The universal von Neumann algebra of smooth four-manifolds revisited

Gábor Etesi

Budapest University of Technology and Economics
Department of Geometry
HUNGARY

New York City Noncommutative Geometry Seminar
26th October 2022 (online from Budapest)

PLAN

* Rapid introduction to smooth 4-manifolds
* Construction of a von Neumann algebra
* Construction of a new smooth 4-manifold invariant (sketch)
* Application to quantization of gravity in 4 dimensions (if time remains)

Smooth 4-manifolds

The basic problem: Assume that a topological space X admits a finite dimensional manifold structure. There are two essentially different possibilities: either X carries a C^0 or topological manifold structure X or a C^∞ or smooth manifold structure M. Given a toplogical manifold X, can it be refined in a compatible way (i.e., "smoothened") to an M, or equivalently: Does X admit any compatible smooth structure M (existence)? If yes, is this structure unique (uniqueness)?

Why is four dimensions so special?

(i) If $\dim_{\mathbb{R}} X \leqq 3$ then X always admits a smooth structure M which is unique (classical fact);

(ii) If $\dim_{\mathbb{R}} X \geqq 5$ and X is compact then it admits at most finitely (including zero) many different smooth structures (Sullivan);

(iii) If $\dim_{\mathbb{R}} X = 4$ then there exists a plethora of smooth structures and the situation is very complicated (Akbulut, Donaldson, Freedman, Gompf, Kirby, Taubes,...). It may happen that: X is compact and not smoothable at all (e.g. the simply connected space with intersection form E_8); it carries countably infinitely many smooth structures (e.g. the $K3$ surface); or we do not know yet how many (the case of S^4); X is not compact and the cardinality of different smooth structures reaches that of the continuum in ZFC set theory. Perhaps the most striking phenomenon:

Theorem

Let M be a smooth manifold which is homeomorphic to $\mathbb{R}^m$. Then M is diffeomorphic to $\mathbb{R}^m$ if $m \neq 4$. If $m = 4$ then there exist many non-countable families $\{R^4\}, \dots$ of pairwise non-diffeomorphic smooth 4-manifolds which are all homeomorphic but not diffeomorphic to $\mathbb{R}^4$ (such an R^4 is called a fake or exotic $\mathbb{R}^4$). $\diamond$

How to recognize or distinguish smoothness in four dimensions? (This question is important both mathematically and physically.) Try to construct computable but sensitive smooth 4-manifold invariants. Interestingly, four is the dimension of macroscopic physical space-time and the strongest invariants arrive from contemporary theoretical particle physics: The Donaldson invariant (from Yang–Mills or gauge theory) and the Seiberg–Witten invariant SW (from supersymmetric gauge theory). Why gravity theories (e.g. general relativity) cannot exhibit smooth invariants?

Remark

Two unsatisfactory properties of the Seiberg–Witten invariant (roughly): (i) If $\mathbb{C}P^2$ denotes the complex projective space then $SW(\mathbb{C}P^2) = 0$; (ii) If $\#$ denotes the connected sum operation on 4-manifolds then $SW(M\#N) = 0$. $\diamond$

Emergence of the II_1 hyperfinite factor

From the superabundance of smooth 4-manifolds one can destillate a single von Neumann algebra as follows:

Theorem (Etesi, 2017, 2022)

Let M be a connected oriented smooth 4-manifold. Making use of its smooth structure only, a von Neumann algebra $\mathfrak{R}$ can be constructed which is geometric in the sense that it contains algebraic (i.e., formal or coming from a metric) curvature tensors on M and $\mathfrak{R}$ itself is a hyperfinite factor of type II_1 (hence is unique up to abstract isomorphism of von Neumann algebras). $\diamond$

The construction is based on two simple steps and looks like this:

Step 1. Let M be a connected oriented smooth 4-manifold and consider $T^{(p,q)}M$, the bundle of (p,q)-type tensors over M. Among these bundles $\wedge^2 T^*M \subset T^{(0,2)}M$, the bundle of 2-forms, is the only one which admits a natural (i.e. defined without any additional structure) pairing over M: Writing $\Omega^2_c(M) := C^\infty_c(M; \wedge^2 T^*M)$ for the infinite dimensional real vector space of compactly supported smooth 2-forms, then for every $\alpha, \beta \in \Omega^2_c(M)$, an $\mathbb{R}$-bilinear symmetric pairing

$$\langle \alpha, \beta \rangle_{L^2(M)} := \int\limits_M \alpha \wedge \beta \tag{1}$$

exists provided by the orientation and the smooth structure of M only.

Observe that this pairing is non-degenerate however is *indefinite* in general thus can be regarded as an indefinite scalar product on $\Omega^2_c(M)$. It therefore induces an indefinite real quadratic form Q given by $Q(\alpha) := \langle\alpha,\alpha\rangle_{L^2(M)}$. Let $C(M)$ denote the complexified infinite dimensional Clifford algebra associated with $(\Omega^2_c(M), Q)$. If $\mathfrak{M}_k(\mathbb{C})$ denotes the algebra of $k\times k$ complex matrices then

$$C(M)\cong\bigcup_{n=0}^{+\infty}\mathfrak{M}_{2^n}(\mathbb{C}) \tag{2}$$

or equivalently

$$C(M)\cong\mathfrak{M}_2(\mathbb{C})\otimes\mathfrak{M}_2(\mathbb{C})\otimes\dots$$

and note that being (1) a non-local operation, this $C(M)$ is a genuine global infinite dimensional object.

The isomorphism (2) shows that $C(M)$ is a complex $*$-algebra whose $*$-operation (provided by taking Hermitian matrix transpose, a non-local operation) is written as $A \mapsto A^{\circledast}$ (note the difference in notation between $*$ and $\circledast$ at various places from now on) moreover $C(M)$ possesses a unit and its center comprises the scalar multiples of the unit only. If $A \in C(M)$ then one can pick the smallest $n \in \mathbb{N}$ such that $A \in \mathfrak{M}_{2^n}(\mathbb{C})$ consequently A has a finite trace defined by

$$\tau(A) := 2^{-n}\mathrm{Trace}(A)$$

i.e., taking the usual normalized trace of the corresponding $2^n \times 2^n$ complex matrix. It is straightforward that $\tau(A) \in \mathbb{C}$ does not depend on n. We can then define a sesquilinear inner product on $C(M)$ by

$$(A, B) := \tau(AB^{\circledast})$$

which is non-degenerate.

Step 2. Thus the completion of $C(M)$ with respect to the norm $\|\cdot\|$ induced by $(\cdot\,,\,\cdot)$ renders $C(M)$ a complex Hilbert space what we shall write as $\mathscr{H}$ and its Banach algebra of all bounded linear operators as $\mathfrak{B}(\mathscr{H})$.
Multiplication in $C(M)$ from the left on itself is continuous hence gives rise to a representation $\pi : C(M) \to \mathfrak{B}(\mathscr{H})$. Finally our central object effortlessly emerges as the weak closure of the image of $C(M)$ under π within $\mathfrak{B}(\mathscr{H})$ or equivalently, by referring to von Neumann's bicommutant theorem we put

$$\mathfrak{R} := (\pi(C(M)))'' \subset \mathfrak{B}(\mathscr{H}) \ .$$

This is a von Neumann algebra and of course admits a unit $1 \in \mathfrak{R}$ moreover continues to have trivial center i.e., it is a factor. Moreover by construction it is hyperfinite. The trace τ as defined has the form

$$\tau(A) = (A, 1) \tag{3}$$

and extends from $C(M)$ to $\mathfrak{R}$ and satisfies $\tau(1) = 1$. Moreover this trace is unique on $\mathfrak{R}$.

Consequently from an M a hyperfinite II_1-type factor von Neumann algebra $\mathfrak{R}$ has been extracted. Moreover whatever M was, this $\mathfrak{R}$ is unique up to abstract isomorphisms of von Neumann algebras.

How the operator algebra $\mathfrak{R}$ *looks like?* One can show that

$$\Omega^2_c(M;\mathbb{C}) \subset C(M) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$$

(the first embedding is canonical, the second is not) which permits to explore at least the $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$ part:

(i) (Some non-local operators) Take a Riemannian manifold (M,g) with $\int_M \mu_g = 1$ and let $\{\varphi_1,\dots,\varphi_i,\dots\}$ be an ONB in $\Omega^2_c(M;\mathbb{C})$. Then if $B\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ satisfies

$$\tau(B) = \lim_{n\to+\infty}\frac{1}{2^n}\sum_{i=1}^{2^n}(B\varphi_i\,,\,\varphi_i)_{L^2(M,g)} < +\infty \tag{4}$$

(which is the same as (3) above) we know that $B\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$;

(ii) (Some local operators) A peculiarity of four dimensions is that the subspace $C^\infty(M; \mathrm{End}(\wedge^2 T^*M)) \subset \mathrm{End}(\Omega^2_c(M))$ of bundle morphisms contains the space of algebraic (i.e., formal or coming from a metric) curvature tensors on M. E.g. if (M, g) is an oriented (pseudo-)Riemannian 4-manifold then its Riemannian curvature tensor R_g is a member of this algebra.

continued...

...continued

With respect to the decomposition of complexified 2-forms into their (anti)self-dual parts, as a map

$$R_g : \begin{matrix} \Omega^+_c(M;\mathbb{C}) \\ \bigoplus \\ \Omega^-_c(M;\mathbb{C}) \end{matrix} \longrightarrow \begin{matrix} \Omega^+_c(M;\mathbb{C}) \\ \bigoplus \\ \Omega^-_c(M;\mathbb{C}) \end{matrix}$$

the complexified curvature looks like

$$R_g = \begin{pmatrix} \frac{1}{12}\mathrm{Scal} + \mathrm{Weyl}^+ & \mathrm{Ric}_0 \\ \mathrm{Ric}^*_0 & \frac{1}{12}\mathrm{Scal} + \mathrm{Weyl}^- \end{pmatrix} \qquad (5)$$

that is, $R_g \in C^\infty(M; \mathrm{End}(\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}))$ indeed.

continued...

...continued

Then if a local operator $R \in C^\infty(M; \mathrm{End}(\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}))$ (hence possessing a pointwise trace function $x \mapsto \mathrm{tr}(R_x)$ like for instance a curvature tensor R_g above) satisfies

$$\tau(R) = \frac{1}{6}\int_M \mathrm{tr}(R)\mu_g < +\infty$$

(this is the shape of the previous non-local trace formula (4) for local operators) we know that $R \in C^\infty(M; \mathrm{End}(\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C})) \cap \mathfrak{R}$.

Observe that both (4) or this latter trace formula are satisfied by any $A \in \mathrm{End}(\Omega^2_c(M; \mathbb{C})$ i.e. $A \in \mathrm{End}(\Omega^2_c(M; \mathbb{C}) \cap \mathfrak{R}$ if it is a bounded operator (with respect to any metric g on M).

A new smooth 4-manifold invariant

The rich representation theory of the II_1 hyperfinite factor allows one to construct a smooth invariant as well:

Theorem (Etesi, 2017, 2022)

Let M be a connected oriented smooth 4-manifold and $\mathfrak{R}$ its von Neumann algebra as before. Then $\mathfrak{R}$ admits a representation on a certain Hilbert space over M such that the unitary equivalence class of this representation is invariant under orientation-preserving diffeomorphisms of M. Consequently the Murray–von Neumann coupling constant of this representation gives rise to a smooth invariant $\gamma(M) \in [0, 1)$. It has the following properties:

continued...

...continued

(i) *It satisfies*

$$\gamma(M) = 1 - \frac{1}{x}$$

where $x \in \{4\cos^2\left(\frac{\pi}{n}\right) \mid n \geqq 3\} \cup [4, +\infty)$, *the set of Jones' subfactor indices;*

(ii) *It behaves like* $\gamma(\overline{M}) = \gamma(M)$ *under reversing orientation,* $\gamma(M \setminus Y) = \gamma(M)$ *under excision of homologically trivial submanifolds and*

$$\gamma(M\#N) = \frac{\gamma(M) + \gamma(N)}{1 + \gamma(M)\gamma(N)}$$

under connected sum;

continued...

...continued

(iii) *If M', M'' are connected, simply connected closed smooth 4-manifolds which are homeomorphic then* (unfortunately) $\gamma(M') = \gamma(M'')$. *Moreover if M is a closed, simply connected, smooth 4-manifold then*

$$\gamma(M) = \frac{17^{b_2(M)} - 1}{17^{b_2(M)} + 1}$$

where $b_2(M)$ is the second Betti number of M. $\diamond$

Thus for instance $\gamma(S^4) = 0$, $\gamma(\mathbb{C}P^2) = \frac{8}{9}$ and for the $K3$ surface having $b_2(K3) = 22$ already $1 - \gamma(K3) \approx 1.70 \times 10^{-27}$ and the general asymptotics of γ in the simply connected case is

$$0 < 1 - \gamma(M) \approx \mathrm{e}^{-\mathrm{const.}\, b_2(M)} \ . \quad (6)$$

This indicates that this invariant maps four dimensional smooth structures into $[0, 1)$ in a logarithmic way.

On the proof of this theorem:

First recall that if $\mathfrak{R}$ is a II_1 hyperfinite factor and $\mathscr{K}$ is a (left) $\mathfrak{R}$-module then there exists a map $\mathscr{K} \mapsto \dim_{\mathfrak{R}} \mathscr{K} \in [0, +\infty)$ called the $\mathfrak{R}$-dimension or the Murray–von Neumann coupling constant of the (left) $\mathfrak{R}$-module $\mathscr{K}$. It is a unitary invariant of the representation and is an isomorphism between equivalence classes of all (left, surely not irreducible) $\mathfrak{R}$-modules and $[0, +\infty)$.

Sketch: Basically using the standard GNS technique alone, out of M and $\mathfrak{R}$ and $\mathscr{H}$ as before, one constructs a Hilbert space $\{0\} \subseteqq \mathscr{I}(M)^{\perp} \subsetneqq \mathscr{H}$ and a representation $\rho_M := \pi|_{\mathscr{I}(M)^{\perp}}$ of $\mathfrak{R}$ on this Hilbert space. If $P_M : \mathscr{H} \to \mathscr{I}(M)^{\perp}$ is the orthogonal projection then $P_M \in \mathfrak{R}$ and $\dim_{\mathfrak{R}} \mathscr{I}(M)^{\perp} = \tau(P_M) \in [0, 1)$ hence is an invariant of the representation. Finally one defines the invariant as

$$\gamma(M) := \tau(P_M) \ .$$

More details: Pick a pair (Σ, ω) consisting of an (immersed) closed orientable surface $\Sigma \looparrowright M$ with induced oriantation and $\omega \in \Omega^2_c(M;\mathbb{C})$ which is also closed i.e., $\mathrm{d}\omega = 0$. Consider the $\mathbb{C}$-linear functional $F_{\Sigma,\omega} : \mathfrak{R} \to \mathbb{C}$ by continuously extending

$$A \longmapsto \int_{\Sigma} A\omega$$

from $\mathrm{End}(\Omega^2_c(M;\mathbb{C})) \cap \mathfrak{R}$. This extension is unique because $\mathrm{End}(\Omega^2_c(M;\mathbb{C})) \cap \mathfrak{R}$ is norm-dense in $\mathfrak{R}$. Let $\{0\} \subseteqq I(M) \subseteqq \mathfrak{R}$ be the subset of elements $A \in \mathfrak{R}$ satisfying $F_{\Sigma,\omega}(A^*A) = 0$. In fact for all pairs (Σ, ω) we obviously find $\{0\} \subsetneqq I(M)$. It turns out that $I(M)$ is a multiplicative left-ideal in $\mathfrak{R}$ which is independent of (Σ, ω) and satisfies $I(M) \cap \mathbb{C}1 = \{0\}$ hence is non-trivial if $F_{\Sigma,\omega}(1) \neq 0$; and $I(M) = \mathfrak{R}$ hence is again independent of (Σ, ω) but trivial in this sense if $F_{\Sigma,\omega}(1) = 0$.

Consider the space $\{0\} \subsetneqq I(M)I(M)^* \subseteqq \mathfrak{R}$ consisting of all finite sums $A_1B_1 + \cdots + A_kB_k \in \mathfrak{R}$ where $A_i \in I(M)$ and similarly $B_j \in I(M)^*$. (This subset is self-adjoint by construction). It gives rise to a closed linear subspace $\{0\} \subsetneqq \mathscr{I}(M) \subseteqq \mathscr{H}$ by closing of $C(M) \cap I(M)I(M)^*$ within $\mathscr{H} \supset C(M)$. Therefore $\mathscr{I}(M)$ is a well-defined closed subspace of $\mathscr{H}$ which is non-trivial if $F_{\Sigma,\omega}(1) \neq 0$ and coincides with $\mathscr{H}$ whenever $F_{\Sigma,\omega}(1) = 0$. Take its orthogonal complementum $\mathscr{I}(M)^\perp$. Note that $\mathscr{I}(M)^\perp$ is isomorphic to $\mathscr{H}/\mathscr{I}(M)$. One checks that the standard representation π of $\mathfrak{R}$ restricts to this subspace. Then define

$$\rho_M : \mathfrak{R} \to \mathfrak{B}(\mathscr{I}(M)^\perp)\text{to be}\begin{cases} \pi|_{\mathscr{I}(M)^\perp} \text{ on } \mathscr{I}(M)^\perp \neq \{0\} \text{ if possible,} \\ \pi|_{\mathscr{I}(M)^\perp} \text{ on } \mathscr{I}(M)^\perp = \{0\} \text{ otherwise.} \end{cases}$$

The choice is unambigously determined by the topology of M.

Then, as noted before, if $P_M \in \mathfrak{R}$ is the orthogonal projection from $\mathscr{H}$ to $\mathscr{I}(M)^\perp$ one puts $\gamma(M) := \tau(P_M)$.

Jones' subfactor theory: The subset $I(M)I(M)^* \subseteqq \mathfrak{R}$ is self-adjoint hence

$$\mathfrak{I}(M) := (\pi(C(M) \cap I(M)I(M)^*))'' \subset \mathfrak{B}(\mathscr{H})$$

is a von Neumann subalgebra of $\mathfrak{R}$; moreover $I(M)I(M)^*$ is a left-ideal too hence $\mathfrak{I}(M)$ is a factor and it acts on $\mathscr{I}(M) \subseteqq \mathscr{H}$ by the standard representation. These imply that

$$\gamma(M) = 1 - \frac{1}{[\mathfrak{R} : \mathfrak{I}(M)]}$$

where $[\mathfrak{R} : \mathfrak{I}(M)]$ is the Jones index of $\mathfrak{I}(M)$ as a subfactor of the hyperfinite II_1 factor $\mathfrak{R}$. Jones' substantial observation in 1983 was that the set of Jones indices is equal to

$$\left\{ 4\cos^2\left(\frac{\pi}{n}\right) \;\middle|\; n \geqq 3 \right\} \cup [4, +\infty) \ .$$

Computations in the simply connected closed case: Take $x \in [0,1)$ and by the connected sum formula put recursively $R_0(x) := 0, R_1(x) := x, \dots, R_k(x) := \frac{x+R_{k-1}(x)}{1+xR_{k-1}(x)}, \dots$ implying

$$R_k(x) = \frac{(1+x)^k - (1-x)^k}{(1+x)^k + (1-x)^k} \ .$$

Put $y := \gamma(\mathbb{C}P^2) = \gamma(\overline{\mathbb{C}P^2}) \neq 0$. Then for every connected, simply connected, closed 4-manifold M one finds

$$\gamma(M) = R_{b_2(M)}(y) \ .$$

(*Proof*: For every pair (M,N) of connected, simply connected closed 4-manifolds there exist integers k_1, l_1 and k_2, l_2 such that $M\# k_1\mathbb{C}P^2\# l_1\overline{\mathbb{C}P^2} \cong N\# k_2\mathbb{C}P^2\# l_2\overline{\mathbb{C}P^2}$. Then put M arbitrary and $N := S^4$ and apply the connected sum formula above.)

Finally one computes by hand using subfactor considerations that $[\mathfrak{R} : \mathfrak{I}(\mathbb{C}P^2)] = 9$ hence

$$y = 1 - \frac{1}{9} = \frac{8}{9}$$

as stated in the theorem.

Interesting question: All subfactors indices provided by smooth 4-manifolds do belong to the wild range $[4, +\infty)$ of Jones' indices?

Application to the cosmological constant problem

Let (M, g) be an oriented (pseudo-)Riemannian 4-manifold satisfying the vacuum Einstein equation

$$\mathrm{Ric} = \Lambda g \tag{7}$$

where $\Lambda \approx 2.89 \times 10^{-122}\, \ell_{\mathrm{Planck}}^{-2}$ is the cosmological constant. Let $* : \Omega^2_c(M;\mathbb{C}) \to \Omega^2_c(M;\mathbb{C})$ be the complexified Hodge operator having the shape $* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ hence $* \in \mathfrak{R}$. If the full complexified Riemannian curvature also happens to satisfy $R_g \in \mathfrak{R}$ then referring to (5) it is easy to see that (7) is equivalent to

$$\begin{cases} R_g^{\circledast} & = R_g \quad \text{(the curvature is symmetric)} \\ \tau(R_g *) & = 0 \quad \text{(algebraic Bianchi identity)} \\ *R_g* & = R_g \quad \text{(Einstein condition).} \end{cases}$$

The cosmological constant itself is recovered by taking trace:

$$\tau(R_g) = \frac{\Lambda}{3} \ .$$

Switching to a quantum mechanical language this latter equation formally means that Λ arises as the expectation value of measurements carried out on the space-time curvature (whatever this means).

These serve as motivations for the following operator-theoretic reformulation of the classical vacuum Einstein equation (with bounded curvature tensor):

Definition

Let M be a connected oriented smooth 4-manifold and $\mathfrak{R}$ its II_1-type hyperfinite factor von Neumann algebra as before.

(i) A refinement of $\mathfrak{R}$ is a pair $(\mathfrak{R}, *)$ where $1 \neq * \in \mathfrak{R}$ and satisfies $*^2 = 1$;

(ii) An operator $Q \in \mathfrak{R}$ solves the quantum vacuum Einstein equation with respect to $(\mathfrak{R}, *)$ if

$$\begin{cases} Q^{*} & = Q \quad \text{(self-adjointness)} \\ \tau(Q*) & = 0 \quad \text{(algebraic Bianchi identity)} \\ *Q* & = Q \quad \text{(Einstein condition).} \end{cases}$$

(iii) The trace $\tau(Q) =: \frac{\Lambda}{3} \in \mathbb{R}$ is called the corresponding quantum cosmological constant.

Unlike the classical Einstein equation (7) which is a non-linear 2nd order PDE on metrics hence is extremely difficult to solve, the quantum one as defined here is a linear equation on bounded operators hence is easily solved. Indeed, given $B \in \mathfrak{R}$ then $S = \frac{1}{2}(B + B^*)$ is self-adjoint and then picking an arbitrary refinement $(\mathfrak{R}, *)$ and taking into account that $*$ is always self-adjoint, $Q := \frac{1}{2}\big(S + *S*\big) - \tau(S*)* \in \mathfrak{R}$ is automatically a solution of the quantum vacuum Einstein equation; moreover using $\tau(*) = 0$ its trace $\frac{\Lambda}{3}$ is equal to $\tau(B)$ hence is independent of the particular refinement.

Taking the (self-adjoint) projections $P_M \in \mathfrak{R}$ used to construct $\gamma(M)$ we find that the average Q_M of for instance $1 - P_M$ i.e.

$$\begin{aligned} Q_M &:= \frac{1}{2}(1 - P_M + *(1 - P_M)*) - \tau((1 - P_M)*)* \\ &= 1 - \frac{1}{2}(P_M + *P_M*) + \tau(P_M*)* \end{aligned}$$

solves the quantum vacuum Einstein equation moreover

$$\begin{aligned} \frac{\Lambda}{3}\ell^2_{\mathrm{Planck}} &= \tau(Q_M) = 1 - \gamma(M) \\ &\in \left(0, \frac{1}{4}\right] \bigcup \left\{ \frac{1}{4}\cos^{-2}\left(\frac{\pi}{n}\right) \,\middle|\, n = \dots 5, 4, 3 \right\} \subset (0, 1] \end{aligned}$$

is automatically a small but strictly positive number!

Assuming that the space-time manifold M underlying the (observable) Universe is simply connected and interpreting $b_2(M)$ as the number of primordial black holes around the Planck era one can use (6) together with the observed value of Λ to obtain an estimate

$$\begin{aligned} b_2(M) &\approx -\mathrm{const.}\log(1-\gamma(M)) = -\mathrm{const.}\log\left(\frac{\Lambda}{3}\ell^2_{\mathrm{Planck}}\right) \\ &\approx 10^2 \ . \end{aligned}$$

This demonstrates that within our model the existence of very early primordial black holes is negligable. This result is consistent with other estimates based on the Press–Schechter mechanism.

Further details to be worked out...

References:

G. Etesi: *The universal von Neumann algebra of smooth four-manifolds revisited*, preprint, 18pp, (2022), available at: `math.bme.hu/~etesi/qgravity55-revisited.pdf`;

G. Etesi: *The universal von Neumann algebra of smooth four-manifolds*, Adv. Theor. Math. Phys. **25**, 995-1024 (2021), arXiv: `1712.01828 [math-ph]`.

Thank you!