 27a^4 = 0,$$

which has non-zero root $\alpha = \frac{4}{27}$. So if F_1, F_2, F_3 are above $[0:1], [u_1:1], [u_2:1]$ in $\mathbb{C}P^1$, and if $q_0(u,v) = u^2v^2$, and $q_1(u,v) = v(u+\frac{4}{27}v)^3$, then $C(q_0), C(q_1), C_\infty$ are in the same pencil, which contains fiber types I_1^*, I_3, II.

Case 4. (fibration with a type III fiber) Here $C_1 = l_1 + 2l_2$ as usual, while C_2 in the pencil on $\mathbb{C}P^2$ now consists of two curves: L_1 line and Q quadric, where L_1 passes through $l_1 \cap l_2$, and Q is tangent to L_1 (in a point, different from $l_1 \cap l_2$), and intesects l_1, l_2 in two-two general points (see Figure 16). On $\mathbb{F}_2$ let C_∞ be as usual, while let $C(q_0) = \Sigma$ be a cuspidal curve, with a cusp point on the fiber F_1. This is now a double section of $\mathcal{O}(4)$.

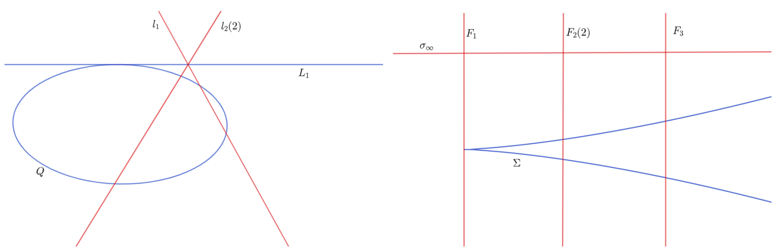

Figure 16: Continuing with the same conventions as earlier, the two diagram shows the pencils, which yield fibrations with I_1^* and III fibers in $\mathbb{C}P^2$ and in $\mathbb{F}_2$).

We will get the I_1^* fiber from the red curves, and easy to see the presence of III type fiber, since the tangency is already given in

$\mathbb{C}P^2$ between L_1 and Q, and in $\mathbb{F}_2$, we will blow up the cuspidal curve at its cusp point, so it will yield two tangent lines. There are three possibilities of the further singular fibers: an I_2 fiber, or a cusp (II), or two fishtails ($2I_1$), let's see them.

a) ($I_1^* + III + I_2$) Take $q_0(u,v) = u(u+v)^3$, and $q_1(u,v) = 4u^2v(u+v)$ degree-4 polynomials. The base points of the pencil they generate, are $[u:v] = [0:1], [-1:1], [1:1]$, because $v=0$ is not a solution, so we can take $v=1$, and now the equation $4u^2(u+1) = u(u+1)^3$ has three different roots ($u=-1,0,1$). This means, that we implement blow up over $[-1:1]$, where $C(q_0)$ has the cusp singularity, and over $[0:1]$ as well, where $C(q_1)$ has the node singularity. Hence the $C(q_0)$ cuspidal curve will turn to an III type fiber, and the $C(q_1)$ nodal curve will turn to an I_2 fiber. None of q_0, q_1 are complete squares, so they will determine double sections in $\mathcal{O}(4)$, with pairs of base points over $[0:1], [-1:1], [1:1] \in \mathbb{C}P^1$, and we have to consider the fibers F_1, F_2, F_3 over these points. It is important, that the 2-multiplicity fiber F_2 must be over $[1:1]$, becasue over $[0:1]$, and $[-1:1]$ we need exactly one-one blow up, to ensure the fibers we described above. By Lemma 4.1, we can see a pencil resulting fibration with fiber types I_1^*, III, and I_2.

b) ($I_1^* + III + 2I_1$) By taking polynomials $q_0(u,v) = vu^3$, and $q_1(u,v) = vu(u+v)^2$, we can see, that $C(q_0)$, and $C(q_1)$ are cuspidal and nodal curves, similarly as in the previous case. The difference is, that now we don't blow up over the point $[-1:1]$, where $C(q_1)$ has its nodal point, so this curve remains a fishtail fiber in the pencil. That is, because the base points are $[u:v] = [1:0], [0:1]$, and $[-\frac{1}{2}:1]$, since the solutions of $u^2 = (u+1)^2$ is only $-\frac{1}{2}$. The curve $C(q_0)$ has its cusp point over $[0:1] \in \mathbb{C}P^1$, so after the blow up, it will evolve to an III type fiber. Similarly to case a), if we consider the fibers F_1, F_2, F_3 on $\mathbb{F}_2$ over $[1:0], [0:1], [-\frac{1}{2}:1]$ (by watching that the multiplicity-2 fiber F_2 is not over $[0:1]$, where the cuspidal singularity turns to two tangent lines under the blow up process), then Lemma 4.1 shows the pencil containing singular fibers I_1^*, III, I_1, and also one more I_1 type fiber, dictated by the classification of possible configurations.

c) $(I_1^* + III + II)$ Now consider the two cuspidal curves $C(q_0)$, $C(q_1)$, where $q_0(u,v) = vu^3$, and $q_1(u,v) = u(u+\alpha v)^3$, for some $\alpha \in \mathbb{C}^\times$. Their cusp points are over $[0:1], [-\alpha:1]$. The pencil they generate, has base point over $[0:1]$, but not over $[-\alpha:1]$, if $\alpha \neq 0$, that is the first curve will yield an III type fiber after the blow ups, while the second curve remains an II type fiber in the pencil. Both curves provide double sections in $\mathcal{O}(4)$, and their pencil shall have its base points over three different points, so that C_∞, which gives the I_1^* fiber, is in the pencil of $C(q_0)$ and $C(q_1)$. To guarantee this, we have to choose the parameter α the way, that $u^2 - (u+\alpha)^3 = 0$ cubic equation has only two different roots (over $[u:v] = [1:0]$ is no base point, so we can assume $v = 1$). The discriminant of this cubic is
$$\Delta_u(\alpha) = 4\alpha^3 - 27a^4 = 0,$$
so if $\alpha = \frac{4}{27}$, then the equation has two different roots (u_1, u_2) as we saw earlier. Choose this value for α, and lay F_i-s over $[0:1], [u_1:1], [u_2:1]$ (F_2 is not over $[0:1]$ similarly to the previous cases), then this configuration will lead to pencil with fibers I_1^*, III, II.

Case 5. (fibration with an I_2 fiber) The pencil providing an I_1^* and an I_2 fiber on $\mathbb{C}P^2$ can be seen in Figure 17.

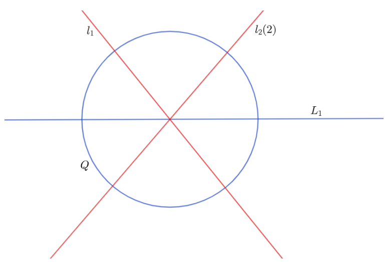

Figure 17: The pencil gives rise to a fibration with I_1^* and I_2 singular fibers on $\mathbb{C}P^2$.

The red curve C_1 is the same as earlier, while C_2 is the union of L_1 line passing through $l_1 \cap l_2$, and Q general positioned quadric curve, transverse to L_1. These two transverse curves (blue) will yield the I_2 fiber after the blow ups, and the red curve will yield the I_1^* fiber as usual. On the Hirzebruch surface the situation is more complicated, since there is two possibilities to arrange curves, so that their pencil admits these two singular fibers (see Figure 18).

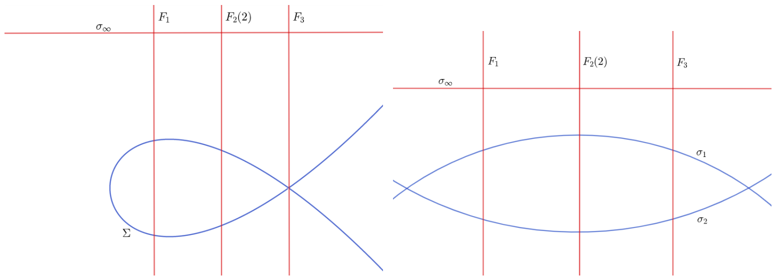

Figure 18: Both of this pencils give rise to fibration with I_1^* and I_2 fibers in $\mathbb{F}_2$.

On the left picture C_0 is a nodal curve, with F_3 going through its node point. This means, that we blow up the blue curve at its node, so it will became an I_2 singular fiber. On the right picture there are two sections of the ruling: σ_1, σ_2, and the F_i-s do not go through the intersection points of the two section. So the σ_1, σ_2, as two components of a curve, produce an I_2 fiber, which do not change under the blow ups. As usual, in both pictures the red C_∞ curve provides the I_1^* fiber. There are three cases for the further singular fibers: one more I_2 and an I_1, or a cusp and a fishtail $(II + I_1)$, or three fishtails $(3I_1)$.

a) $(I_1^* + 2I_2 + I_1)$ In this case, we will refer to the left picture of Figure 18, and consider $q_0(u, v) = u^2v(u + v)$ polynomial, and $C(q_0)$ curve with node point over $[0 : 1] \in \mathbb{C}P^1$. On the other hand taking $q_1(u, v) = uv^2(3u - v)$, the curve $C(q_1)$ has a node over $[1 : 0] \in \mathbb{C}P^1$. We can see, that both $[0 : 1]$, and $[1 : 0]$ are solutions of $u^2v(u + v) = uv^2(3u - v)$, so they are base points of the pencil, which $C(q_0), C(q_1)$ generate. Thus both nodal curves will became an I_2 fiber after blowing up at their node points. Besides these two, the further base points are determined by the solution of $u(u + 1) = 3u - 1$, which has only one root, $u = 1$. So by setting F_2 multilicity-2 fiber over $[1:1]$, and F_1, F_3 fibers over $[0:1], [1:0]$, we can apply Lemma 4.1, which shows us a pencil with singular fibers I_1^* and two I_2. There is only one possible configuration with these fibers, the $I_1^* + 2I_2 + I_1$ one, exactly what we wanted.

b) $(I_1^* + I_2 + II + I_1)$ If $q_0(u,v) = u^2(u+v)(u+\alpha v)$ for some $\alpha \in \mathbb{C}^\times$, and $q_1(u,v) = uv^3$, the curves $C(q_0), C(q_1)$ are a nodal and a cuspidal curve. Although $C(q_0)$ has its node over $[0:1]$, which is a base point, so this curve will yield an I_2 fiber after the blow ups. The $C(q_1)$ curve has its cusp over $[1:0]$, which isn't a base point, so this curve will provide a type II fiber in the pencil. Similarly to the earlier cases, we need to set α the way, that $u(u+1)(u+\alpha) = 1$ cubic equation has two different solutions. Check its discriminant:

$$\Delta_u(\alpha) = \alpha^4 + 2\alpha^3 - 5\alpha^2 - 6\alpha - 23,$$

and we can compute, that $\Delta_u(\alpha) = 0$ satisfies for example if $\alpha = \frac{1}{2}(-1-\sqrt{13+16\sqrt{2}})$, and the above equation has exactly two u_1, u_2 solutions. With this α-value, the pencil contains I_2 and II fibers, and I_1^* by the C_∞ curve, if F_1 is over $[0:1]$, and F_2, F_3 are over $[u_1:1], [u_2:1]$. The pencil with I_1^*, I_2, II fibers has necessarily an I_1 as well, by the classification.

c) $(I_1^* + I_2 + 3I_1)$ The a)- and b)-cases correspond to the left diagram of Figure 18, with C_0 double section of $\mathcal{O}(4)$ on the Hirzebruch surface. This case corresponds to the right diagram of Figure 18, by choosing $q_0(u,v) = u^2v^2$, which has two components σ_1, σ_2 as sections of $\mathcal{O}(2)$. Let $q_1(u,v) = (u+\alpha v)^2(u+v)(u-v)$, hence $C(q_1)$ is a nodal curve. One can easily see, that neither of $[0:1], [1:0]$, the intersection points of the components of $C(q_0)$, nor $[-\alpha:1]$, where $C(q_1)$ has its node, determines base points of the pencil ($\alpha \neq 0$). The base points can be found by solving $u^2 = (u+\alpha)^2(u+1)(u-1)$ degree-4 equation ($v \neq 0$, so assume $v = 1$). We need to find an α, such that the below discriminant of the equation is 0.

$$\Delta_u(\alpha) = 16\alpha^8 - 96\alpha^6 - 240\alpha^4 - 128\alpha^2,$$

which is zero e.g. if $\alpha = 2\sqrt{2}$. One can easily check, that with this value the above equation has 3 different roots u_1, u_2, u_3, thus F_1, F_2, F_3 must be above $[u_1 : 1], [u_2 : 1], [u_3 : 1]$ points in $\mathbb{C}P^1$ to satisfy the conditions of Lemma 4.1. All together, we got a pencil with singular fibers of type I_1^*, I_2, I_1. The problem is, that there are more possible configurations with these fibers, for example the a)- and b)-cases are such. In order to find out, what configuration we get with these curves, consider the general element of the pencil.

$$q_\lambda = \lambda_0 q_0 + \lambda_1 q_1 = \lambda_0 u^2 + \lambda_1 (u + 2\sqrt{2})^2 (u+1)(u-1),$$

where $\lambda = [\lambda_0 : \lambda_1] \in \mathbb{C}P^1$ the parameter of the pencil, and we take $v = 1$. This is the general fiber of the fibration determined by the pencil, which is a singular fiber, if the discriminant the polynomial is 0. The discriminant of the general element is

$$\Delta_u(\lambda_0, \lambda_1) = 8\sqrt{2}\lambda_0^3\lambda_1 + 97\lambda_0^2\lambda_1^2 + 232\sqrt{2}\lambda_0\lambda_1^3 + 196\lambda_1^4.$$

For example, if $\lambda_1 = 0$, then the discriminant is 0, and we get the $C(q_0)$ singular fiber of type I_2. Besides this, we can assume, that $\lambda_1 = 1$, hence the discriminant is

$$\Delta_u(\lambda_0) = 8\sqrt{2}\lambda_0^3 + 97\lambda_0^2 + 232\sqrt{2}\lambda_0 + 196.$$

This is a cubic polynomial, and one can easily check, that it is general, i.e. it has three distinct roots. This means, that next to the I_1^* and I_2 fibers, there must be three more singular fibers in the pencil, so it can be only the $I_1^* + I_2 + 3I_1$ case (in the other cases with I_1^* and I_2, there are only 2 more singular fibers).

The three remaing cases are the most general ones, i.e. the involved cusp and fishtail fibers do not require special position with respect to the C_∞ curve on $\mathbb{F}_2$, while in the previous cases, there needed to be a F_i fiber going through the singular points of these curves. For this general positions see Figure 19 (cusp case left, fishtail case right). This also means, that we will not apply any blow up over the cuspidal or nodal points, therefore the II and I_1 singular fibers will directly appear in the pencils, independently from the implemented blow ups. With this arrangement we can only ensure the existence of one-one II or I_1 fiber next to I_1^*, but these fiber types appear in many configurations. So we will have similar arguments as in Case 5 c), with respect to the number of singular fibers in the pencil, from which we can conclude the certain combination of singular fibers.

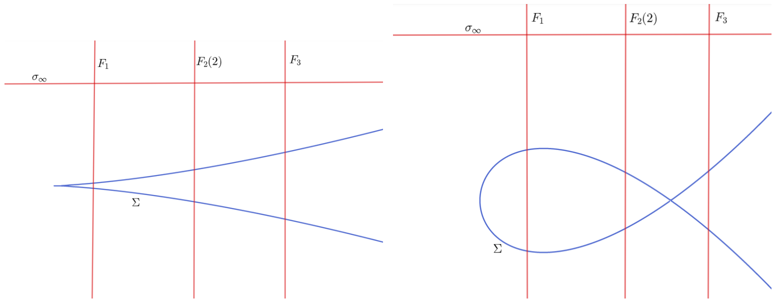

Figure 19: In the following cases the cuspidal and nodal curves will have general position in the pencil on the Hirzebruch surface.

Case 6. (fibration with one or two II type fibers)

a) $(I_1^* + 2II + I_1)$ In order to provide two cusp singularities, consider the polynomials $q_0(u,v) = u^3v$, and $q_1(u,v) = (u+1)^3(u+\alpha v)$. It's easy to see, that the base points of the pencil generated by $C(q_0)$, $C(q_1)$ can be found by solving $u^3 = (u+1)^3(u+\alpha)$, and points $[0:1], [-1:1]$ (where the cusp singularities are) are not among them. This corresponds to the above description. The discriminant of the equation is

$$\Delta_u(\alpha) = -27\alpha^4 + 162\alpha^3 + 27\alpha^2 + 90\alpha - 23,$$

which is zero e.g. at the value $\alpha = \frac{3}{2} + \sqrt{3} - \frac{1}{6}\sqrt{153 + 100\sqrt{3}}$. Choosing this value for α, the pecil will have its base points over three different points of $\mathbb{C}P^1$, thus we can arrange the C_∞ curve the way, that it appears in the pencil. This pencil now gives rise to singular fibers I_1^*, and two II-type fibers, and by the list of possible configuration this can be only the $I_1^* + 2II + I_1$.

b) $(I_1^* + II + 3I_1)$ This case will go just the same, as we described before the beginning of Case 6. A nodal and a cuspidal curve is determined by the homogenous degree-4 polynomials $q_0(u,v) = u^3v$, and $q_1(u,v) = (u+\alpha v)^2(u+v)(u-v)$ ($\alpha \in \mathbb{C}^\times$), which generate a pencil with II, and I_1 fibers. Similarly to the previous cases, we need to settle α, so that $u^3 = (u+\alpha)^2(u+1)(u-1)$ has three different roots. Its discriminant is

$$\Delta_u(\alpha) = 20\alpha^3 - 167\alpha^4 + 132\alpha^5 - 296\alpha^6 + 48\alpha^7 + 4\alpha^8 - 16\alpha^9.$$

We can numerically compute, that if $\alpha \approx 0.129214$, then it is zero, and then the above equation has three different solution. We get a fibration with I_1^*, II, I_1 fibers, hence we still have to enumerate the other singularities in the pencil, by investigating its general element. For $\lambda = [\lambda_0 : \lambda_1] \in \mathbb{C}P^1$ parameter

$$q_\lambda = \lambda_0 q_0 + \lambda_1 q_1 = \lambda_0 u^3 + \lambda_1 (u+\alpha)^2(u+1)(u-1),$$

has the following discriminant (we can assume $\lambda_1 = 1$)

$$\Delta_u(\lambda_0) = \lambda_0^4 27\alpha^4 + \lambda_0^3(-180\alpha^5 - 4\alpha^3) + \\ +\lambda_0^2(4\alpha^8 - 296\alpha^6 - 140\alpha^4) + \lambda_0(16\alpha^9 - 48\alpha^7 + 48\alpha^5 - 16\alpha^3).$$

Substituting the numerical $\alpha \approx 0.129214$ value, we get a degree-4 polynomial in λ_0, and one can easily check with the help of computer, that it has four different roots (which is not surprising, because that is the general case). This shows, that the pencil now has 4 singular fibers next to I_1^*, one of them is a cusp, and one of them is a fishtail, and by browsing the list of possible configuration, we can notice, that only the $I_1^* + II + 3I_1$ have these features.

Case 7. (fibration with five I_1 fibers) This case goes the same, as the previous one, the difference is, that here we choose two general nodal curves, and

$$q_0(u,v) = u^2v(u+v)$$
$$q_1(u,v) = (u+\alpha v)^2(u+2v)(u-v)$$

polynomials providing them. Now we will omit the tidious, computer supported computations, which show, that for $\alpha \approx -3.25852$, the base points of the pencil are over three distinc $\mathbb{C}P^1$ points (and none of them is such, where the curves have their node points). By investigating the discriminant of the q_λ general element of the pencil (which we will now also omit, becasue it is just the same as in the previous computation), we can see that there are five more singular fibers next to I_1^*, so it can be only the $I_1^* + 5I_1$ configuration.

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