A quantum gravity from the universal von Neumann algebra of smooth four-manifolds

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* A comment on smooth structures in 4 dimensions
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References:

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Emergence of the II_1 hyperfinite factor

From the superabundance of smooth 4-manifolds one can destillate a single von Neumann algebra as follows:

Theorem (Etesi, 2017, 2022)

Let M be a connected oriented smooth 4-manifold. Making use of its smooth structure only, a von Neumann algebra $\mathfrak{R}$ can be constructed which is geometric in the sense that it contains algebraic (i.e., formal or coming from a metric) curvature tensors on M and $\mathfrak{R}$ itself is a hyperfinite factor of type II_1 (hence is unique up to abstract isomorphism of von Neumann algebras). $\diamond$

The construction is based on two simple steps and looks like this:

Step 1. Let M be a connected oriented smooth 4-manifold and consider $T^{(p,q)}M$, the bundle of (p,q)-type tensors over M. Among these bundles $\wedge^2 T^*M \subset T^{(0,2)}M$, the bundle of 2-forms, is the only one which admits a natural (i.e. defined without any additional structure) pairing over M: Writing $\Omega^2_c(M) := C^\infty_c(M; \wedge^2 T^*M)$ for the infinite dimensional real vector space of compactly supported smooth 2-forms, then for every $\alpha, \beta \in \Omega^2_c(M)$, an $\mathbb{R}$-bilinear symmetric pairing

$$\langle \alpha, \beta \rangle_{L^2(M)} := \int_M \alpha \wedge \beta \tag{1}$$

exists provided by the orientation and the smooth structure of M only.

Observe that this pairing is non-degenerate however is *indefinite* in general thus can be regarded as an indefinite scalar product on $\Omega^2_c(M)$. It therefore induces an indefinite real quadratic form Q given by $Q(\alpha) := \langle \alpha, \alpha \rangle_{L^2(M)}$. Let $C(M)$ denote the complexified infinite dimensional Clifford algebra associated with $(\Omega^2_c(M), Q)$. If $\mathfrak{M}_k(\mathbb{C})$ denotes the algebra of $k \times k$ complex matrices then

$$C(M) \cong \bigcup_{n=0}^{+\infty} \mathfrak{M}_{2^n}(\mathbb{C}) \tag{2}$$

or equivalently

$$C(M) \cong \mathfrak{M}_2(\mathbb{C}) \otimes \mathfrak{M}_2(\mathbb{C}) \otimes \dots$$

and note that being (1) a non-local operation, this $C(M)$ is a genuine global infinite dimensional object.

The isomorphism (2) shows that $C(M)$ is a complex $*$-algebra whose $*$-operation (provided by taking Hermitian matrix transpose, a non-local operation) is written as $A \mapsto A^{\circledast}$ (note the difference in notation between $*$ and $\circledast$ at various places from now on) moreover $C(M)$ possesses a unit and its center comprises the scalar multiples of the unit only. If $A \in C(M)$ then one can pick the smallest $n \in \mathbb{N}$ such that $A \in \mathfrak{M}_{2^n}(\mathbb{C})$ consequently A has a finite trace defined by

$$\tau(A) := 2^{-n}\mathrm{Trace}(A)$$

i.e., taking the usual normalized trace of the corresponding $2^n \times 2^n$ complex matrix. It is straightforward that $\tau(A) \in \mathbb{C}$ does not depend on n. We can then define a sesquilinear inner product on $C(M)$ by

$$(A, B) := \tau(AB^{\circledast})$$

which is non-degenerate.

Step 2. Thus the completion of $C(M)$ with respect to the norm $\|\cdot\|$ induced by $(\cdot,\cdot)$ renders $C(M)$ a complex Hilbert space what we shall write as $\mathscr{H}$ and its Banach algebra of all bounded linear operators as $\mathfrak{B}(\mathscr{H})$.
Multiplication in $C(M)$ from the left on itself is continuous hence gives rise to a representation $\pi : C(M) \to \mathfrak{B}(\mathscr{H})$. Finally our central object effortlessly emerges as the weak closure of the image of $C(M)$ under π within $\mathfrak{B}(\mathscr{H})$ or equivalently, by referring to von Neumann's bicommutant theorem we put

$$\mathfrak{R} := (\pi(C(M)))'' \subset \mathfrak{B}(\mathscr{H}) \ .$$

This is a von Neumann algebra and of course admits a unit $1 \in \mathfrak{R}$ moreover continues to have trivial center i.e., it is a factor. Moreover by construction it is hyperfinite. The trace τ as defined has the general form

$$\tau(A) = (A, 1) \tag{3}$$

in terms of the scalar product on $C(M)$, extends from $C(M)$ to $\mathfrak{R}$ and satisfies $\tau(1) = 1$. Moreover this trace is unique on $\mathfrak{R}$. Consequently from an M a hyperfinite II_1-type factor von Neumann algebra $\mathfrak{R}$ has been extracted. Moreover whatever M was, this $\mathfrak{R}$ is unique up to abstract isomorphisms of von Neumann algebras.

How the operator algebra $\mathfrak{R}$ *looks like?* One can show that

$$\begin{array}{ccccc} & & \mathfrak{R} & & \\ & & \cup & & \\ \Omega^2_c(M;\mathbb{C}) & \subset & C(M) & \subset & \mathrm{End}(\Omega^2_c(M;\mathbb{C})) \\ & & \cap & & \\ & & \mathscr{H} & & \end{array}$$

(the second horizontal embedding is not canonical). Two examples:

1. Embedding spaces with non-local operators. If $x \in M$ then let $\Omega^2_c(M,x;\mathbb{C}) \subset \Omega^2_c(M;\mathbb{C})$ be the subspace of 2-forms vanishing in x. Let P_x be the projection to $\overline{\Omega^2_c(M,x;\mathbb{C})} \subset \mathscr{H}$. We know that $P_x \in \mathfrak{R}$ hence $x \mapsto P_x$ gives $i_M : M \to \mathfrak{R}$ which is a continuous embedding. Since $\mathfrak{R}$ is universal, all M's (various classical spaces) embed into $\mathfrak{R}$ (a unique noncommutative space).

2. Curvatures as local operators. A peculiarity of four dimensions is that the subspace $C^\infty(M; \mathrm{End}(\wedge^2 T^*M)) \subset \mathrm{End}(\Omega^2_c(M))$ of bundle morphisms i.e. local operators contains the space of algebraic (i.e., formal or coming from a metric) curvature tensors on M.

E.g. if (M, g) is an oriented Riemannian 4-manifold then its Riemannian curvature tensor R_g is a member of this algebra.

With respect to the decomposition of real 2-forms into their (anti)self-dual parts and taking their complexifications, the curvature as a map

$$R_g : \begin{matrix} \Omega^+_c(M;\mathbb{C}) \\ \bigoplus \\ \Omega^-_c(M;\mathbb{C}) \end{matrix} \longrightarrow \begin{matrix} \Omega^+_c(M;\mathbb{C}) \\ \bigoplus \\ \Omega^-_c(M;\mathbb{C}) \end{matrix}$$

arises as the $\mathbb{C}$-linear extension of the well-known real decomposition

$$R_g = \begin{pmatrix} \frac{1}{12}\mathrm{Scal} + \mathrm{Weyl}^+ & \mathrm{Ric}_0 \\ \mathrm{Ric}_0^* & \frac{1}{12}\mathrm{Scal} + \mathrm{Weyl}^- \end{pmatrix} \tag{4}$$

that is, $R_g \in C^\infty(M; \mathrm{End}(\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}))$ indeed.

If an operator is local i.e. $R \in C^\infty(M; \mathrm{End}(\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C}))$ then it possesses a pointwise trace function $x \mapsto \mathrm{tr}(R_x)$. The trace (3) of R (if exists) looks like

$$\tau(R) = \frac{1}{6} \int\limits_M \mathrm{tr}(R)\mu_g$$

with respect to some (M, g) with $\int_M \mu_g = 1$ hence τ on $\mathfrak{R}$ is the generalization of the total scalar curvature. It turns out that if $|\tau(R)| < +\infty$ then $R \in \mathfrak{R}$. Hence e.g. bounded curvature tensors belong to $\mathfrak{R}$.

Application to gravity

If $A, B \in \mathfrak{R}$ then $\tau(AB) \in \mathbb{C}$ is well-defined hence interpreting this quantity formally as "the expectation value of the observable A in state B" we get a quantumlike description of gravity in 4 dimensions. This theory at least mathematically (i.e. at the level of its syntax) is correct. However its physical meaning (i.e. its semantics) is yet to understand.

Two examples:

1. Comparison of geometries. Take two (pseudo-)Riemannian manifolds (M, g) and (N, h) with curvatures R_g and R_h respectively. Taking into account the universality of $\mathfrak{R}$ more precisely the existence of canonical continuous embeddings

$$i_M : M \longrightarrow \mathfrak{R} \longleftarrow N : i_N$$

talking about $M \cap N$ etc. within $\mathfrak{R}$ is correct. If $x \in M$ and $y \in N$ then $\tau(P_x P_y) \in \mathbb{C}$ is correct. Even more, $\tau(R_g R_h) \in \mathbb{C}$ (if exists) is also correct. Hence these sort of numbers look like describing a sort of "expectation value of (M, g) with respect to (N, h)" (whatever it means).

2. The quantum Einstein equation. Let (M, g) be an oriented Riemannian 4-manifold satisfying the vacuum Einstein equation

$$\mathrm{Ric} = \Lambda g \tag{5}$$

where $\Lambda \approx 2.89 \times 10^{-122}\, \ell_{\mathrm{Planck}}^{-2}$ is the cosmological constant. Let $* : \Omega^2_c(M;\mathbb{C}) \to \Omega^2_c(M;\mathbb{C})$ be the complexified Hodge operator having the shape $* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ hence $* \in \mathfrak{R}$. If the full complexified Riemannian curvature also happens to satisfy $R_g \in \mathfrak{R}$ then referring to (4) it is easy to see that (5) is equivalent to

$$\begin{cases} R_g^{\circledast} & = R_g \quad \text{(the curvature is symmetric)} \\ \tau(R_g *) & = 0 \quad \text{(algebraic Bianchi identity)} \\ * R_g * & = R_g \quad \text{(Einstein condition).} \end{cases}$$

The cosmological constant itself is recovered by taking trace:

$$\tau(R_g) = \frac{\Lambda}{3} \ .$$

Switching to a quantum mechanical language this latter equation formally says "Λ arises as the expectation value of measurements carried out on the space-time curvature" (whatever this means).

These serve as motivations for the following operator-theoretic reformulation of the classical vacuum Einstein equation (with bounded curvature tensor):

Definition

Let M be a connected oriented smooth 4-manifold and $\mathfrak{R}$ its II_1-type hyperfinite factor von Neumann algebra as before.

(i) A refinement of $\mathfrak{R}$ is a pair $(\mathfrak{R}, *)$ where $1 \neq * \in \mathfrak{R}$ and satisfies $*^2 = 1$;

(ii) An operator $Q \in \mathfrak{R}$ solves the quantum vacuum Einstein equation with respect to $(\mathfrak{R}, *)$ if

$$\begin{cases} Q^{\circledast} & = Q \quad \text{(self-adjointness)} \\ \tau(Q*) & = 0 \quad \text{(algebraic Bianchi identity)} \\ *Q* & = Q \quad \text{(Einstein condition).} \end{cases}$$

(iii) The trace $\tau(Q) =: \frac{\Lambda}{3} \in \mathbb{R}$ is called the corresponding quantum cosmological constant.

Unlike the classical Einstein equation (5) which is a non-linear 2nd order PDE on metrics hence is extremely difficult to solve, the quantum one as defined here is a linear equation on bounded operators hence is easily solved. Indeed, given $B \in \mathfrak{R}$ then $S = \frac{1}{2}(B + B^{*})$ is self-adjoint and then picking an arbitrary refinement $(\mathfrak{R}, *)$ and taking into account that $*$ is always self-adjoint, $Q := \frac{1}{2}\left(S + *S*\right) - \tau(S*)* \in \mathfrak{R}$ is automatically a solution of the quantum vacuum Einstein equation; moreover using $\tau(*) = 0$ its trace $\frac{\Lambda}{3}$ is equal to $\tau(B)$ hence is independent of the particular refinement.

Assuming that the space-time manifold M underlying the (observable) Universe is simply connected and interpreting $b_2(M)$ as the number of primordial black holes around the Planck era one can use the observed value of Λ to obtain an estimate

$$b_2(M) \approx -\mathrm{const.} \log\left(\frac{\Lambda}{3}\ell^2_{\mathrm{Planck}}\right) \approx 10^2$$

within this framework. This demonstrates that within our model the existence of very early primordial black holes is negligable. This result is consistent with other estimates based on the Press–Schechter mechanism.

Further details to be worked out...

References:

G. Etesi: *The universal von Neumann algebra of smooth four-manifolds revisited*, 24 pp., to appear in the AMS Contemporary Mathematics volume "Advances in Functional Analysis and Operator Theory" (2024); available at:

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Thank you!