Four dimensional Riemannian general relativity as a spontaneous symmetry breaking

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Abstract

In this paper 4 dimensional Riemannian (or Euclidean) vacuum general relativity is recovered from a phase transition by spontaneous symmetry breaking within a quantum field theory (all in the sense of the algebraic approach [3, 14]) provided by the unique II_1 hyperfinite factor von Neumann algebra. The formal temperature of this phase transition is $T = \frac{1}{2}T_{\text{Planck}} \approx 7.06 \times 10^{31}\text{K}$.

More precisely, first we recall [12, 13] that making use of its smooth structure only, out of a connected oriented smooth 4-manifold M a von Neumann algebra $\mathfrak{R}$ can be constructed which is geometric in the sense that it is generated by local operators and as a special four dimensional phenomenon it contains all algebraic (i.e., formal or coming from a metric) curvature tensors of the underlying 4-manifold. The von Neumann algebra $\mathfrak{R}$ itself is a hyperfinite II_1 factor hence is unique up to abstract isomorphisms of von Neumann algebras. Interpreting $\mathfrak{R}$ as the operator algebra of a (bosonic) quantum field theory, on the one hand, the unique tracial state on $\mathfrak{R}$ gives rise to a phase whose only state is the corresponding infinitely high temperature KMS state of the theory with trivial Tomita–Takesaki or modular dynamics. On the other hand, M induces further states on $\mathfrak{R}$ having the property that they are of finite temperature, are stationary and stable under a new non-trivial dynamics on $\mathfrak{R}$ what we call the Hodge dynamics. Hence it is reasonable to call these states as finite temperature equilibrium states too (however they are not KMS states in the sense of [7, 3, 14]). Furthermore, M embeds into $\mathfrak{R}$ which induces a Riemannian 4-manifold (M,g) and it is observed that (M,g) satisfies the vacuum Einstein equation if and only if both $M \subset \mathfrak{R}$ and its Riemannian curvature tensor $R_g \in \mathfrak{R}$ are invariant under the corresponding Hodge dynamics. Thus the other phase of the proposed quantum field theory is built up from the huge moduli space of states provided by all smooth 4-manifold structures and the transition between these phases can be considered as a spontaneous symmetry breaking by cooling (perhaps about the Big Bang times).

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1 Introduction and summary

A longstanding problem of contemporary theoretical physics is how to unify the obviously successful and mathematically consistent *theory of general relativity* with the obviously successful but yet mathematically problematic *relativistic quantum field theory*. It has been generally believed that these two fundamental pillars of modern theoretical physics are in a clash not only because of the different mathematical tools they use but are in tension even at a deep conceptional level: classical notions of general relativity such as the space-time event, the light cone or the event horizon of a black hole are too "sharp" objects from a quantum theoretic viewpoint whereas relativistic quantum field theory is not background independent from the aspect of general relativity.

The demand by general relativity summarized as the *principle of general covariance* is perhaps one of the two main obstacles why general relativity has remained outside of the mainstream classical and quantum field theoretic expansion in the 20$^{\text{th}}$ century. Indeed, an implementation of this inherent principle of general relativity forces that a robust group, namely the full diffeomorphism group of the underlying space-time manifold must belong to the symmetry group of a field theory compatible with general relativity. However an unexpected consequence of the vast diffeomorphism symmetry is that it even allows one to transform time itself away from the theory (known as the "problem of time" in general relativity, see e.g. [25, Appendix E] for a technical presentation as well as e.g. [5, Chapter 2] and [16, Subsection 2.1] for a broader philosophical survey on this problem) making it problematic to apply standard canonical quantization methods—based on Hamiltonian formulation hence on an essential explicit reference to an "auxiliary time"—in case of general relativity. The other reason is the as well in-built core idea, the *equivalence principle* which renders general relativity a strongly self-interacting classical field theory in the sense that precisely in four dimensions the "free" and the "interaction-with-itself" modes of the gravitational field have energetically the same magnitudes, obfuscating perturbative considerations. In fact the equivalence principle says that there is no way to make a physical distinction between these two modes of gravity. Heisenberg and Pauli were still optimistic concerning canonical and perturbative quantization of gravity with respect to a fixed time or more generally a reference or ambient space-time in their 1929 paper [17]; however these initial hopes quickly evaporated already in the 1930's by recognizing the *essential impossibility* of quantizing general relativity via canonical quantization and exhibit it as a perturbatively renormalizable quantum field theory in a coherent way. This was clearly observed by Bronstein [4] first; as he wrote in his 1936 paper: "[...] the elimination of the logical inconsistencies [requires] rejection of our ordinary concepts of space and time, modifying them by some much deeper and nonevident concepts." (also cited by Smolin [23, p. 85]).

Roughly the thinking about gravity has split into two main branches since the 1950-60's [16]. The first older and more accepted direction postulates that gravity should be quantized akin to other fundamental forces but with more advanced methods including (super)string theoretic [15], Feynman integral, loop quantum gravity or some further techniques—or at least one should construct it as a low energy effective field theory of an unknown high energy theory; the other newer and yet less-accepted attitude declares that gravity is an emergent macroscopic phenomenon in the sense that it always involves a huge amount of physical degrees-of-freedom (beyond the obvious astronomical evidences, also supported by various theoretical discoveries during the 1970-80's such as Hawking's area theorem, black hole radiation, all resembling thermodynamics) hence is not subject to quantization at all. Nevertheless, as a matter of fact in the 2020's, we have to admit that an overall accepted quantum theory of gravity does not exist yet and even general relativity as a classical field theory persists to keep its conceptionally isolated position within current theoretical physics [16]. Perhaps it is worth mentioning here that general relativity receives further challenges from low dimensional differential topology too by recent discoveries which were unforeseenable earlier, cf. e.g. [11, Section 1] for a brief summary.

Therefore, respecting its long resistance against quantization, we do not make a new attempt here to quantize general relativity. Rather the purpose of this paper is to occupy an in-between position in the battlefield and in part motivated by [4, 8] to take a fresh look on precisely four dimensional and Riemannian general relativity by identifying it as a spontaneously broken *classical* phase of a proposed *quantum* field theory based upon the famous unique hyperfinite factor von Neumann algebra of II_1-type and its rich representation theory (also cf. [6]). The remaining part of this introduction is an informal trailer to this idea and we refer for the physical details to Section 2 while for the mathematical subtleties to Section 3 below.

As it is known [1] the hyperfinite II_1 factor von Neumann algebra $\mathfrak{R}$ is unique up to abstract isomorphisms but admits lot of inequivalent representations on various Hilbert spaces. These representations are classified by their so-called $\mathfrak{R}$-dimensions (or Murray–von Neumann coupling constants) taking values in the non-negative real line $[0,+\infty)$ and these are their appropriately defined dimensions as (left-)modules over $\mathfrak{R}$. For instance the representation having $\mathfrak{R}$-dimension precisely $1\in[0,+\infty)$ is its standard represention on itself by (left-)multiplications. This is the best-known representation and within the framework of the GNS-construction it can be obtained from the distinguished faithful state provided by the unique finite *trace* on the hyperfinite II_1 factor. However many other new representations can be constructed using further non-faithful *differential-geometric* states provided by connected oriented smooth 4-manifolds as observed in our earlier efforts [12, 13]. The $\mathfrak{R}$-dimensions of these representations always belong to the interval $[0,1)\subset[0,+\infty)$ hence they are never equivalent to the standard representation. It was Haag's remarkable observation long time ago [14, Subsection II.1.1] that the cardinality of the degrees-of-freedom in a quantum system can be characterized by the existence of inequivalent representations of its algebra of observables. Since $\mathfrak{R}$ admits lot of representations it is reasonable to expect that the hyperfinite II_1 factor has something to do with quantum statistical mechanics of infinite systems.[1] Indeed, the common feature of the various states on $\mathfrak{R}$ to be introduced here (and giving rise to representations by the GNS-construction) is that they represent thermal equlibrium states at various temperatures. The temperature can be assigned to them in a uniform way by computing the periodicity $\hbar\beta$ where $\beta=\frac{1}{k_BT}$ is the inverse temperature (note that $[\hbar\beta]=$time) of the imaginary-time dynamics they are related with. The connection between periodicity in imaginary time and the existence of a temperature in real time is well-known cf. e.g. [18].

First let us consider the well-known case of a KMS state. Recall that the KMS condition is as follows. Take a C^*-algebra and consider a dynamics on this algebra given by a 1-real-parameter continuous family of its $*$-automorphisms $t\mapsto\alpha_t$. A state ω on this C^*-algebra *satisfies the KMS condition* if and only if for every operator A the function $t\mapsto\omega(\alpha_t(A))$ can be extended to $z\mapsto\omega(\alpha_z(A))$ which is complex analytic in the interiors and continuous up to the boundaries of two strips above and below the real line of width $0\leqq\hbar\beta\leqq+\infty$ i.e. for $\mathrm{Re}\,z\in\mathbb{R}$ and $0\leqq\pm\mathrm{Im}\,z\leqq\hbar\beta$ such that for every $t\in\mathbb{R}$ and every A,B from the C^*-algebra

$$\omega(\alpha_t(A)B)=\omega(B\alpha_{t\pm\sqrt{-1}\hbar\beta}(A))$$

holds. However this implies that the corresponding "pseudo-dynamics" in the imaginary-time direction[2] is periodic with period $\hbar\beta$. Indeed, the KMS state is faithful [14, Subsection V.1.4] consequently

[1] As it is known, the hyperfinite II_1 factor was von Neumann's favourite operator algebra and he wanted to use it to bring quantum mechanics on an even conceptionally acceptable basis as a statistical theory relative to an unsorted or absolute statistical ensemble provided by this von Neumann algebra (cf. e.g. [20]).

[2] Actually this is not a dynamics in the strict sense because in this case automorphisms fail to be $*$-automorphisms hence for instance $(\alpha_{\sqrt{-1}\hbar\beta}(A))^*=\alpha_{-\sqrt{-1}\hbar\beta}(A^*)$ only. In general observe that on a C^*-algebra among the two automorphisms $t\mapsto\alpha_t$ and $t\mapsto\alpha_{\sqrt{-1}t}$ with $t\in\mathbb{R}$ at most one can be a $*$-automorphism hence represent a dynamics. Moreover we can term the one which is periodic as "imaginary-time (pseudo-)dynamics" and the other one as "real-time (pseudo-)dynamics".

$\omega(TT^*) = 0$ implies $T = 0$; putting $T := A - \alpha_{\sqrt{-1}\hbar\beta}(A)$ and using the KMS condition couple of times at $t = 0$ with straightforward substitutions we compute

$$\begin{aligned}\omega\big((A-\alpha_{\sqrt{-1}\hbar\beta}(A))(A-\alpha_{\sqrt{-1}\hbar\beta}(A))^*\big) &= \omega\big((A-\alpha_{\sqrt{-1}\hbar\beta}(A))(A^*-\alpha_{-\sqrt{-1}\hbar\beta}(A^*))\big)\\ &= \omega(AA^*)-\omega\big(A^*A\big)-\omega\big(A^*A\big)+\omega\big(AA^*\big)\end{aligned}$$

demonstrating that if A is self-adjoint then the right hand side is zero hence so is the other side consequently $A = \alpha_{\sqrt{-1}\hbar\beta}(A)$; since every element in a C^*-algebra is a sum (a $\mathbb{C}$-linear combination) of two self-adjoint operators we find that given a "real-time dynamics" the corresponding KMS condition indeed implies the periodicity of the "imaginary-time pseudo-dynamics". On the one hand we can apply this observation on the hyperfinite II_1 factor $\mathfrak{R}$ in its standard representation by picking the positive self-adjoint modular operator Δ and taking the Tomita–Takesaki or modular "real-time dynamics" $t \mapsto \mathrm{Ad}_{\Delta^{\sqrt{-1}t}}$ as well as the corresponding "imaginary-time pseudo-dynamics" $t \mapsto \mathrm{Ad}_{\Delta^t}$ (Footnote 2). However in this case $\Delta = 1$ hence the former dynamics is trivial likewise the latter one yielding $\hbar\beta = 0$ consequently the temperature of the corresponding KMS state τ, which is nothing but the unique trace on $\mathfrak{R}$, is infinite. Hence this tracial state (and its corresponding standard representation π on a Hilbert space $\mathscr{H}$ whose $\mathfrak{R}$-dimension therefore satisfies $\dim_{\mathfrak{R}}\mathscr{H} = 1$) describes an infinite high temperature thermal equilibrium state on $\mathfrak{R}$.

On the other hand we can consider another dynamics on $\mathfrak{R}$ which is introduced by an oriented Riemannian 4-manifold (M,g) through its Hodge-operator acting on 2-forms over M. This operator turns out to represent a unitary element $* \in \mathfrak{R}$ hence gives rise to a dynamics $t \mapsto \mathrm{Ad}_{*^t}$ on $\mathfrak{R}$. This dynamics is already non-trivial however the well-known property $*^2 = 1$ makes it periodic; therefore here this "Hodge dynamics" should be termed as the "imaginary-time dynamics" while the corresponding $t \mapsto \mathrm{Ad}_{*^{\sqrt{-1}t}}$ as the "real-time pseudo-dynamics" (cf. Footnote 2). Nevertheless a corresponding temperature can be computed from $\hbar\beta = 2t_{\mathrm{Planck}}$ yielding $T = \frac{1}{2}T_{\mathrm{Planck}}$. It turns out that picking a surface $\Sigma \subset M$ and a 2-form ω on M there exists a state $F_{\Sigma,\omega}$ on $\mathfrak{R}$ induced by (M,g) which is stationary under the Hodge dynamics i.e. $F_{\Sigma,\omega}(*^tA*^{-t}) = F_{\Sigma,\omega}(A)$ for every $t \in \mathbb{R}$ and $A \in \mathfrak{R}$ moreover it is stable against small perturbations. Consequently it is reasonable to call this state as a finite temparture thermal equilibrium state on $\mathfrak{R}$ too obtained from a connected oriented smooth 4-manifold M. However this state cannot be KMS because the corresponding representation ρ_M of $\mathfrak{R}$ on a Hilbert space $\mathscr{I}(M)^\perp$ obtained by the GNS construction is not faithful, actually its $\mathfrak{R}$-dimension satisfies $0 \leqq \dim_{\mathfrak{R}}\mathscr{I}(M)^\perp < 1$.

Our key observation in this paper is that if (M,g) is an oriented Riemannian 4-manifold which satisfies the vacuum Einstein equation then M together with its Riemannian curvature tensor R_g embeds into $\mathfrak{R}$ as a portion of the fixed-point-subalgebra of the corresponding "Hodge dynamics" generated by the Hodge operator of (M,g). Therefore, speaking expressively, "starting" with the infinitely high temperature standard representation of $\mathfrak{R}$ whose corresponding dynamics is trivial hence its fixed-point-set is the whole $\mathfrak{R}$ we can "pass" to one of its finite temperature representation whose dynamics is not trivial anymore and its fixed-point-set "cuts out" the vacuum space-time (M,g) from $\mathfrak{R}$ we began with; and regard this whole story as a phase transition by a spontaneous symmetry breaking because of cooling in the early Big Bang times (this scheme is formally displayed in (1) below). An advantage of this pattern is that it avoids the problem of the Beginning for it dissolves the classical dynamical phase of the Universe in a dynamically trivial abstract quantum phase "just before" reaching the "Big Bang event". We also have to admit that the whole machinery here works in Riemannian (i.e. Euclidean) signature only. But at least the space-time dimension, which must be precisely four here, matches.

The paper is organized as follows. Section 2 contains a detailed presentation of this phase transition approach while Section 3 contains all the (supposed to be complete) mathematical constructions.

2 Recovering Riemannian general relativity

In this section by recalling some results from [12, 13] we will introduce a framework based on the algebraic quantum field theory approach [3, 14] in which the immense class of classical space-times of general relativity is replaced with a single universal "quantum space-time" allowing us to lay down the foundations of a manifestly four dimensional, covariant, non-perturbative and genuinely quantum theory related with gravitation in some way. This construction is natural, simple and self-contained. More precisely here not *one particular Riemannian* 4*-manifold*—physically regarded as a particular classical space-time—but *the unique hyperfinte* II_1 *factor von Neumann algebra*—physically viewed as the universal quantum space-time—is declared to be the primarily given object and 4 dimensional Riemannian general relativity is derived by a phase transition within this theory from an infinitely high temperature pure quantum phase to a finite temperature classical mixed phase carried out by a spontaneous symmetry breaking. Let us see how it works!

A quantum theory and its quantum phase. To begin with, let $\mathscr{H}$ be an abstractly given infinite dimensional complex separable Hilbert space and $\mathfrak{R} \subset \mathfrak{B}(\mathscr{H})$ be a type II_1 hyperfinite factor hence tracial von Neumann algebra acting on $\mathscr{H}$ by the standard representation π. Its unique trace is denoted by τ and satisfies $\tau(1) = 1$. We call $\mathfrak{R}$ the *algebra of observables*, its tangent space $T_1\mathfrak{R} \supset \mathfrak{R}$ consisting of the Fréchet derivatives of 1-parameter families of observables at the unit $1 \in \mathfrak{R}$ the *algebra of fields*, while $\mathscr{H}$ the *state space*. The subgroup $\mathrm{U}(\mathscr{H}) \cap \mathfrak{R}$ of the unitary group of $\mathscr{H}$ operating as inner automorphisms on $\mathfrak{R}$ is the *gauge group*. Note that the gauge group acts on both $\mathfrak{R}$ and $\mathscr{H}$ but in a different way. If the unique standard representation $\pi : \mathfrak{R} \to \mathfrak{B}(\mathscr{H})$ is constructed through the GNS construction then it is obtained from the faithful state τ on $\mathfrak{R}$ hence the resulting $\mathscr{H}$ possesses a cyclic and separating vector; thus π renders $\mathfrak{R}$ a dynamical system under its own *Tomita–Takesaki* or *modular* automorphisms [3, 7, 8, 14] which dynamics is however trivial—or rephrasing, the corresponding imaginary-time pseudo-dynamics is periodic with period $\hbar\beta = 0$—because in addition τ is tracial [3, Vol. I, p. 90]. Nevertheless τ is the corresponding KMS state which is therefore of infinitely high temperature as well as stationary and stable in the sense of [14, Sections V.1-3]. The cardinality of the physical degrees-of-freedom in a quantum system can be characterized by the existence of inequivalent representations of its algebra of observables cf. e.g. [14, Subsection II.1.1]. It is well-known [1, Chapter 8] that beyond its standard representation $\mathfrak{R}$ admits lot of further inequivalent representations too hence this operator algebra is capable to describe an infinite degree-of-freedom physical system e.g. a field. Thus we can say that by default the theory exhibited by $\mathfrak{R}$ together with its state τ and corresponding standard representation π on $\mathscr{H}$ describes some infinite degree-of-freedom quantum statistical system in a pure phase whose only state is an infinitely high temperature thermal equilibrium state. We call from now on this infinitely high temperature (physically rather equal to $T_{\mathrm{Planck}} \approx 1.42 \times 10^{32}\mathrm{K}$) equilibrium phase as the *quantum phase* of the proposed quantum field theory.

However before proceeding further we ask ourselves what kind of quantum theory is the one in which $\mathfrak{R}$ plays the role of the algebra of (bounded) *physical observables*? The starting point to answer this question in particular and for a more detailed analysis of this quantum theory in general is the following result taken from [13] whose proof for convenience (as well as to fix notation to be used throughout) is elaborated here in Section 3 again:

Theorem 2.1. *(i) Let M be a connected oriented smooth* 4*-manifold. Making use of its smooth structure only, a von Neumann algebra $\mathfrak{R}$ can be constructed which is geometric in the sense that it is generated by local operators including all bounded complexified algebraic (i.e., formal or stemming from a metric) curvature tensors of M and $\mathfrak{R}$ itself is a hyperfinite factor of type* II_1 *hence is unique up to abstract isomorphism of von Neumann algebras.*

(ii) Furthermore M admits an embedding $M \subset \mathfrak{R}$ via projections and this embedding induces a Riemannian structure (M, g) whose Riemannian curvature, if bounded, satisfies $R_g \in \mathfrak{R}$. Two connected oriented smooth 4-manifolds M, N with corresponding embeddings have abstractly isomorphic von Neumann algebras however not canonically. Nevertheless different abstract isomorphisms between these von Neumann algebras induce orientation-preserving diffeomorphisms of M and N respectively i.e. leave their embeddings unchanged. Hence up to diffeomorphisms every connected oriented smooth 4-manifold admits an embedding into a commonly given abstract von Neumann algebra $\mathfrak{R}$.

(iii) Finally taking M its von Neumann algebra $\mathfrak{R}$ admits a non-faithful representation on a certain complex separable Hilbert space such that the unitary equivalence class of this representation is invariant under orientation-preserving diffeomorphisms of M (hence gives rise to a smooth 4-manifold invariant).

We see by item (i) of Theorem 2.1 that the operator algebra $\mathfrak{R}$, when regarded to operate on a differentiable manifold, contains all bounded algebraic curvature tensors along it whenever the real dimension of this manifold is precisely *four*. Moreover $\mathfrak{R}$ is generated by local operators more precisely by algebraic operators R acting like *multiplication* operators and by operators L_X acting like *first order differential* operators. Picking canonically conjugate pairs i.e. necessarily unbounded self-adjoint operators $R, \sqrt{-1}L_X \in T_1\mathfrak{R}$ satisfying $[R, \sqrt{-1}L_X] = \hbar$ shows that $T_1\mathfrak{R}$ contains a CCR algebra hence revealing the *bosonic* character of $\mathfrak{R}$. Recall that in classical general relativity *local* gravitational phenomena are caused by the curvature of space-time. Hence interpreting $\mathfrak{R}$ physically as an operator algebra of local observables the corresponding quantum theory is expected to be a *four dimensional quantum theory of pure gravity* (hence a bosonic theory). In this way we fulfill the *Heisenberg dictum* that a quantum theory should completely and unambigously be formulated and interpreted in terms of its local physical observables (and not the other way round). In modern understanding by a *physical theory* one means a two-level description of a branch of natural phenomena which are interrelated phenomenologically: the theory possesses a *syntax* provided by its mathematical core structure and a *semantics* which is the meaning i.e. interpretation of the bare mathematical model in terms of physical concepts. In this context our quantum theory is not merely a mathematical theory anymore but a physical theory. This is because the bare mathematical structure $\mathfrak{R}$ (together with a representation π on $\mathscr{H}$) is dressed up i.e., interpreted by assigning a physical (in fact, gravitational) meaning to the experiments consistently performable by the aid of this structure (i.e., the usual quantum measurements of operators $A \in \mathfrak{R}$ in pure states $v \in \mathscr{H}$ or in more general ones, see below). In our opinion it is of particular interest that the geometrical dimension—equal to four—is fixed at the semantical level only and it matches the known phenomenological dimension of space-time. This is in sharp contrast to e.g. string theory where the geometrical dimension of the theory is fixed already at its syntactical level i.e., by its mathematical structure (namely, demaninding the theory to be free of conformal anomaly) and turns out to be much higher than the phenomenological dimension of space-time. As a further clarification it is stressed that the aim here is *not* to quantize general relativity in one or another way but rather conversely: being already a successful theory at the classical level, general relativity should be somehow derived from this abstractly given quantum theory by taking an appropriate "classical limit" as naturally as possible.

Observables as the universal space of all space-times. Item (ii) of Theorem 2.1 says that any connected oriented smooth 4-manifold M embeds into $\mathfrak{R}$. This embedding is given by (8) and is "physical" for it maps a point of $x \in M$ i.e. a *classical elementary event* to a projection $P_x \in \mathfrak{R}$ i.e. a *quantum elementary event*. Moreover, as emphasized in Theorem 2.1 too, due to these inclusions and the uniqueness of the hyperfinite II_1 factor, $\mathfrak{R}$ can be considered as the collection of *all* classical space-time manifolds and we can interpret the appearance of the gauge group here as the manifestation of the diffeomorphism gauge symmetry of classical general relativity in this quantum theory. Indeed, orientation-preserving diffeomorphisms of M interchange its points as well as act on the corresponding operators (projections) in $\mathfrak{R}$ by unitary inner automorphisms i.e. $\mathrm{Diff}^+(M)$ embeds into $\mathrm{Aut}\,\mathfrak{R}$. Reformulating this in a more geometric language we can say that classical space-times within $\mathfrak{R}$ appear as special orbits of its gauge group. The operators in $\mathfrak{R}$ representing geometric points are the simplest possible operators namely projections. Consequently the full non-commutative algebra $\mathfrak{R}$ is not exhausted by operators representing points of space-time alone; it certainly contains much more operators—e.g. various projections which are not of geometric origin—therefore this "universal quantum space-time" is more than a bunch of all classical space-times. As a comparison, in *algebraic quantum field theory* [14] one starts with a particular smooth m-manifold M and considers an assignment $U \mapsto \mathfrak{R}(U)$ describing local algebras of observables along all open subsets $\emptyset \subsetneqq U \subsetneqq M$. However in our case, quite conversely, space-times are secondary structures only and all of them are injected into the unique observable algebra $\mathfrak{R}$ which is considered to be primary.

More examples of observables and fields. Let us take a closer look of the elements of $\mathfrak{R}$ and of $T_1\mathfrak{R} \supset \mathfrak{R}$. Referring to item (ii) of Theorem 2.1 again we see that the embedding of a connected oriented smooth 4-manifold M into $\mathfrak{R}$ canonically improves it to a Riemannian manifold (M,g). The metric arises simply by pulling back the standard scalar product on $\mathfrak{R}$ provided by τ (see Section 3 for details and concerning the notation below). Remember that given a connected oriented Riemannian 4-manifold (M,g) one can introduce the Hodge operator $*$ which satisfies $*^2 = 1$ on 2-forms hence induces a splitting

$$\Omega^2_c(M) = \Omega^+_c(M) \oplus \Omega^-_c(M)$$

into (anti)self-dual or ± 1-eigenspaces. Thus the Hodge star as an operator on $\Omega^2_c(M)$ with respect to this splitting has the form $* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ hence is symmetric; likewise the curvature tensor R_g of (M,g) looks like (5) i.e., is also a pointwise symmetric operator. We summarize these facts by writing that as pointwise algebraic operators $*, R_g \in C^\infty(M; S^2\wedge^2 T^*M) \subset C^\infty(M; \mathrm{End}(\wedge^2 T^*M))$. The complex-linear extension of $*$ whose trace satisfies $\tau(*) = 0$ i.e. is finite, (by considerations in Section 3) always gives an element $* \in \mathfrak{R}$ which is self-adjoint. Likewise if R_g is a bounded operator (in the sense explained in Section 3) then its complex-linear extension gives rise to $R_g \in \mathfrak{R}$ which is moreover self-adjoint i.e., $R^*_g = R_g$ because R_g is symmetric. Therefore its trace is also real. Proceeding further R_g satisfies the algebraic Bianchi identity. It has the form $b(R_g) = 0$ where b is a certain fiberwise averaging operator on symmetric bundle morphisms satisfying $b^2 = b$ and $b^* = b$ with respect to the metric. This induces a g-orthogonal decomposition $S^2\wedge^2 T^*_xM \otimes \mathbb{C} = \ker b_x \oplus \mathrm{im}\, b_x$ at every point $x \in M$. A subtlety of *four dimensions* is that $\mathrm{im}\, b_x \subset S^2\wedge^2 T^*_xM \otimes \mathbb{C}$ is spanned by the symmetric operator $* \in S^2\wedge^2 T^*_xM \otimes \mathbb{C}$ alone cf. [22]. Consequently $b(R_g) = 0$ is equivalent to saying that $g(R_g, *) = 0$ i.e., $R_g, * \in S^2\wedge^2 T^*_xM \otimes \mathbb{C}$ are such that R_g is g-orthogonal to $*$ at every $x \in M$. By exploiting that $*$ is a pointwise symmetric operator we can use the pointwise equality $g(R_g, *) = \mathrm{tr}(R_g*)$ and then (7) to compute the trace of the product of R_g and $*$ in $\mathfrak{R}$ as follows:

$$\tau(R_g*) = \frac{1}{6}\int\limits_M \mathrm{tr}(R_g*)\mu_g = 0$$

which geometrically means that $(\hat{R}_g, \hat{*}) = 0$ i.e., $\hat{R}_g$ is perpendicular to $\hat{*}$ as vectors within $\mathscr{H}$. Hence the Bianchi identity.

The vacuum Einstein equation with cosmological constant $\Lambda \in \mathbb{R}$ reads as $\mathrm{Ric} = \Lambda g$ therefore it is equivalent to the vanishing of the traceless Ricci part Ric_0 of R_g in (5). From these it readily follows, as noticed in [22], that (M,g) is Einstein i.e., satisfies the vacuum Einstein equation with cosmological constant if and only if $*R_g = R_g*$ or equivalently $*R_g*^{-1} = R_g$ within $\mathfrak{R}$. Moreover taking into account that $\mathrm{Weyl}^{\pm} \in C^\infty(M; S^2\wedge^2 T^*M\otimes\mathbb{C})$ in (5) are traceless their pointwise scalar products with the identity of $C^\infty(M; S^2\wedge^2 T^*M\otimes\mathbb{C})$ look like $g(\mathrm{Weyl}^{\pm}, 1) = \mathrm{tr}(\mathrm{Weyl}^{\pm}) = 0$. Using (7) this shows again that the corresponding vectors $\widehat{\mathrm{Weyl}}^{\pm} \in \mathscr{H}$ are perpendicular to $\hat{1} \in \mathscr{H}$ i.e., $(\widehat{\mathrm{Weyl}}^{\pm}, \hat{1}) = 0$. Moreover $\mathrm{Scal} = 4\Lambda$ consequently via (5) we get $\tau(R_g) = (\hat{R}_g, \hat{1}) = \frac{1}{12}(\widehat{\mathrm{Scal}}, \hat{1}) = \frac{4\Lambda}{12}(\hat{1}, \hat{1}) = \frac{\Lambda}{3}$. All of these serve as a motivation for the following operator-algebraic reformulation or generalization of the classical vacuum Einstein equation with cosmological constant [13, Definition 3.1]:

Definition 2.1. *Let M be a connected oriented smooth 4-manifold and $\mathfrak{R}$ its II_1-type hyperfinite factor von Neumann algebra as before.*

(i) A refinement of $\mathfrak{R}$ *is a pair $(\mathfrak{R}, *)$ where $1 \neq * \in \mathfrak{R}$ and satisfies $*^2 = 1$;*

(ii) An operator $Q \in \mathfrak{R}$ solves the quantum vacuum Einstein equation *with respect to $(\mathfrak{R}, *)$ if*

$$\begin{cases} Q^{\circledast} &= Q \quad \textit{(self-adjointness)} \\ \tau(Q*) &= 0 \quad \textit{(algebraic Bianchi identity)} \\ *Q*^{-1} &= Q \quad \textit{(Einstein condition)}; \end{cases}$$

(iii) The trace $\tau(Q) =: \frac{\Lambda}{3} \in \mathbb{R}$ is called the corresponding quantum cosmological constant.

Note that the curvature tensor R_g of an Einstein manifold (M, g) if bounded as an operator always solves the quantum vacuum Einstein equation with respect to the canonical refinement provided by metric (anti)self-duality. However in sharp contrast to the classical Einstein equation whose solution is a metric therefore is non-linear, its quantum generalization is linear hence easily solvable. Of course this is beacuse in the generalization we do not request the solution (which can be any non-local operator) to originate from a metric. Indeed, given $B \in \mathfrak{R}$ then $S = \frac{1}{2}(B + B^{\circledast})$ is self-adjoint and then picking an arbitrary refinement $(\mathfrak{R}, *)$ and taking into account that $*$ is always self-adjoint, the averaged operator $Q := \frac{1}{2}\big(S + *S*\big) - \tau(S*)* \in \mathfrak{R}$ is automatically a solution of the quantum vacuum Einstein equation; moreover using $\tau(*) = 0$ its trace $\frac{\Lambda}{3}$ is equal to $\tau(B)$ hence is independent of the particular refinement. It is important to note that, contrary to smooth solutions, many singular solutions of classical general relativity theory cannot be interpreted as observables because their curvatures lack being bounded operators hence do not belong to $\mathfrak{R}$. As a result, we expect that the classical (Riemannian) Schwarzschild or Kerr black hole solutions and more generally gravitatioal fields of isolated bodies (cf. [9, 10]) give rise not to *observables* in $\mathfrak{R}$ but rather *fields* in $T_1\mathfrak{R} \supset \mathfrak{R}$.

The classical phase of the quantum theory. Now we are in a position to introduce another phase of the proposed quantum field theoretic interpretation of $\mathfrak{R}$ which—unlike its pure quantum phase above consisting of a single infinitely high temperature thermal equilibrium state τ with the corresponding unique standard representation π of $\mathfrak{R}$ only—is built up from a huge moduli of finite temperature thermal equilibrium states $F_{\Sigma,\omega}$ with corresponding new representations ρ_M of $\mathfrak{R}$. Moreover we want to understand these states as classical (Riemannian) vacuum space-times (M, g). These will be achieved by introducing a new dynamics on $\mathfrak{R}$.

Definition 2.2. *Let $(\mathfrak{R}, *)$ be a refinement of $\mathfrak{R}$ as in Definition 2.1. The condition $* \neq 1$ and $*^2 = 1$ implies that there exists a basis in which $* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ therefore introducig the skew-Hermitian operator $\log * := \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{-1}\pi \end{pmatrix}$ for every $t \in \mathbb{R}$ we can set $*^t := \mathrm{e}^{t\log *}$ which is a unitary within $\mathfrak{R}$. The corresponding 1-parameter family of unitary inner automorphisms on $\mathfrak{R}$ given by*

$$A \longmapsto *^t A *^{-t}$$

for all $A \in \mathfrak{R}$ and $t \in \mathbb{R}$ introduces a dynamics on $\mathfrak{R}$ what we call the Hodge dynamics. *Accordingly $\big(\mathfrak{R}, \{\mathrm{Ad}_{*^t}\}_{t\in\mathbb{R}}\big)$ is a* Hodge dynamical system on the hyperfinite II_1 factor.

How these dynamics on $\mathfrak{R}$ look like? Observe first that contrary to the unique and trivial Tomita–Takesaki dynamics these highly non-unique Hodge dynamics are already non-trivial however $*^2=1$ says that they are uniformly periodic with physical duration $\hbar\beta = 2t_{\rm Planck}$ consequently also give rise to a uniform *temperature* equal to $\frac{\hbar}{2k_Bt_{\rm Planck}} = \frac{1}{2}T_{\rm Planck}$. In particular an embedding $M\subset\mathfrak{R}$ induces an oriented Riemannian 4-manifold (M,g) hence thanks to its Hodge operator $*\in\mathfrak{R}$ it generates a Hodge dynamics on $\mathfrak{R}$ too. What kind of operators are its (non-trivial) fixed points?

Lemma 2.1. *Let M be a connected oriented smooth 4-manifold and consider its embedding $M\subset\mathfrak{R}$ given by $x\mapsto P_x$ and the induced oriented Riemannian 4-manifold (M,g) as in Theorem 2.1. Take its Hodge and curvature operators $*, R_g$ respectively. We know that $*\in\mathfrak{R}$ and assume that $R_g\in\mathfrak{R}$ too.*

Then $M\subset\mathfrak{R}$ is invariant under the induced Hodge dynamics on $\mathfrak{R}$ i.e. the corresponding projections satisfy $*^tP_x*^{-t}=P_x$ for every $x\in M$ and $t\in\mathbb{R}$; moreover (M,g) is Einstein if and only if its Riemannian curvature operator is also invariant i.e. $*^tR_g*^{-t}=R_g$ for all $t\in\mathbb{R}$.

Proof. Concerning $M\subset\mathfrak{R}$ given by $x\mapsto P_x$ where $P_x\in\mathfrak{R}$ is a projection, it follows from its construction (see the construction of the map (8) in Section 3) that P_x commutes with $*$ hence is invariant under the corresponding Hodge dynamics. Consequently the dynamics on $\mathfrak{R}$ generated by $M\subset\mathfrak{R}$ leaves this embedding unchanged.

Concerning R_g the assertion is straightforward as by recalling [22] we already noticed that (M,g) is Einstein if and only if $*$ commutes with R_g (see Definition 2.1). □

Therefore classical Riemannian general relativity in vacuum i.e. an oriented Riemannian 4-manifold (M,g) satisfying the vacuum Einstein equation (with cosmological constant) $R_g=\Lambda g$ can be characterized quantum mechanically as a subset $\{P_x\}_{x\in M}\cup\{R_g\}\subset\mathfrak{R}$ which belongs to the fixed-point-set of a natural non-trivial dynamics on this operator algebra generated by this subset.[3]

Having introduced non-trivial dynamics on $\mathfrak{R}$ then following the KMS analysis of [3, 14] now we would like to find states on $\mathfrak{R}$ which are stationary and stable under these finite temperature dynamics hence could be termed as "thermal equilibrium states". This is possible if we recall now item (iii) of Theorem 2.1 which is based on Lemma 3.1 in Section 3. Consider again M and take a pair (Σ,ω) consisting of an (immersed) closed orientable surface $\Sigma\looparrowright M$ with induced orientation and $\omega\in\Omega^2_c(M;\mathbb{C})$ which is also closed i.e. $\mathrm{d}\omega=0$. Consider the differential geometric $\mathbb{C}$-linear functional $A\mapsto\int_\Sigma A\omega$ and continuously extend it from $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$ to a state $F_{\Sigma,\omega}:\mathfrak{R}\to\mathbb{C}$. This extension is unique because $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$ is norm-dense in $\mathfrak{R}$.

Lemma 2.2. *Consider an embedding $M\subset\mathfrak{R}$ with induced oriented Riemannian 4-manifold (M,g) and corresponding Hodge dynamical system $\left(\mathfrak{R},\{\mathrm{Ad}_{*^t}\}_{t\in\mathbb{R}}\right)$. Assume that both $\Sigma\looparrowright M$ and $\omega\in\Omega^2_c(M;\mathbb{C})$ are self-dual in the sense that the (real and not necessarily compactly supported) Poincaré-dual 2-form $\eta_\Sigma\in\Omega^2(M)\subset\Omega^2(M;\mathbb{C})$ of Σ is self-dual i.e. $*\eta_\Sigma=\eta_\Sigma$ and likewise $\omega\in\Omega^2_c(M;\mathbb{C})$ satisfies $*\omega=\omega$.*

Then the corresponding state $F_{\Sigma,\omega}:\mathfrak{R}\to\mathbb{C}$ is stationary under the induced Hodge dynamics i.e. $F_{\Sigma,\omega}(*^tA*^{-t})=F_{\Sigma,\omega}(A)$ for every $A\in\mathfrak{R}$ and $t\in\mathbb{R}$.

Moreover take $*'\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$ and let $\left(\mathfrak{R},\{\mathrm{Ad}_{(*')^t}\}_{t\in\mathbb{R}}\right)$ be a "nearby" dynamical system in the sense that $*'$ commutes with $*$ (therefore preserves $M\subset\mathfrak{R}$ too by Lemma 2.1) and $(*')^{2\pm\varepsilon}=1$ (i.e. having slightly disturbed periodicity). Then there exist $\Sigma'\looparrowright M$ and $\omega'\in\Omega^2_c(M;\mathbb{C})$ such that the "nearby" state $F_{\Sigma',\omega'}$ on $\mathfrak{R}$ is stationary under the "nearby" dynamics.

[3] Actually the full fixed-point-set of any dynamics is a von Neumann subalgebra which always contains the center of $\mathfrak{R}$. It is a normal subalgebra if and only if the dynamics is inner and periodic [24]. We conjecture that in our case this normal von Neumann subalgebra is generated by the self-adjoint subset $\{1\}\cup\{P_x\}_{x\in M}\cup\{*,R_g\}\subset\mathfrak{R}$. It seems there is a connection between 4 dimensional Einstein structures and normal von Neumann subalgebras of the hyperfinte II_1 factor.

Proof. Assume that $A \in \mathrm{End}(\Omega^2_c(M;\mathbb{C})) \cap \mathfrak{R}$. By its definition and the requested properties of the Poincaré-dual the restricted functional can be re-written as a unitary L^2 scalar product on the space $\Omega^2_c(M;\mathbb{C}) \times \Omega^2(M;\mathbb{C})$ over (M,g) as follows:

$$F_{\Sigma,\omega}(A) = \int\limits_{\Sigma} A\omega = \int\limits_{M} A\omega \wedge \eta_\Sigma = \int\limits_{M} A\omega \wedge *\overline{\eta}_\Sigma = \left(A\omega\,,\,\eta_\Sigma\right)_{L^2(M,g)}\,.$$

Now let $t \in \mathbb{R}$ be arbitrary and taking into account that $*^t$ is unitary with respect to this L^2 scalar product we can rearrange things as $\left(*^t A *^{-t}\omega, \eta_\Sigma\right)_{L^2(M,g)} = \left(A *^{-t}\omega, *^{-t}\eta_\Sigma\right)_{L^2(M,g)}$. However

$$\frac{\mathrm{d}}{\mathrm{d}t}F_{\Sigma,\omega}(*^t A *^{-t}) = -\left(A *^{-t}(\log *)\omega\,,\, *^{-t}\eta_\Sigma\right)_{L^2(M,g)} - \left(A *^{-t}\omega\,,\, *^{-t}(\log *)\eta_\Sigma\right)_{L^2(M,g)} = 0$$

because we know that in the (anti)self-dual basis $\log * = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \sqrt{-1}\pi \end{smallmatrix}\right)$ hence its action on the self-dual elements $\left(\begin{smallmatrix} \omega \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \eta_\Sigma \\ 0 \end{smallmatrix}\right)$ vanishes. This implies by continuity the stationarity of $F_{\Sigma,\omega}$ for all $A \in \mathfrak{R}$.

Concerning stability since by assumption $A \mapsto *'A(*')^{-1}$ is a $*$-automorhism of $\mathfrak{R}$ we necessarily find that $*' \in \mathrm{End}(\Omega^2_c(M;\mathbb{C})) \cap \mathfrak{R}$ is unitary and maps $\Omega^2_c(M;\mathbb{C})$ into itself like $*$ does; thus there exists a unitary operator $U \in \mathrm{End}(\Omega^2_c(M;\mathbb{C})) \cap \mathfrak{R}$ such that $*' = *U$. It readily follows from $*'* = *U*$ and $**' = U$ that the assumption that $*$ and $*'$ commute implies $*$ also commutes with U hence the perturbed dynamics decomposes as $(*')^t = *^t U^t$. The further condition $(*')^{2\pm\varepsilon} = 1$ eventually yields that in the (anti)self-dual basis we can write $U = z\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pm 1 \end{smallmatrix}\right)$ where the phase $z = z(\varepsilon) \in \mathbb{C}$ satisfies $z(0) = 1$ but $|z(\varepsilon)| = 1$ in general. Consequently the two dynamical systems differ by a simple inner automorphism Ad_U of $\mathfrak{R}$ only. Therefore putting simply $\Sigma' := \Sigma$ hence $\eta_{\Sigma'} := \eta_\Sigma$ and $\omega' := \omega$ we can introduce a new state which simply satisfies $F_{\Sigma',\omega'}(B) = F_{\Sigma,\omega}(B)$ i.e. coincides with the old state. Then we compute

$$\begin{aligned} F_{\Sigma',\omega'}((*')^t A (*')^{-t}) &= F_{\Sigma,\omega}(*^t U^t A U^{-t} *^{-t}) \\ &= \left(AU^{-t} *^{-t}\omega\,,\, U^{-t} *^{-t}\eta_\Sigma\right)_{L^2(M,g)} \\ &= |z|^{-t}\left(Az^t U^{-t} *^{-t}\omega\,,\, z^t U^{-t} *^{-t}\eta_\Sigma\right)_{L^2(M,g)} \\ &= \left(A(z^{-1}U)^{-t} *^{-t}\omega\,,\, (z^{-1}U)^{-t} *^{-t}\eta_\Sigma\right)_{L^2(M,g)} \end{aligned}$$

hence using again that $(\log *)\omega = 0 = (\log *)\eta_\Sigma$ and noticing that $\log(z^{-1}U) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \text{something} \end{smallmatrix}\right)$ hence likewise $\log(z^{-1}U)\omega = 0 = \log(z^{-1}U)\eta_\Sigma$ we can differentiate the last expression and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}F_{\Sigma',\omega'}((*')^t A (*')^{-t}) = 0$$

demonstrating that $F_{\Sigma',\omega'}$ is stationary as stated. $\square$

Thus summarizing: in addition to its infinitely high temperature pure *quantum phase* above (consisting of the single KMS state τ and corresponding standard representation π on $\mathscr{H}$), we hereby introduce the so-called *classical phase* of the quantum field theory based on $\mathfrak{R}$ to be the moduli space comprising all finite temperature thermal equilibrium states like $F_{\Sigma,\omega}$ on $\mathfrak{R}$ provided by all possible embeddings $M \subset \mathfrak{R}$ of connected oriented smooth 4-manifolds into this algebra such that the induced Riemannian 4-manifold (M,g) is Einstein. The uniform temperature of these states is $\frac{1}{2}T_{\mathrm{Planck}}$. Application of the GNS construction to states in the classical phase gives further representations of $\mathfrak{R}$. More precisely it is proved in Lemma 3.1 below that given an embedding $M \subset \mathfrak{R}$ and a functional $F_{\Sigma,\omega} : \mathfrak{R} \to \mathbb{C}$ the isomorphism class of the corresponding representation ρ_M on a complex separable Hilbert space

$\mathscr{I}(M)^\perp$ is independent of the particular $\Sigma \looparrowright M$ and $\omega \in \Omega^2_c(M;\mathbb{C})$ i.e. these representations depend only on M. Moreover (unlike the standard representation π) all of these representations ρ_M are not faithful. Consequently the states $F_{\Sigma,\omega}$ (unlike the tracial state τ) are *not* KMS states.

Vacuum general relativity from a spontaneous symmetry breaking. The modern physical and mathematical basis for thinking about space, time, gravity and the structure and history of the Universe (cosmology) has been provided by classical general relativity in the form of (pseudo-)Riemannian 4-manifolds (M,g) subject to Einstein's field equation. As we have summarized in Theorem 2.1 on the one hand out of M alone an operator algebra $\mathfrak{R}$ and a Hilbert space $\mathscr{I}(M)^\perp$ carrying a representation ρ_M of $\mathfrak{R}$ can be constructed; on the other hand $M \subset \mathfrak{R}$ holds which automatically induces a Riemannian structure (M,g) which is moreover Einstein if and only if this embedding together with its Riemannian tensor R_g is invariant with respect to the Hodge dynamics on $\mathfrak{R}$ also induced by the embedding. Remember the plain technicality that a curvature tensor of (M,g) must be *bounded* in order to belong to the operator algebra. However it has been known for a long time that in general relativity the gravitational field of an isolated massive configuration cannot be regular everywhere [9, 10]. Thus we are forced to move towards a global non-singular cosmological context when trying to work out a physical interpretation of the present quite circular interaction between $\mathfrak{R}$ and M. Therefore hereby we make a *working hypothesis*: the unique abstract triple $(\mathfrak{R},\mathscr{H},\pi)$ as a quantum theory describes the Universe from the Big Bang till about $2t_{\rm Planck} \approx 1.07\times 10^{-43}$ sec while a highly non-unique triple $(\mathfrak{R},\mathscr{I}(M)^\perp,\rho_M)$ giving rise to a 4 dimensional vacuum space-time (M,g) in classical general relativity describes its evolution from $2t_{\rm Planck}$ onwards. A usual choice for (M,g) is the FLRW solution with or without cosmological constant. The "moment" $2t_{\rm Planck}$ would therefore formally or symbolically label the emergence or onset of a space-time structure in the course of the history of the Universe[4] establishing a physical correspondence between the two representations of $\mathfrak{R}$. We display symbolically this passage as

$$\begin{array}{ccc} (\mathfrak{R},\mathscr{H},\pi) & \Longrightarrow & \big(\text{A Hodge-invariant subset } \{P_x\}_{x\in M}\cup\{R_g\}\subset\mathfrak{R},\mathscr{I}(M)^\perp\subset\mathscr{H},\rho_M=\pi|_{\mathscr{I}(M)^\perp}\big) \\ \Updownarrow & & \Updownarrow \\ \text{The "quantum space-time"} & & \text{A particular 4 dimensional Riemannian vacuum space-time } (M,g) \\ \text{at infinite temperature} & & \text{at temperature } \frac{1}{2}T_{\rm Planck} \end{array} \tag{1}$$

having in mind a sort of phase transition at $2t_{\rm Planck}$ from the quantum to the classical phase of the theory via spontaneous symmetry breaking driven by cooling (or by a spontaneous jump from the unique Tomita–Takesaki to a particular Hodge dynamics on $\mathfrak{R}$). Note that this transition from the unique quantum regime $(\mathfrak{R},\mathscr{H},\pi)$ to a particular 4 dimensional classical vacuum regime (M,g) given by another representation $(\mathfrak{R},\mathscr{I}(M)^\perp,\rho_M)$ has been captured in the framework of algebraic quantum field theory [3, 14] as switching from the unique representation π to a different particular representation ρ_M of the same algebra $\mathfrak{R}$. One can also formally label the transition with $\frac{1}{2}T_{\rm Planck}\approx 7.06\times 10^{31}$ K; this high temperature is reasonable if we make a further technical observation that π is induced by the unique tracial state τ on $\mathfrak{R}$ hence one can imagine that $(\mathfrak{R},\mathscr{H},\pi)$ describes a quantum statistical mechanical system in an infinitely high temperature KMS equilibrium state τ (hence having trivial modular dynamics in the sense of [8]).

Finally let us mention that although this picture is unphysical because uses imaginary time it avoids the problem of "initial singularity" in a natural way for it dissolves the dynamically non-trivial classical

[4] We are aware of, but cannot do better, how nonsense it is, strictly speaking, to talk about the emergence of space and time at the moment $2t_{\rm Planck}$; this assertion requires the contradictory pre-supposition of time before $t_{\rm Planck}$.

evolutionary phase of the Universe in a dynamically trivial quantum phase about $2t_{\mathrm{Planck}}$ i.e. "just before the initial singularity".

A comment on the measurement problem. It is dictated by the mathematical structure of the II_1 hyperfinite factor that if $A \in \mathfrak{R}$ represents a (bounded) observable while $B \in \mathfrak{R}$ a state (regarded as a "density matrix") describing a physical situation in this theory then without further specifications on A, B the expectation value of A in B should be given by $\tau(AB) \in \mathbb{C}$ i.e. a complex number in general. We may then ask ourselves: after performing the physical experiment to measure the quantity represented by A in the state represented by B should we expect that the state will necessarily "collapse" to an eigenstate B_λ of A? In our opinion *no* and this is a *fundmental difference* between gravity and quantum mechanics [4]. Namely, in quantum mechanics an ideal observer compared to the physical object to be observed is *infinitely large* hence the immense physical interaction accompanying the measurement procedure drastically disturbs the entity leading to the collapse of its state. However, in sharp contrast to this, in gravity an ideal observer is *infinitely small* hence it is reasonable to expect that measurements might *not* alter gravitational states. This is in accordance with our old experience concerning measurements in astronomy.

As an example we can compute the energy content of a classical space-time (M, g) occupied by pure gravitation only i.e. without matter. As we have seen in the classical phase the valid dynamics on $\mathfrak{R}$ is the Hodge dynamics generarted by (M, g). It is therefore reasonable to suppose that the energy should be captured by measuring the Hamilton operator $H(M, g)$ of the Hodge dynamics in the state represented by the curvature operator R_g satisfying the vacuum Einstein equation. Let us suppose that this latter operator belongs to $\mathfrak{R}$ then it is self-adjoint. Moreover recalling that $*^t = \mathrm{e}^{t \log *}$ we can introduce the corresponding self-adjoint Hamilton operator $H(M, g) := \frac{1}{\sqrt{-1}} \log *$ which looks like $H(M, g) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \pi \end{smallmatrix}\right)$ in the (anti)self-dual basis hence belongs to $\mathfrak{R}$ (cf. Definition 2.2). In addition the curvature in this basis has the shape (5). Taking into account that both R_g and $H(M, g)$ act pointwise on M more precisely are bundle morphisms over $\wedge^2 T^*M \otimes \mathbb{C}$ hence so is their product $H(M, g)R_g \in \mathfrak{R}$ we can use (7) to compute the trace. Recalling that $\mathrm{Scal} = 4\Lambda$ and using the normalized measure μ_g (which surely exists if M is compact) we obtain

$$E(M, g) = \tau(H(M, g)R_g) = \frac{1}{6} \int\limits_M \mathrm{tr}(H(M, g)R_g)\mu_g = \frac{1}{6} \cdot 3\pi \cdot \frac{4\Lambda}{12} = \frac{\pi}{6}\Lambda$$

consequently the quantum mechanical energy content of a vacuum space-time is proportional to the cosmological constant as one would expect on physical grounds. This also helps to recover the resulting cosmological constant from "inside" the spontaneously broken state.

This is a good point to exit our yet admittedly incomplete trial to set up some physical picture behind the unique II_1 hyperfinite factor.

3 Appendix: mathematical constructions

In this section for completeness and the Reader's convenience we recall and partly extend further the mathematical exposition in [13] and give a detailed proof of Theorem 2.1. First we shall exhibit a simple self-contained two-step construction of a von Neumann algebra attached to any oriented smooth 4-manifold. Then the structure of this algebra will be explored in some detail. Finally we exhibit a new (i.e. not the standard) representation of this von Neumann algebra induced by the whole contruction. For clarity we emphasize that the forthcoming constructions are rigorous in the sense that *no* physical ideas, considerations, steps, etc. to be used.

Construction of an algebra. Take the isomorphism class of a connected oriented smooth 4-manifold (without boundary) and from now on let M be a once and for all fixed representative in it carrying the action of its own orientation-preserving group of diffeomorphisms $\mathrm{Diff}^+(M)$. Among all tensor bundles $T^{(p,q)}M$ over M the 2^{nd} exterior power $\wedge^2 T^*M \subset T^{(0,2)}M$ is the only one which can be endowed with a pairing in a natural way i.e., with a pairing extracted from the smooth structure (and the orientation) of M alone. Indeed, consider its associated vector space $\Omega^2_c(M) := C^\infty_c(M; \wedge^2 T^*M)$ of compactly supported smooth 2-forms on M. Define a pairing $\langle \cdot\, , \cdot \rangle_{L^2(M)} : \Omega^2_c(M) \times \Omega^2_c(M) \to \mathbb{R}$ via integration:

$$\langle \alpha, \beta \rangle_{L^2(M)} := \int\limits_M \alpha \wedge \beta \tag{2}$$

and observe that this pairing is non-degenerate however is *indefinite* in general thus can be regarded as an indefinite scalar product on $\Omega^2_c(M)$. It therefore induces an indefinite real quadratic form Q on $\Omega^2_c(M)$ given by $Q(\alpha) := \langle \alpha, \alpha \rangle_{L^2(M)}$. Let $C(M)$ denote the complexification of the infinite dimensional real Clifford algebra associated with $(\Omega^2_c(M), Q)$. Because Clifford algebras are usually constructed out of definite quadratic forms, we summarize this construction [19, Section I.§3] to make sure that the resulting object $C(M)$ is well-defined i.e. is not sensitive for the indefiniteness of (2). To begin with, let $V_m \subset \Omega^2_c(M)$ be an m dimensional real subspace and assume that $Q_{r,s} := Q|_{V_m}$ has signature (r,s) on V_m that is, the maximal positive definite subspace of V_m with respect to $Q_{r,s}$ has dimension r while the dimension of the maximal negative definite subspace is s such that $r+s=m$ by the non-degeneracy of $Q_{r,s}$. Then out of the input data $(V_m, Q_{r,s})$ one constructs in the standard way a finite dimensional real Clifford algebra $C_{r,s}(M)$ with unit $1 \in C_{r,s}(M)$ and an embedding $V_m \subset C_{r,s}(M)$ with the property $\alpha^2 = Q_{r,s}(\alpha)1$ for every element $\alpha \in V_m$. This real algebra depends on the signature (r,s) however fortunately its complexification $C_m(M) := C_{r,s}(M) \otimes \mathbb{C}$ is already independent of it. In fact, if $\mathfrak{M}_k(\mathbb{C})$ denotes the algebra of $k \times k$ complex matrices, then it is well-known [19, Section I.§3] that $C_0(M) \cong \mathfrak{M}_1(\mathbb{C})$ while $C_1(M) \cong \mathfrak{M}_1(\mathbb{C}) \oplus \mathfrak{M}_1(\mathbb{C})$ and the higher dimensional cases follow from the complex periodicity $C_{m+2}(M) \cong C_m(M) \otimes \mathfrak{M}_2(\mathbb{C})$. Consequently depending on the parity $C_m(M)$ is isomorphic to either $\mathfrak{M}_{2^{\frac{m}{2}}}(\mathbb{C})$ or $\mathfrak{M}_{2^{\frac{m-1}{2}}}(\mathbb{C}) \oplus \mathfrak{M}_{2^{\frac{m-1}{2}}}(\mathbb{C})$. These imply that 2-step-chains of successive embeddings of real subspaces $V_m \subset V_{m+1} \subset V_{m+2} \subset \Omega^2_c(M)$ starting with $V_0 = \{0\}$ and given by iterating $\omega \mapsto \binom{\omega}{0}$ provide us with injective algebra homomorphisms $\mathfrak{M}_{2^{\frac{m}{2}}}(\mathbb{C}) \hookrightarrow \mathfrak{M}_{2^{\frac{m}{2}+1}}(\mathbb{C})$ having the shape $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$. Therefore $C(M)$ is isomorphic to the injective limit of this directed system, that is there exists a linear-algebraic isomorphism

$$C(M) \cong \bigcup_{n=0}^{+\infty} \mathfrak{M}_{2^n}(\mathbb{C}) \tag{3}$$

or equivalently

$$C(M) \cong \mathfrak{M}_2(\mathbb{C}) \otimes \mathfrak{M}_2(\mathbb{C}) \otimes \dots$$

because this injective limit is also isomorphic to the infinite tensor product of $\mathfrak{M}_2(\mathbb{C})$'s. For clarity note that being (2) a non-local operation, $C(M)$ is a genuine global infinite dimensional object.

It is well-known (cf. [7, Section I.3]) that *any* complexified infinite Clifford algebra like $C(M)$ above generates *the* II_1-type hyperfinite factor von Neumann algebra. Let us summarize this procedure too (cf. [1, Section 1.1.6]). It readily follows that $C(M)$ possesses a unit $1 \in C(M)$ and its center comprises the scalar multiples of the unit only. Moreover $C(M)$ continues to admit a canonical embedding $\Omega^2_c(M; \mathbb{C}) \subset C(M)$ satisfying $\omega^2 = Q(\omega)1$ where now Q denotes the quadratic form induced by the complex-bilinear extension of (2). We also see via (3) already that $C(M)$ is a complex $*$-algebra whose $*$-operation (provided by taking Hermitian matrix transpose, a non-local operation) is written as $A \mapsto A^{\circledast}$. The isomorphism (3) also shows that if $A \in C(M)$ then one can pick the smallest $n \in \mathbb{N}$ such that $A \in \mathfrak{M}_{2^n}(\mathbb{C})$ consequently A has a finite trace defined by $\tau(A) := 2^{-n}\mathrm{Trace}(A)$ i.e., taking the usual normalized trace of the corresponding $2^n \times 2^n$ complex matrix. It is straightforward that $\tau(A) \in \mathbb{C}$ does not depend on n. We can then define a sesquilinear inner product on $C(M)$ by $(A,B) := \tau(AB^{\circledast})$ which is non-degenerate thus the completion of $C(M)$ with respect to the norm $\|\cdot\|$ induced by $(\cdot\,,\cdot)$ renders $C(M)$ a complex Hilbert space what we shall write as $\mathscr{H}$ and its Banach algebra of all bounded linear operators as $\mathfrak{B}(\mathscr{H})$. Multiplication in $C(M)$ from the left on itself is continuous hence gives rise to a representation $\pi : C(M) \to \mathfrak{B}(\mathscr{H})$. Finally our central object effortlessly emerges as the weak closure of the image of $C(M)$ under π within $\mathfrak{B}(\mathscr{H})$ or equivalently, by referring to von Neumann's bicommutant theorem [1, Theorem 2.1.3] we put

$$\mathfrak{R} := (\pi(C(M)))'' \subset \mathfrak{B}(\mathscr{H}) \ .$$

This von Neumann algebra of course admits a unit $1 \in \mathfrak{R}$ moreover continues to have trivial center i.e., is a factor. Moreover by construction it is hyperfinite. The trace τ as defined extends from $C(M)$ to $\mathfrak{R}$ and satisfies $\tau(1) = 1$. Moreover [1, Proposition 4.1.4] this trace is unique on $\mathfrak{R}$. Likewise we obtain by extension a representation $\pi : \mathfrak{R} \to \mathfrak{B}(\mathscr{H})$. The canonical inclusion $\Omega^2_c(M;\mathbb{C}) \subset C(M)$ recorded above extends to both $\Omega^2_c(M;\mathbb{C}) \subset \mathscr{H}$ and $\Omega^2_c(M;\mathbb{C}) \subset \mathfrak{R}$. Thus in order to carefully distinguish the two different completions $\mathfrak{R}$ and $\mathscr{H}$ of one and the same object $C(M)$ we shall write $A \in \mathfrak{R}$ but $\hat{B} \in \mathscr{H}$ from now on as usual. This is necessary since $\mathfrak{R}$ and $\mathscr{H}$ are very different for example as $\mathrm{U}(\mathscr{H})$-modules: given a unitary operator $V \in \mathrm{U}(\mathscr{H})$ then $A \in \mathfrak{R}$ is acted upon as $A \mapsto VAV^{-1}$ but $\hat{B} \in \mathscr{H}$ transforms as $\hat{B} \mapsto V\hat{B}$. Using this notation and introducing $\hat{A} := \pi(A)\hat{1}$ the trace always can be written as a scalar product with the image of the unit in $\mathscr{H}$ that is, for every $A \in \mathfrak{R}$ we have

$$\tau(A) = (\hat{A}, \hat{1})$$

yielding a general and geometric expression for the trace.

Exploring the algebra $\mathfrak{R}$. Before proceeding further let us make a digression here to gain a better picture. This is desirable because taking the weak closure like $\mathfrak{R}$ of some explicitly known structure like $C(M)$ often involves a sort of loosing control over the latter. Nevertheless we already know promisingly that $\mathfrak{R}$ is a hyperfinite factor von Neumann algebra of II_1-type. Let us now exhibit some of its elements.

1. Our first examples are the 2-forms themselves as it follows from the already mentioned canonical embedding $\Omega^2_c(M;\mathbb{C}) \subset C(M)$ combined with $C(M) \subset \mathfrak{R}$. This also implies that in fact $\mathfrak{R} \supset C(M)$ is weakly generated by $1 \in C(M)$ and all finite products of 2-forms $\omega_1\omega_2\dots\omega_n$ within the associative algebra $C(M)$(and likewise, $\mathscr{H} \supset C(M)$ is the closure of the unit and all finite products too). We might call this as the *first picture* on $\mathfrak{R}$ provided by the embedding $\Omega^2_c(M;\mathbb{C}) \subset C(M)$ however this description is not very informative.

2. To see more examples, let us return to the Clifford algebra in (3) for a moment. We already know that there exists a canonical embedding $\Omega^2_c(M;\mathbb{C}) \subset C(M)$. In addition to this let us find a Clifford module for $C(M)$. Consider again any finite *even* dimensional approximation $C_m(M) = C_{r,s}(M) \otimes \mathbb{C}$ constructed from $(V_m, Q_{r,s})$ where now $V_m \subset \Omega^2_c(M)$ is a real even $m = r+s$ dimensional subspace. Choose any $2^{\frac{m}{2}}$ dimensional complex vector subspace S_m within $\Omega^2_c(M;\mathbb{C})$. If $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ denotes the associative algebra of *all* $\mathbb{C}$-linear transformations of $\Omega^2_c(M;\mathbb{C})$ then $S_m \subset \Omega^2_c(M;\mathbb{C})$ induces an embedding $\mathrm{End}\, S_m \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ moreover we know that $\mathrm{End}\, S_m \cong \mathfrak{M}_{2^{\frac{m}{2}}}(\mathbb{C}) \cong C_m(M)$. Therefore we obtain a non-canonical inclusion $C_m(M) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ for every fixed $m \in 2\mathbb{N}$. Furthermore $S_m \subset S_{m+2} \subset \Omega^2_c(M;\mathbb{C})$ given by $\omega \mapsto \binom{\omega}{0}$ induces a sequence $C_m(M) \subset C_{m+2}(M) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ for Clifford algebras which is compatible with the previous ascending chain of their matrix algebra

realizations. Consequently taking the limit $m \to +\infty$ we come up with a non-canonical injective linear-algebraic homomorphism

$$C(M) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C})) \tag{4}$$

and this embedding gives rise to the *second picture* on $\mathfrak{R}$. Of course, unlike the first picture above, this second one does not exist in the finite dimensional case.

Although the algebra $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ is yet too huge, we can at least exhibit some of its elements. The simplest ones are the 2-forms themselves because $\Omega^2_c(M;\mathbb{C}) \subset C(M) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ holds as we already know. Furthermore orientation-preserving diffeomorphisms act $\mathbb{C}$-linearly on $\Omega^2_c(M;\mathbb{C})$ via pullbacks thus we conclude that $\mathrm{Diff}^+(M) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$. Likewise $\mathrm{Lie}(\mathrm{Diff}^+(M)) \cong C^\infty_c(M;TM)$ consisting of compactly supported real vector fields acts $\mathbb{C}$-linearly on $\Omega^2_c(M;\mathbb{C})$ through Lie derivatives hence we also find that $\mathrm{Lie}(\mathrm{Diff}^+(M)) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$.

Moreover $C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes\mathbb{C})) \subset \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ i.e., bundle morphisms are also included. These are local algebraic operators but are important because they allow to make a contact with local *four dimensional* differential geometry.[5] A peculiarity of four dimensions is that the space $C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes\mathbb{C}))$ contains curvature tensors (more precisely their complex linear extensions) on M. If (M,g) is an oriented Riemannian 4-manifold then its Riemannian curvature tensor R_g is indeed a member of this subalgebra: with respect to the splitting of complexified 2-forms into their (anti)self-dual parts its complex linear extension looks like (cf. [22])

$$R_g = \begin{pmatrix} \frac{1}{12}\mathrm{Scal}+\mathrm{Weyl}^+ & \mathrm{Ric}_0 \\ \mathrm{Ric}_0^* & \frac{1}{12}\mathrm{Scal}+\mathrm{Weyl}^- \end{pmatrix} \ : \ \begin{matrix}\Omega^+_c(M;\mathbb{C})\\ \bigoplus \\ \Omega^-_c(M;\mathbb{C})\end{matrix} \longrightarrow \begin{matrix}\Omega^+_c(M;\mathbb{C})\\ \bigoplus \\ \Omega^-_c(M;\mathbb{C})\end{matrix} \tag{5}$$

and more generally, $C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes\mathbb{C}))$ contains the complex linear extensions of all algebraic (i.e. formal only, not stemming from a metric) curvature tensors R over M.

How to decide whether or not these elements of $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ belong to $\mathfrak{R}$? The key concept here is the trace. Compared with the above trace expression $\tau(A) = (\hat{A},\hat{1})$ generally valid on $\mathfrak{R}$, more specific trace formulata are obtained if M is endowed with a normalized Riemannian metric g i.e., the corresponding volume form $\mu_g = *1$ satisfies $\int_M \mu_g = 1$. The unique sesquilinear extension of g induces a positive definite sesquilinear L^2-scalar product

$$(\varphi,\psi)_{L^2(M,g)} := \int\limits_{x\in M} g(\varphi(x),\psi(x))\mu_g(x) = \int\limits_M \varphi\wedge *\overline{\psi}$$

on $\Omega^2_c(M;\mathbb{C})$. If $\{\varphi_1,\varphi_2,\dots\}$ is a smooth orthonormal frame in $\Omega^2_c(M;\mathbb{C})$ then it readily follows that the trace of any $B\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ formally looks like

$$\tau(B) = \lim_{n\to+\infty}\frac{1}{2^n}\sum_{i=1}^{2^n}(B\varphi_i\,,\,\varphi_i)_{L^2(M,g)} \tag{6}$$

and, if exists, is independent of the frame used. Obviously $B\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap C(M)$ if and only if the sum on the right hand side is constant after finitely many terms; and an inspection of this trace

[5] In fact all the constructions so far work for an arbitrary oriented and smooth $4k$-manifold with $k=0,1,2,\dots$ (note that in $4k+2$ dimensions the indefinite pairing (2) gives rise to a symplectic structure on $2k+1$-forms).

expression at finite stages shows that in general $B\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$ if and only if $\tau(B)$ exists.[6] As a consequence note that $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$ is already independent of the particular inclusion (4).

An example: $\Phi\in\mathrm{Diff}^+(M)$ acts on $\Omega^2_c(M;\mathbb{C})$ via $(\Phi^{-1})^*$ however this action induces a conjugate action on $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ too consequently $(\Phi^{-1})^*$ is unitary yielding $|\tau((\Phi^{-1})^*)|=1$; thus it extends to $\mathscr{H}$ (cf. Footnote 6) as a unitary operator demonstrating $\mathrm{Diff}^+(M)\subset\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$. Another example: likewise if $X\in C^\infty_c(M;TM)$ and its corresponding Lie derivative L_X operating on 2-forms satisfies $\tau(L_X)<+\infty$ then it extends to $\mathscr{H}$ (cf. again Footnote 6) such that $L_X\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$.

Furthermore the curvature R_g of (M,g) as an operator in (5) acts on $\Omega^2_c(M;\mathbb{C})$. If in addition it is bounded which means that

$$\sup_{\|\omega\|_{L^2(M,g)}=1}\|R_g\omega\|_{L^2(M,g)}\leqq K<+\infty$$

then

$$0\leqq|\tau(R_g)|\leqq\lim_{n\to+\infty}\frac{1}{2^n}\sum_{i=1}^{2^n}|(R_g\varphi_i\,,\,\varphi_i)_{L^2(M,g)}|\leqq\lim_{n\to+\infty}\frac{1}{2^n}\sum_{i=1}^{2^n}\|R_g\varphi_i\|_{L^2(M,g)}\leqq K$$

thus extends over $\mathscr{H}$ (cf. again Footnote 6) and satisfies $R_g\in C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes\mathbb{C}))\cap\mathfrak{R}$ and more generally any $R\in C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes\mathbb{C}))\cap\mathfrak{R}$ if it is bounded.

Actually when $R\in C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes\mathbb{C}))$ the previous trace formula can be further specified because one can compare the global trace $\tau(R)$ and the local trace function $x\mapsto\mathrm{tr}(R(x))$ given by the pointwise traces of the local operators $R(x):\wedge^2T^*_xM\otimes\mathbb{C}\to\wedge^2T^*_xM\otimes\mathbb{C}$ at every $x\in M$. Recall that $\mathfrak{R}$ has been constructed as the weak closure of the Clifford algebra (3). In fact [1, Section 1.1.6] the universality of $\mathfrak{R}$ permits to obtain it from other matrix algebras too, like for instance from $\bigcup\limits_{n=0}^{+\infty}\mathfrak{M}_{6^n}(\mathbb{C})$ whose weak closure therefore is again $\mathfrak{R}$. By the aid of this altered construction we can formally start with

$$\tau(R)=\lim_{n\to+\infty}\frac{1}{6^n}\sum_{i=1}^{6^n}(R\varphi_i\,,\,\varphi_i)_{L^2(M,g)}\ .$$

Fix $n\in\mathbb{N}$, write $M_n:=\bigcap\limits_{i=1}^{6^n}\mathrm{supp}\varphi_i\subseteqq M$ and take a point $x\in M_n$. Since $\dim_{\mathbb{C}}(\wedge^2T^*_xM\otimes\mathbb{C})=\binom{4}{2}=6$ the maximal number of completely disjoint linearly independent sub-6-tuples in $\{\varphi_1(x),\varphi_2(x),\dots,\varphi_{6^n}(x)\}$ is equal to $\frac{6^n}{6}=6^{n-1}$. Moreover it follows from Sard's lemma that with a generic smooth choice for $\{\varphi_1,\varphi_2,\dots\}$ the subset of those points $y\in M_n$ where this number is less than 6^{n-1} has measure zero in M_n with respect to the measure μ_g. Consequently

$$\sum_{i=1}^{6^n}(R\varphi_i,\varphi_i)_{L^2(M_n,g)}=\int\limits_{x\in M_n}\sum_{i=1}^{6^n}g(R(x)\varphi_i(x),\varphi_i(x))\,\mu_g(x)=6^{n-1}\int\limits_{x\in M_n}\mathrm{tr}(R(x))\mu_g(x)\ .$$

Since $\{\varphi_1,\varphi_2,\dots\}$ is a basis in $\Omega^2_c(M;\mathbb{C})$ therefore $M\setminus\bigcup\limits_{n=0}^{+\infty}M_n$ has measure zero as well we can let

[6] By definition $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$ contains those operators in $\mathfrak{R}\subset\mathfrak{B}(\mathscr{H})$ which are defined on the whole $\mathscr{H}$ but map $\Omega^2_c(M;\mathbb{C})\subset\mathscr{H}$ into itself; obviously such operators have finite trace. Conversely, given $B\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ we may try to extend it from $\Omega^2_c(M;\mathbb{C})$ to $\mathscr{H}$ step-by-step as follows. For a fixed n take the sub-basis $\{\varphi_1,\dots,\varphi_{2^n}\}$ in $\Omega^2_c(M;\mathbb{C})$ and consider the restriction B_n to the corresponding 2^n dimensional complex subspace; then iterating $B_n\mapsto\left(\begin{smallmatrix}B_n&0\\0&B_n\end{smallmatrix}\right)$ embed it into $C(M)$ and define the action of B_n on $\mathscr{H}$ by continuously extending over $\mathscr{H}$ the multiplication from the left on $C(M)\subset\mathscr{H}$ with this image; it is clear that $B_n\to B$ weakly as $n\to+\infty$ yielding a well-defined action of B on $\mathscr{H}$ such that $B\in\mathfrak{R}$ if $\tau(B)<+\infty$ i.e. in this case $B\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\cap\mathfrak{R}$ indeed.

$n \to +\infty$ to end up with

$$\tau(R) = \frac{1}{6}\int\limits_M \mathrm{tr}(R)\mu_g \tag{7}$$

and observe that τ in this form is nothing else than the generalization of the total scalar curvature of a Riemannian manifold. Moreover if and only if (7) exists R extends to $\mathscr{H}$ (cf. Footnote (6) as usual) and gives $R \in C^\infty(M;\mathrm{End}(\wedge^2 T^*M \otimes \mathbb{C})) \cap \mathfrak{R}$. So if we start with (4) i.e. the *second picture* we can use several useful tracial criteria for checking whether or not an operator in $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ extends to an operator in $\mathfrak{R}$.

3. Now we are ready to exhibit an especially important class of elements in $\mathfrak{R}$ through a natural regular embedding

$$i_M : M \longrightarrow \mathfrak{R} \tag{8}$$

of any connected oriented smooth 4-manifold M into its $\mathfrak{R}$ by the aid of the *first picture* as follows. To every sufficiently nice closed subset $\emptyset \subsetneqq X \subsetneqq M$ there exists an associated linear subspace $\Omega^2_c(M,X;\mathbb{C}) \subset \Omega^2_c(M;\mathbb{C}) \subset C(M) \subset \mathscr{H}$ consisting of compactly supported smooth 2-forms vanishing at least along X. In this way to every point $x \in M$ one can attach a subspace $V_x \subset \mathscr{H}$ defined by using the multiplication in $C(M)$ and then taking the closure of the $\mathbb{C}$-linear subspace

$$\mathbb{C}1 + \Omega^2_c(M,x;\mathbb{C}) + \Omega^2_c(M,x;\mathbb{C})\Omega^2_c(M;\mathbb{C}) + \Omega^2_c(M,x;\mathbb{C})\Omega^2_c(M;\mathbb{C})\Omega^2_c(M;\mathbb{C}) + \dots$$

within $\mathscr{H}$. Let $P_x : \mathscr{H} \to V_x$ be the corresponding orthogonal projection. Observe that *a priori* $P_x \in \mathfrak{B}(\mathscr{H})$ however in fact $P_x \in \mathfrak{R}$. This is because V_x arises as the image of an operator $A_x \in \mathfrak{R}$ which for instance simply looks like $\omega_1 \dots \omega_n \mapsto \frac{1}{Q(\psi_x)}\psi_x\omega_1 \dots \omega_n$ on the dense subset $C(M) \subset \mathscr{H}$ i.e. multiplication from the left within $C(M)$ by a fixed 2-form $\psi_x \in \Omega^2_c(M,x;\mathbb{C})$ satisfying $Q(\psi_x) \neq 0$. Indeed, obviously $A_xC(M) \subsetneqq V_x$ however $A_x(\psi_x(V_x \cap C(M))) = V_x \cap C(M)$ hence $A_x\mathscr{H} = V_x$. It readily follows that the resulting map in (8) defined to be $x \mapsto P_x$ is injective and continuous in the norm topology hence gives rise to a continuous embedding of M into $\mathfrak{R}$ via projections.

Proceeding further take a connected oriented smooth 4-manifold M and consider its embedding into $\mathfrak{R}$ via (8). Orientation-preserving diffeomorphisms act transitively on M thus for any two points $x, y \in M$ there exists $\Phi \in \mathrm{Diff}^+(M)$ such that $\Phi(x) = y$ which implies $(\Phi^{-1})^*V_x = V_y$ consequently for their corresponding projections $P_y = (\Phi^{-1})^*P_x\Phi^*$ holds hence they are unitary equivalent (yielding τ is constant along $i_MM \subset \mathfrak{R}$). Let $\{\gamma(t)\}_{t\in(-\varepsilon,+\varepsilon)}$ be a smooth curve in M satisfying $\gamma(0) = x \in M$ and $\dot\gamma(0) = X \in T_xM$; then putting $P(t) := i_M\gamma(t)$ to be the corresponding family of projections in $\mathfrak{R}$ we find $P(0) = P_x$ and there exists a 1-parameter family of diffeomorphisms such that $P(t) = \Phi^*_{-t}P_x\Phi^*_t$. Therefore formally $\dot P(0) = [P_x\,,\,L_Z]$ where $L_Z \in \mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ is the corresponding Lie derivative representing the infinitesimal generator of $\{\Phi_t\}_{t\in(-\varepsilon,+\varepsilon)}$. Without loss of generality we can assume $\Phi_t = \mathrm{id}_M$ outside a small neighbourhood of $x \in M$ hence the support of $Z \in C^\infty_c(M;TM)$ is also small. Therefore we can suppose $\tau(L_Z) < +\infty$ allowing a unique extension $L_Z \in \mathrm{End}(\Omega^2_c(M;\mathbb{C})) \cap \mathfrak{R}$ which implies that the commutator exists and $\dot P(0) = [P_x, L_Z] \in \mathfrak{R}$. Consequently $T_{P_x}(i_MM) \subset \mathfrak{R}$ moreover in a manner such that, taking into account (as a special *four dimensional* phenomenon) that in a von Neumann algebra every element is a sum of at most four unitary operators, the whole tangent space at $x \in M$ is actually spanned over $\mathbb{R}$ by four fixed unitary operators from $\mathfrak{R}$ i.e. which depend only on x. Thus having $X \in T_xM$ define $i_{M*}X := \dot P(0) \in T_{P_x}(i_MM) \subset \mathfrak{R}$. Assume that $X \neq 0$ hence $L_Z \neq 0$ however $\dot P(0) = [P_x, L_Z] = 0$. This would imply that the image V_x of P_x is invariant under L_Z however this is not possible for if e.g. φ_x is any 2-form vanishing at $x \in M$ hence belongs to V_x then in general $L_Z\varphi_x$ does not vanish there hence is not in V_x. All of these finally allow us to use the non-degenerate scalar product $(A,B) := \tau(AB^*)$ on $\mathfrak{R}$ which completes it to $\mathscr{H}$ as above to obtain a non-degenerate Riemannian metric g on M by the pullback formula

$$g(X,Y) := \mathrm{Re}\big(i_{M*}X\,,\,i_{M*}Y\big)\ .$$

Therefore embedding M via (8) into $\mathfrak{R}$ canonically enhances it an oriented Riemannian 4-manifold (M,g). Observe that if $*\in C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes\mathbb{C}))$ denotes the induced complexified Hodge operator on (M,g) then $\tau(*)=0$ implies $*\in\mathfrak{R}$ and $*^2=1$ that it is self-adjoint. Hence $*$ commutes with the projection $P_x\in\mathfrak{R}$ for every $x\in M$. To see this, take $\mathscr{H}=V_x\oplus V_x^\perp$ with respect to P_x; then $*$ obviously preserves $\Omega^2_c(M,x;\mathbb{C})\subset\Omega^2_c(M;\mathbb{C})$ thus if $A_x\in\mathfrak{R}$ is as before the multiplication from the left by the element $\psi_x\in\Omega^2_c(M,x;\mathbb{C})$ hence $A_x\mathscr{H}=V_x$ then $B_x:=*A_x\in\mathfrak{R}$ is multiplication from the left with $*\psi_x\in\Omega^2_c(M,x;\mathbb{C})$ hence $B_x\mathscr{H}=V_x$ too consequently $*V_x=*A_x\mathscr{H}=B_x\mathscr{H}=V_x$; likewise $0=(V_x,V_x^\perp)=(*^2V_x,V_x^\perp)=(*V_x,*V_x^\perp)=(V_x,*V_x^\perp)$ consequently $*V_x^\perp=V_x^\perp$ yielding $[P_x,*]=0$. Finally we remark that (8) is analogous to embedding Riemannian manifolds into Hilbert spaces via heat kernel techniques [2].

Having understood $M\subset\mathfrak{R}$ for a given 4-manifold let us compare these embeddings for different spaces. So let M,N be two connected oriented smooth 4-manifolds and consider their corresponding embeddings into their von Neumann algebras via (8) respectively. Regardless what M or N are, their abstractly given algebras are both hyperfinite factors of II_1-type, therefore these latter structures are isomorphic [1, Theorem 11.2.2] however not in a canonical fashion. Indeed, if $F':\mathfrak{R}\to\mathfrak{R}$ is an isomorphism between the $\mathfrak{R}$'s for M and N respectively then any other isomorphism between them has the form $F''=\beta^{-1}F'\alpha$ where α and β are inner automorphisms of the abstractly given $\mathfrak{R}$ for M and the abstractly given $\mathfrak{R}$ for N respectively.[7] Therefore to understand the freedom how operator algebras for different 4-manifolds are identified we need a description of the inner automorphism group of $\mathfrak{R}$.

Since $C(M)$ is weakly dense in $\mathfrak{R}$ by construction and the former is invariant against the inner automorphisms of the latter, it is enough to understand how the inner automorphisms of $C(M)$ look like. However these are simply conjugations with unitaries from $C(M)$ and referring to (4) they belong to $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))\cap C(M)$ where the former component is the group of all invertible $\mathbb{C}$-linear transformations of $\Omega^2_c(M;\mathbb{C})$. If $x\in M$ recall that $\Omega^2_c(M,x;\mathbb{C})\subset\Omega^2_c(M;\mathbb{C})$ is a complex subspace and it is easy to show by invertability that subspaces of this kind are permuted by elements of $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$. Consequently any member of $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$ induces an orientation-preserving diffeomorphism Φ of M and a fiberwise $\mathbb{C}$-linear diffeomorphism f of $\wedge^2T^*M\otimes\mathbb{C}$ such that

$$\begin{tikzcd}
\wedge^2T^*M\otimes\mathbb{C} \arrow[r, "f"] \arrow[d, "\pi"] & \wedge^2T^*M\otimes\mathbb{C} \arrow[d, "\pi"] \\
M \arrow[r, "\Phi"] & M
\end{tikzcd}$$

is commutative i.e. yields a bundle isomorphism of $\wedge^2T^*M\otimes\mathbb{C}$. Conversely it is straightforward that every bundle isomorphism of $\wedge^2T^*M\otimes\mathbb{C}$ gives rise to an element of $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$. Thus there exists an isomorphism of groups $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))\cong\mathrm{Iso}(\wedge^2T^*M\otimes\mathbb{C})$. Consider the short exact sequence $1\to\mathscr{G}(\wedge^2T^*M\otimes\mathbb{C})\to\mathrm{Iso}(\wedge^2T^*M\otimes\mathbb{C})\to\mathrm{Diff}^+(M)\to1$ involving the fiberwise automorphism group (the gauge group) $\mathscr{G}(\wedge^2T^*M\otimes\mathbb{C})\cong C^\infty(M;\mathrm{Aut}(\wedge^2T^*M\otimes\mathbb{C}))$ the global automor-

[7] If one wants to consider $\mathfrak{R}$ not abstractly i.e. in itself but spatially i.e. together with $\mathfrak{R}\subset\mathfrak{B}(\mathscr{H})$ provided by its construction then in this case isomorphisms are to be parameterized not only by inner automorhisms of $\mathfrak{R}$ but by all automorphisms of $\mathfrak{R}\subset\mathfrak{B}(\mathscr{H})$. However one can prove [21] that inner automorphisms are dense in the strong*-topology in this larger group hence in this sense all outer automorphisms of $\mathfrak{R}\subset\mathfrak{B}(\mathscr{H})$ can be approximated by inner ones of $\mathfrak{R}$. For further characterization of inner automorphisms cf. [24].

phism group $\mathrm{Iso}(\wedge^2T^*M\otimes\mathbb{C})\cong\mathrm{Aut}(C^\infty(M;\wedge^2T^*M\otimes\mathbb{C}))=\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$, both of the vector bundle $\wedge^2T^*M\otimes\mathbb{C}$, and the diffeomorphism group $\mathrm{Diff}^+(M)$ of the underlying space M respectively. This short exact sequence can therefore be re-written as

$$1\longrightarrow C^\infty(M;\mathrm{Aut}(\wedge^2T^*M\otimes\mathbb{C}))\longrightarrow\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))\longrightarrow\mathrm{Diff}^+(M)\longrightarrow 1\ . \qquad (9)$$

In addition the map $\Phi\mapsto(\Phi^{-1})^*$ gives rise to group injection $\mathrm{Diff}^+(M)\to\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$ consequently (9) can be supplemeted to

$$1\longrightarrow C^\infty(M;\mathrm{Aut}(\wedge^2T^*M\otimes\mathbb{C}))\longrightarrow\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))\rightleftarrows\mathrm{Diff}^+(M)\longrightarrow 1$$

inducing a set-theoretical splitting which means that at least as a *set* there exists a decomposition $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))=C^\infty(M;\mathrm{Aut}(\wedge^2T^*M\otimes\mathbb{C}))\,\mathrm{Diff}^+(M)$. Therefore an inner automotphism α of $C(M)$ as a conjugation with an element of $\mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$ admits a unique splitting $\alpha=\mathrm{Ad}_g\mathrm{Ad}_{(\Phi^{-1})^*}$ where g is a gauge transformation of $\wedge^2T^*M\otimes\mathbb{C}$ hence leaves M pointwise fixed and Φ is an orientation-preserving diffeomorphism of M. Likewise, $\beta=\mathrm{Ad}_h\mathrm{Ad}_{(\Psi^{-1})^*}$ is the shape of an inner automorphism of $C(N)$. These make sure that given an isomorphism $F':\mathfrak{R}\to\mathfrak{R}$ between the von Neumann algebras for M and N respectively then any other isomorphism has the form $F''=\mathrm{Ad}_{\Psi^*}(\mathrm{Ad}_{h^{-1}}F'\mathrm{Ad}_g)\mathrm{Ad}_{(\Phi^{-1})^*}$ between them. Consequently isomorphisms between a pair of abstractly given von Neumann algebras differ only by inner automorphisms which preserve their underlying 4-manifolds as embedded within their algebras respectively; i.e. differences between identifications are inessential in this sense. Our overall conclusion therefore is that *up to diffeomorphisms every connected oriented smooth* 4*-manifold M admits an embedding into a commonly given abstract von Neumann algebra $\mathfrak{R}$ via (8) and these embeddings induce Riemannian structures (M,g) on them.*

4. We close the partial comprehension of $\mathfrak{R}$ with an observation regarding its general structure. The shape of (9) at the Lie algebra level looks like

$$0\longrightarrow C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes\mathbb{C}))\longrightarrow\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\longrightarrow\mathrm{Lie}(\mathrm{Diff}^+(M))\longrightarrow 0\ .$$

We already have an embedding (4). In addition to this there exists an isomorphism of Lie algebras $L:C^\infty_c(M;TM)\to\mathrm{Lie}(\mathrm{Diff}^+(M))$ such that $X\mapsto L_X$ is nothing but taking Lie derivative with respect to a compactly supported real vector field where the first-order $\mathbb{C}$-linear differential operator L_X is supposed to act on 2-forms hence $\mathrm{Lie}(\mathrm{Diff}^+(M))\subset\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ as we know already too. Therefore the intersection of this sequence with $C(M)\subset\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ is meaningful and gives

$$0\longrightarrow C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes\mathbb{C}))\cap C(M)\longrightarrow C(M)\longrightarrow\mathrm{Lie}(\mathrm{Diff}^+(M))\cap C(M)\longrightarrow 0\ .$$

The second term consists of fiberwise algebraic hence local operators having finite trace (7) and likewise the fourth term consists Lie derivatives having finite trace via (6) thus belongs to the class of local operators too. Since the vector spaces underlying $C(M)$ considered either as an associative or a Lie algebra are isomorphic we conclude, as an important structural observation, that the overall construction here is geometric in the sense that *the algebra $\mathfrak{R}$ is generated by local operators*.

Summing up all of our findings so far: $\mathfrak{R}$ is a hyperfinite factor von Neumann algebra of II_1-type associated to M such that the solely input in its construction has been the pairing (2). Hence $\mathfrak{R}$ depends only on the orientation and the smooth structure of M in a functorial way. It contains, certainly among many other non-local operators, the space M itself as projections, its orientation-preseving diffeomorphisms as well as its space of bounded algebraic curvature tensors. Nevertheless $\mathfrak{R}$ is geometric in the sense that it is generated by M's local operators alone. It is remarkable that despite the plethora of smooth

4-manifolds detected since the 1980's their associated von Neumann algebras here are unique offering a sort of justification terming $\mathfrak{R}$ as "universal". Moreover one is allowed to say that every connected oriented smooth 4-manifold M togeter with its curvature tensor embeds into a common $\mathfrak{R}$ and to look upon this von Neumann algebra as a natural common non-commutative space generalization of all oriented smooth 4-manifolds or all 4-geometries. This universality also justifies the simple notation $\mathfrak{R}$ used throughout the text.

A new representation of the algebra $\mathfrak{R}$. After these preliminary considerations we are in a position to exhibit lot of new representations of the hyperfinite II_1 factor. The following lemma is repeated verbatim from [13, Lemma 2.1]:

Lemma 3.1. *Let M be a connected oriented smooth 4-manifold and $\mathfrak{R}$ its von Neumann algebra with trace τ as before. Then there exists a complex separable Hilbert space $\mathscr{I}(M)^\perp$ and a representation $\rho_M : \mathfrak{R} \to \mathfrak{B}(\mathscr{I}(M)^\perp)$ with the following properties. If $\pi : \mathfrak{R} \to \mathfrak{B}(\mathscr{H})$ is the standard representation constructed above then $\{0\} \subseteqq \mathscr{I}(M)^\perp \subsetneqq \mathscr{H}$ and $\rho_M = \pi|_{\mathscr{I}(M)^\perp}$ holds; therefore, although ρ_M can be the trivial representation, it is surely not unitary equivalent to the standard representation. Moreover the unitary equivalence class of ρ_M is invariant under orientation-preserving diffeomorphisms of M.*

Thus the Murray–von Neumann coupling constant[8] of ρ_M is invariant under orientation-preserving diffeomorphisms. Writing $P_M : \mathscr{H} \to \mathscr{I}(M)^\perp$ for the orthogonal projection the coupling constant is equal to $\tau(P_M) \in [0,1) \subset \mathbb{R}_+$ consequently $\gamma(M) := \tau(P_M)$ is a smooth 4-manifold invariant.

Proof. First let us exhibit a representation of $\mathfrak{R}$; this construction is inspired by the general Gelfand–Naimark–Segal technique however exploits the special features of our construction so far as well. Pick a pair (Σ, ω) consisting of an (immersed) closed orientable surface $\Sigma \looparrowright M$ with induced oriantation and $\omega \in \Omega^2_c(M;\mathbb{C})$ which is also closed i.e., $\mathrm{d}\omega = 0$. Consider the differential geometric $\mathbb{C}$-linear functional $F_{\Sigma,\omega} : \mathfrak{R} \to \mathbb{C}$ by continuously extending

$$A \longmapsto \int_\Sigma A\omega$$

from $\mathrm{End}(\Omega^2_c(M;\mathbb{C})) \cap \mathfrak{R}$. This extension is unique because $\mathrm{End}(\Omega^2_c(M;\mathbb{C})) \cap \mathfrak{R}$ is norm-dense in $\mathfrak{R}$. In case of $F_{\Sigma,\omega}(1) \neq 0$ let $\{0\} \subseteqq I(M) \subseteqq \mathfrak{R}$ be the subset of elements $A \in \mathfrak{R}$ satisfying $F_{\Sigma,\omega}(A^*A) = 0$. In fact for all pairs (Σ,ω) we obviously find $\{0\} \subsetneqq I(M)$ and $I(M) \cap \mathbb{C}1 = \{0\}$ hence $I(M) \subsetneqq \mathfrak{R}$ too. We assert that $I(M)$ is a multiplicative left-ideal in $\mathfrak{R}$ which is independent of (Σ,ω). In the case of $F_{\Sigma,\omega}(1) = 0$ we put $I(M) = \mathfrak{R}$ hence it is again independent of (Σ,ω) but trivially in this way.

Consider the case when $F_{\Sigma,\omega}(1) \neq 0$. Then we can assume that $F_{\Sigma,\omega}(1) = 1$ hence $F_{\Sigma,\omega}$ is a positive functional; applications of the standard inequality $|F_{\Sigma,\omega}(A^*B)|^2 \leqq F_{\Sigma,\omega}(A^*A)F_{\Sigma,\omega}(B^*B)$ show that $I_{\Sigma,\omega}(M)$ defined by the elements satisfying $F_{\Sigma,\omega}(A^*A) = 0$ is a multiplicative left-ideal in $\mathfrak{R}$. Concerning its ω-dependence, let $\omega' \in \Omega^2_c(M;\mathbb{C})$ be another closed 2-form having the property $F_{\Sigma,\omega'}(1) = 1$; since neither ω nor ω' are identically zero, we can pick an *invertible* operator $T \in \mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$ satisfying $\omega'|_\Sigma = T\omega|_\Sigma$. Then by $F_{\Sigma,\omega'}(A^*A) = F_{\Sigma,\omega}(A^*AT)$ and applying the above inequality in the form $|F_{\Sigma,\omega'}(A^*AT)|^2 \leqq F_{\Sigma,\omega}(A^*A)F_{\Sigma,\omega}((AT)^*(AT))$ we find $I_{\Sigma,\omega'}(M) \subseteqq I_{\Sigma,\omega}(M)$. Likewise, making use of $F_{\Sigma,\omega}(A^*A) = F_{\Sigma,\omega'}(A^*AT^{-1})$ and $|F_{\Sigma,\omega}(A^*AT^{-1})|^2 \leqq F_{\Sigma,\omega'}(A^*A)F_{\Sigma,\omega'}((AT^{-1})^*(AT^{-1}))$ imply the converse inequality $I_{\Sigma,\omega'}(M) \supseteqq I_{\Sigma,\omega}(M)$. Consequently $I_{\Sigma,\omega'}(M) = I_{\Sigma,\omega}(M)$.

Concerning the general (Σ,ω)-dependence of $I_{\Sigma,\omega}$ we argue as follows. Let $\eta_\Sigma \in \Omega^2(M;\mathbb{R})$ be a closed real 2-form representing the Poincaré-dual $[\eta_\Sigma] \in H^2(M;\mathbb{R})$ of $\Sigma \looparrowright M$. Equip M with an arbitrary Riemannian metric g; since the corresponding Hodge operator $*$ is an isomorphism on $\wedge^2 T^*M$

[8]Also called the $\mathfrak{R}$-*dimension* of a left $\mathfrak{R}$-module hence denoted $\dim_{\mathfrak{R}}$, cf. [1, Chapter 8].

we can take a 2-form φ_Σ such that $\eta_\Sigma = *\varphi_\Sigma$. Then via $\int_\Sigma \omega = \int_M \omega\wedge\overline{\eta}_\Sigma = \int_M \omega\wedge *\overline{\varphi}_\Sigma = (\omega,\varphi_\Sigma)_{L^2(M,g)}$ the functional can be re-expressed as $F_{\Sigma,\omega}(A^*A) = (A^*A\omega,\varphi_\Sigma)_{L^2(M,g)}$ in terms of the corresponding definite sesquilinear L^2-scalar product on (M,g). Let $\Sigma' \hookrightarrow M$ be another closed surface and ω' another closed 2-form such that $F_{\Sigma',\omega'}(1) = 1$. Altering ω and ω' along Σ and Σ' respectively if necessary (which has no effect on $I(M)$ as we have seen) we can pick some compactly supported 2-form Ω on M such that $\Omega|_\Sigma = \omega$ and $\Omega|_{\Sigma'} = \omega'$. Moreover as before take a representative $*\varphi_{\Sigma'} \in \Omega^2(M;\mathbb{R})$ for the Poincaré-dual of Σ'. We can use again $T\in \mathrm{Aut}(\Omega^2_c(M;\mathbb{C}))$ satisfying $\varphi_{\Sigma'} = T\varphi_\Sigma$. Thus

$$F_{\Sigma',\omega'}(A^*A) = (A^*A\Omega,\varphi_{\Sigma'})_{L^2(M,g)} = (A^*A\Omega,T\varphi_\Sigma)_{L^2(M,g)} = (T^*A^*A\Omega,\varphi_\Sigma)_{L^2(M,g)} = F_{\Sigma,\omega}(T^*A^*A)$$

together with $|F_{\Sigma,\omega}(T^*A^*A)|^2 \leqq F_{\Sigma,\omega}\big((AT)^*AT\big)F_{\Sigma,\omega}(A^*A)$ demonstrate that $I_{\Sigma',\omega'}(M) \subseteqq I_{\Sigma,\omega}(M)$. In the same fashion $F_{\Sigma,\omega}(A^*A) = F_{\Sigma',\omega'}((T^{-1})^*A^*A)$ together with the corresponding inequality convinces us that $I_{\Sigma',\omega'}(M) \supseteqq I_{\Sigma,\omega}(M)$. Thus $I_{\Sigma',\omega'}(M) = I_{\Sigma,\omega}(M)$.

Secondly if (Σ,ω) is such that $F_{\Sigma,\omega}(1) = 0$ then by definition $I_{\Sigma,\omega}(M) = \mathfrak{R}$. Therefore if (Σ',ω') is another pair with $F_{\Sigma',\omega'}(1) = 0$ then $I_{\Sigma,\omega}(M) = I_{\Sigma',\omega'}(M)$ (and equal to $\mathfrak{R}$). We are now convinced that it is correct to write $I_{\Sigma,\omega}(M)$ as $I(M)$. In summary it satisfies $\{0\} \subsetneqq I(M) \subseteqq \mathfrak{R}$.

Let us proceed further by exploiting now the observation made during the construction of $\mathfrak{R}$ that it acts on a Hilbert space $\mathscr{H}$ with scalar product $(\,\cdot\,,\,\cdot\,)$ by the representation π i.e., multiplication from the left. In fact, since $(\hat{A},\hat{B}) = \tau(AB^*)$ we see that π is the standard representation. Consider the space $\{0\} \subsetneqq I(M)I(M)^* \subseteqq \mathfrak{R}$ consisting of all finite sums $A_1B_1 + \cdots + A_kB_k \in \mathfrak{R}$ where $A_i \in I(M)$ and similarly $B_j \in I(M)^*$. It gives rise to a closed linear subspace $\{0\} \subsetneqq \mathscr{I}(M) \subseteqq \mathscr{H}$ by taking the closure of $C(M)\cap I(M)I(M)^*$ within $\mathscr{H} \supset C(M)$. Therefore $\mathscr{I}(M)$ is a well-defined closed subspace of $\mathscr{H}$ which is non-trivial if $F_{\Sigma,\omega}(1) \neq 0$ and coincides with $\mathscr{H}$ whenever $F_{\Sigma,\omega}(1) = 0$. Take its orthogonal complementum $\{0\} \subseteqq \mathscr{I}(M)^\perp \subsetneqq \mathscr{H}$. Note that $\mathscr{I}(M)^\perp$ is isomorphic to $\mathscr{H}/\mathscr{I}(M)$. Taking into account that the subset $I(M)$ is a multiplicative left-ideal $\pi : \mathfrak{R} \to \mathfrak{B}(\mathscr{H})$ given by left-multiplication restricts to $\mathscr{I}(M)$ but even more, since the scalar product on $\mathscr{H}$ satisfies the identity $(\widehat{AB},\widehat{C}) = (\hat{B},\widehat{A^*C})$ the standard representation restricts to $\mathscr{I}(M)^\perp$ as well. This latter representation is either a unique non-trivial representation if $\mathscr{I}(M)^\perp \neq \{0\}$ (provided by a functional over M with $F_{\Sigma,\omega}(1) \neq 0$ if exists), or the trivial one if $\mathscr{I}(M)^\perp = \{0\}$ (provided by a functional with $F_{\Sigma,\omega}(1) = 0$ which always exists). Keeping these in mind, for a given M we define

$$\rho_M : \mathfrak{R} \to \mathfrak{B}(\mathscr{I}(M)^\perp) \;\text{ to be }\; \begin{cases} \pi|_{\mathscr{I}(M)^\perp} \text{ on } \mathscr{I}(M)^\perp \neq \{0\} \text{ if possible,} \\ \pi|_{\mathscr{I}(M)^\perp} \text{ on } \mathscr{I}(M)^\perp = \{0\} \text{ otherwise.} \end{cases}$$

The choice is unambigously determined by the topology of M (see the *Remark* below).

From the general theory [1, Chapter 8] we know that if $P_M : \mathscr{H} \to \mathscr{I}(M)^\perp$ is the orthogonal projection then $P_M \in \mathfrak{R}$ because $\mathscr{H}$ is the standard $\mathfrak{R}$-module. The Murray–von Neumann coupling constant of ρ_M depends only on the unitary equivalence class of ρ_M and is equal to $\tau(P_M) \in [0,1]$. However observing that ρ_M is surely not isomorphic to π since $\mathscr{I}(M)$ is never trivial the case $\tau(P_M) = 1$ is excluded i.e., $\tau(P_M) \in [0,1)$. Let $\Phi : M \to M$ be an orientation-preserving diffeomorphism. It induces an inner automorphism $A \mapsto (\Phi^{-1})^*A\Phi^*$ of $\mathfrak{R}$. Taking into account that the scalar product on $\mathscr{H}$ is induced by the trace which is invariant under cyclic permutations this inner automorphism is unitary. Moreover it transforms $I_{\Sigma,\omega}(M)$ into $I_{\Sigma',\omega'}(M) = I_{\Phi(\Sigma),(\Phi^{-1})^*\omega}(M)$ hence $F_{\Sigma,\omega}(1) = 0$ if and only if $F_{\Sigma',\omega'}(1) = 0$ consequently the Hilbert space $\mathscr{I}(M)^\perp$ is invariant under Φ. Thus Φ transforms ρ_M into a new representation $(\Phi^{-1})^*\rho_M\Phi^*$ on $\mathscr{I}(M)^\perp$ which is unitary equivalent to ρ_M.

We conclude that $\gamma(M) := \tau(P_M) \in [0,1)$ is a smooth invariant of M as stated. $\square$

Remark. Note that $\gamma(M)=0$ corresponds to the situation when ρ_M is the trivial representation on $\mathscr{I}(M)^\perp=\{0\}$. To avoid this we have to demand $F_{\Sigma,\omega}(1)\neq 0$ which by the closedness assumptions on $\Sigma \looparrowright M$ and $\omega\in\Omega^2_c(M;\mathbb{C})$ is in fact a *topological condition*: it is equivalent that

$$\frac{1}{2\pi\sqrt{-1}}F_{\Sigma,\omega}(1)=\frac{1}{2\pi\sqrt{-1}}\int\limits_{\Sigma}\omega=\langle[\Sigma],[\omega]\rangle\in\mathbb{C}$$

as a pairing of $[\Sigma]\in H_2(M;\mathbb{Z})$ and $[\omega]\in H^2(M;\mathbb{C})$ in homology is not trivial. Hence $\gamma(M)=0$ iff $H_2(M;\mathbb{C})=H_2(M;\mathbb{Z})\otimes\mathbb{C}=\{0\}$ (or equivalently, $H^2(M;\mathbb{C})=H^2(M;\mathbb{Z})\otimes\mathbb{C}=\{0\}$). Thus unfortunately $\gamma(M)=0$ for all acyclic or aspherical manifolds (including homology 4-spheres). Examples in the simply connected case are $M=S^4,\mathbb{R}^4,R^4$ (this latter is any exotic or fake $\mathbb{R}^4$) while $M=S^3\times S^1$ is a non-simply connected example. Nevertheless $\gamma(M)$ is not always trivial and some of the topological properties of this new smooth 4-manifold invariant have been investigated in [13, Theorems 1.1-3].

Proof of Theorem 2.1. Parts (i) and (ii) follow from the material presented in Section 3 while part (iii) from Lemma 3.1. □

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