Kazhdan groups have cost 1

Gábor Pete
Alfréd Rényi Institute of Mathematics & Technical University, Budapest
http://www.math.bme.hu/~gabor

Joint work with Tom Hutchcroft
University of Cambridge

Inventiones Math. (2020)
[arXiv:1810.11015 math.GR]

Plan of talk

1. What is cost?
2. What are Kazhdan (T) groups?
3. Infinite Kazhdan groups have cost 1.
4. Open problems.

Please interrupt me with questions at any time.

Measurable cost

Group Γ, generating set S.

$$\text{cost}(\Gamma) := \frac{1}{2}\inf\Big\{\mathbf{E}_\mu[\deg(o)] : \mu \text{ is an invariant probability measure on connected spanning graphs on } \Gamma\Big\}.$$

$$\text{cost}(\Gamma, S) := \frac{1}{2}\inf\Big\{\mathbf{E}_\mu[\deg(o)] : \text{ ... } \textit{sub}\text{graphs of Cay}(\Gamma, S)\Big\}.$$

(Also $\text{cost}\big(\Gamma \curvearrowright (X, \Sigma, \mu)\big)$ and $\text{cost}\big(\Gamma \curvearrowright (X, \Sigma, \mu),\ S\big)$, where we want a measurable spanning (sub)graphing of the orbit-equivalence relation of the probability measure preserving (p.m.p.) action. The above costs are the infimal costs over all actions.)

Defined by Levitt '95, studied extensively by Gaboriau '98 onwards.

Measurable cost: examples

Example 0. On any finite group, any connected spanning graph has at least $|\Gamma|-1$ edges, achieved, e.g., by the uniformly random spanning tree UST of Cay(Γ, S), hence cost$(\Gamma, S) = 1 - \frac{1}{|\Gamma|}$.

Example 1. cost(∞ amenable, S) $= 1$.

Follows from Ornstein-Weiss '87, saying that all pmp actions of all amenable groups are orbit equivalent to each other. **Simple proof** by Benjamini-Lyons-Peres-Schramm '99. Assume finitely generated, for simplicity.

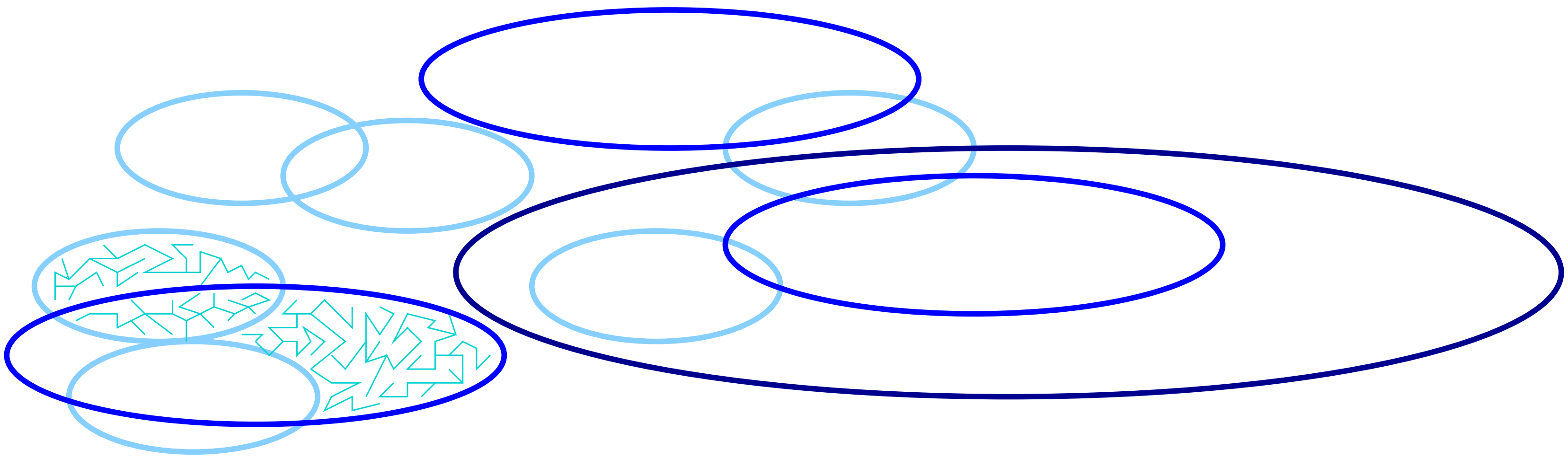
Take Følner sequence F_n in Cay(Γ, S) such that $\frac{|\partial_E F_n|}{|F_n|} \to 0$ fast.
Delete the boundary edges of each xF_n with probability $1/|F_n|$, for each n.
$\mathbf{P}[\, o$ is not separated from ∞ at stage $n\,] \leqslant (1 - 1/|F_n|)^{|F_n|} \sim 1/e$.
$\mathbf{P}[\, o$ is not separated from $\infty\,] = 0$.
The probability of any edge to be deleted is $\sum_n \frac{|\partial_E F_n|}{|F_n|} < \epsilon$.
In each finite component, take UST. Add back the $\partial_E(xF_n)$ edges.
Connected, with average degree $< 2 + \epsilon \deg_S(o)$.

Measurable cost: examples

Example 0. On any finite group, any connected spanning graph has at least $|\Gamma| - 1$ edges, achieved, e.g., by the uniformly random spanning tree UST of Cay(Γ, S), hence cost$(\Gamma, S) = 1 - \frac{1}{|\Gamma|}$.

Example 1. cost$(\infty$ amenable$, S) = 1$.

Follows from Ornstein-Weiss '87, saying that all pmp actions of all amenable groups are orbit equivalent to each other. **Simple proof** by Benjamini-Lyons-Peres-Schramm '99. Assume finitely generated, for simplicity.



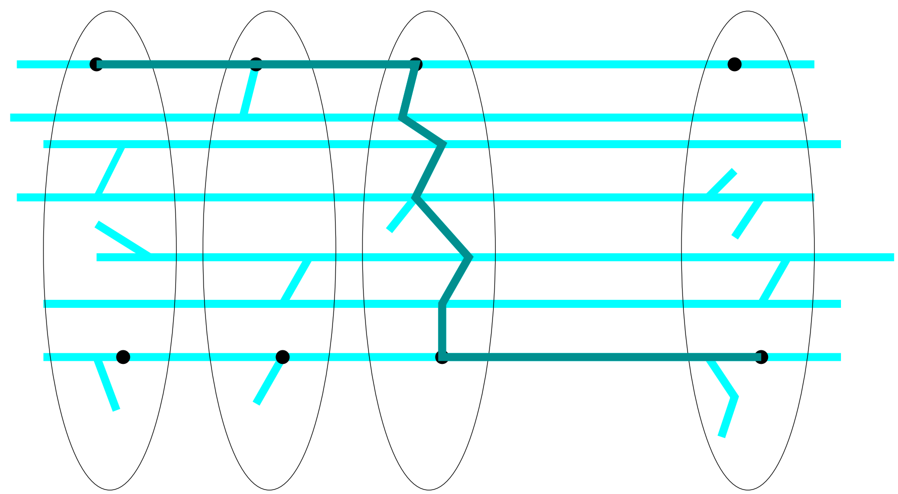
Measurable cost: examples

Example 2. Free groups cost$(F_d) = d$ Gaboriau '98.

So cost shows that there exist orbit-inequivalent actions.

Example 3. cost$(\Gamma \times \mathbb{Z}) = 1$ for any finitely generated Γ.

Proof. Bernoulli(ϵ) percolation on edges of Cay(Γ, S). Plus full $\mathbb{Z}$ copies.



Example 4. cost(SL$(n, \mathbb{Z}), S) = 1$ for $n \geqslant 3$, with the usual generating set where "generators commute with each other in a connected manner".

Cost, first ℓ^2-Betti number, FUSF

Cost is not always easy to determine. Cohomology is often easier.

Gaboriau '02: $\text{cost}(\Gamma) \geqslant 1 + \beta_1^{(2)}(\Gamma)$, where two probabilistic definitions:

$\beta_1^{(2)}(\Gamma)$ = von Neumann dimension of space of harmonic functions $f : \Gamma \longrightarrow \mathbb{R}$ with finite Dirichlet energy $\sum_{(x,y)\in E} |f(x) - f(y)|^2 < \infty$.

Or, for the Free Uniform Spanning Forest, $\mathbf{E}\big[\deg_{\mathsf{FUSF}}(o)\big] = 2 + 2\beta_1^{(2)}(\Gamma)$, in any Cayley graph.

The FUSF is the limit of UST along any exhaustion by finite subgraphs.

Question (Gaboriau '02). Is there = always? E.g., for Kazhdan groups, where $\beta_1^{(2)}(\Gamma) = 0$ is known from Bekka-Valette '97, do we have cost $= 1$?

IF there is a way to add an invariant ϵ-density bond percolation to the FUSF so that it becomes connected, then Yes.

In Kazhdan groups, adding Bernoulli(ϵ) does not work.

Kazhdan's property (T) definitions

Definition 1 (Kazhdan '67). A topological group Γ has property (T) iff *every* unitary representation $\rho : \Gamma \longrightarrow \mathsf{U}(\mathcal{H})$ on a real or complex Hilbert space $\mathcal{H}$ has a spectral gap:

if there are no non-zero invariant vectors (fixed by all $g \in \Gamma$), then there is some $\kappa > 0$ and a compact $K \subset \Gamma$ such that for every nonzero $v \in \mathcal{H}$ exists $k \in K$ with $\|\rho(k)v - v\| > \kappa\|v\|$.

If a countable group has (T), then it is finitely generated, and every finite generating set S works as K above.

Kazhdan proved that $\mathsf{SL}(n, \mathbb{R})$ for $n \geqslant 3$ has (T), extended this to every lattice in them (such as $\mathsf{SL}(n, \mathbb{Z})$), and concluded that all these lattices are finitely generated.

Kazhdan's property (T) definitions

Definition 2 (Connes-Weiss '80). Whenever $\Gamma \curvearrowright (X, \Sigma, \mu)$ is an ergodic p.m.p. action on a probability space (i.e., every Γ-invariant $A \in \Sigma$ is trivial, $\mu(A) \in \{0, 1\}$), it is also strongly ergodic: every asymptotically invariant A_n (i.e., $\mu\big(A_n \triangle g^{-1}(A_n)\big) \to 0$ for every $g \in \Gamma$) is asymptotically trivial: $\mu(A_n)\big(1 - \mu(A_n)\big) \to 0$.

Definition 3 (Glasner-Weiss '97). The set $\mathsf{Erg}(\Gamma \curvearrowright \{0,1\}^\Gamma)$ of ergodic random 2-colorings of Γ is closed in the weak* topology within $\mathsf{Inv}(\Gamma \curvearrowright \{0,1\}^\Gamma)$. In particular, $p\,\delta_{\text{all }0} + (1-p)\,\delta_{\text{all }1}$ cannot be locally approximated by ergodic 2-colorings:

For every $\epsilon > 0$ and finite generating set S, there is a $\delta > 0$ such that whenever $\sigma : \Gamma \longrightarrow \{0, 1\}$ is an ergodic invariant random 2-coloring of the vertices with distribution μ, with marginals $\epsilon < \mu(\sigma(g) = 1) < 1 - \epsilon$, then, for every $s \in S$, we have $\mu\big(\sigma(g) = \sigma(gs)\big) < 1 - \delta$.

The equivalence of Def 1, Def 2, Def 3 is similar to the equivalence of the spectral and isoperimetric definitions of being an expander graph.

Kazhdan's property (T) examples

Example 0. Finite groups.

Example 1. Infinite amenable groups are not. Because:

Let $(\omega_v)_{v\in\Gamma}$ be an iid Bernoulli(1/2) coloring.
Let F_n be a good Følner set. Let $\sigma_n(x) :=$ Maj$\{\omega_v : v \in xF_n\}$.

Example 2. The free groups are not.

One colouring. $F_2 \longrightarrow \mathbb{Z}$ surjection by forgetting one generator. Now pull back the Følner Majority colouring of $\mathbb{Z}$.

Another colouring. Take Bernoulli$(1-\epsilon)$ bond percolation. Colour each cluster by flipping a fair coin. There are infinitely many infinite clusters, hence this is ergodic (Lyons-Schramm '99), for any $\epsilon > 0$. But take $\epsilon \to 0$.

Example 3. SL$(n, \mathbb{Z})$, $n \geqslant 3$, yes.

The cost of Kazhdan groups

Theorem (Hutchcroft-P '20). In any infinite Kazhdan group Γ, any finite generating set S, we have cost$(\Gamma, S) = 1$.

Step 1 (reduction). If there exists, for any $\epsilon > 0$, an invariant ϵ-density site percolation with a unique infinite cluster, then cost$(\Gamma, S) = 1$.

Step 2 (a strange construction). For any $p \in (0, 1)$, an iterative sequence of invariant site percolations μ_n that converge weakly to $p\,\delta_{\text{all } 0} + (1-p)\,\delta_{\text{all } 1}$.

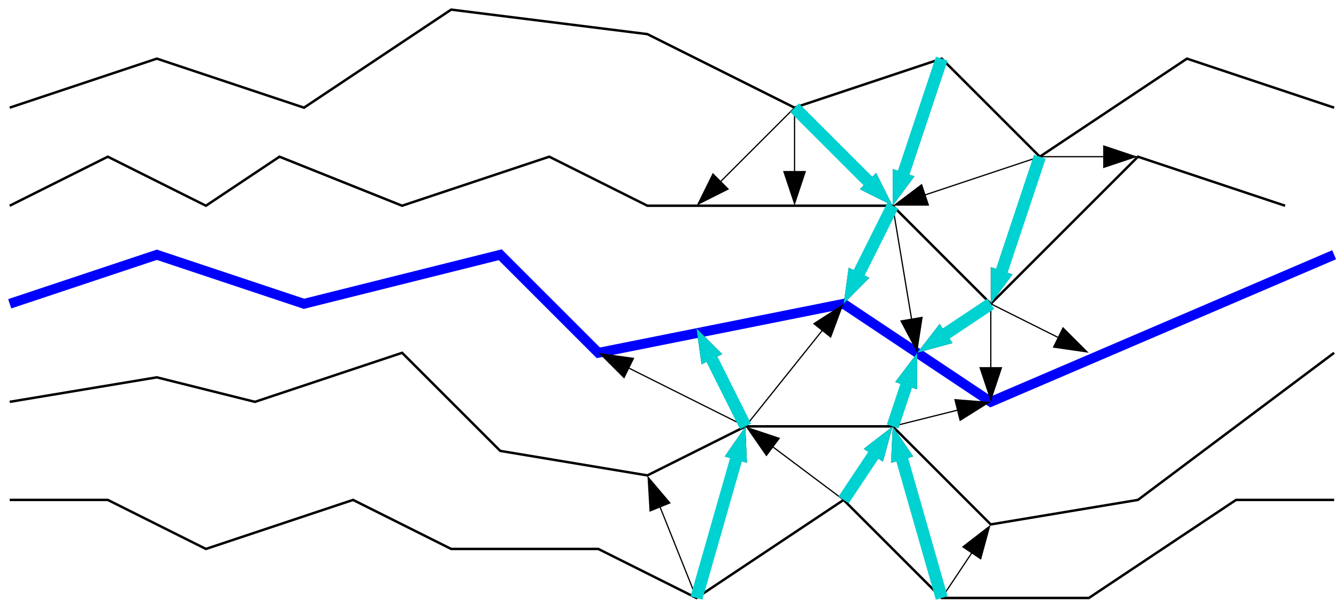
Step 3 (ergodicity). If every cluster of μ_{n-1} has "zero frequency", then μ_n is ergodic.

- In a Kazhdan group, there is some μ_N that is not ergodic (Step 2).
- Hence μ_{N-1} has a cluster of positive frequency. There can be only finitely many clusters of largest frequency — choose one, get unique infinite cluster at density $\leqslant p$ (by Step 3).
- This was for any $p \in (0, 1)$. So Step 1 finishes the proof. □

Step 1 (reduction)

Let η be the ϵ-density infinite cluster. Let η_1 be the vertices at distance 1 from η, and η_{k+1} be the vertices at distance 1 from η_k.

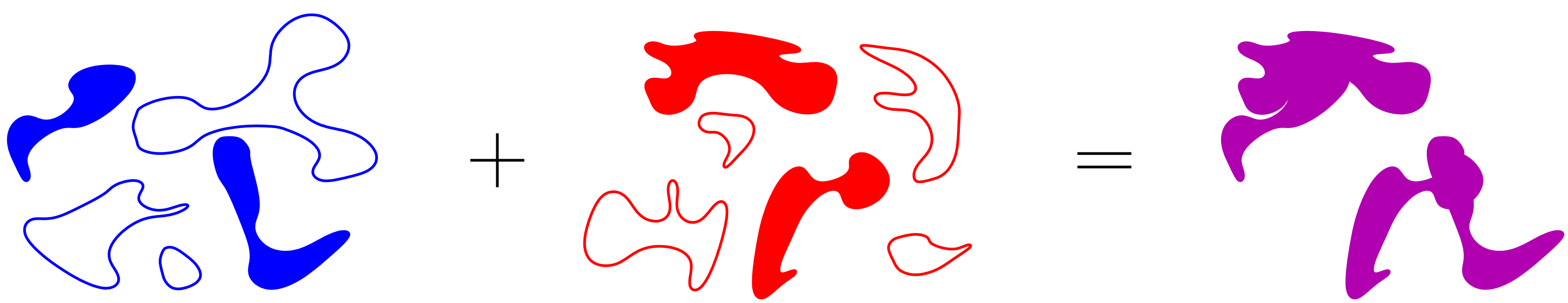
From each vertex in η_{k+1}, take one random edge to η_k.



Outdegree is 1. So expected indegree is also 1. Altogether $2 + \epsilon \deg_S(o)$.

Step 2 (a strange construction)

- μ_1 is Bernoulli(p) site percolation.
- Given μ_i, keep each cluster only with some probability q, independently. This is the q-thinned percolation measure μ_i^q.
- Take two independent copies, and take their union. This is μ_{i+1}.



The q-thinning reduces the density, taking the union of two increases it. With $q = q(p) = \frac{1-\sqrt{1-p}}{p}$, the density remains p in each iteration.

And μ_i(two neighbours agree) $\to 1$ as $i \to \infty$. So $\mu_i \to p\delta_{\text{all } 0} + (1-p)\delta_{\text{all } 1}$.

Step 3 (ergodicity)

This proof is inspired by Lyons-Schramm '99.

Lemma (LS'99). For any invariant site percolation on any Cayley graph, for simple random walk $(X_n)_{n\geqslant 0}$ started at any vertex $X_0 = v$,

$$\mathrm{freq}(C) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_{\{X_n \in C\}}$$

exists and is independent of v, for every percolation cluster C.

Proposition. For any ergodic site percolation μ on any Cayley graph, if $\mathrm{freq}(C) = 0$ for every cluster C, then the q-thinned measure μ^q is ergodic.

Idea of proof. $\mathrm{freq}(C) = 0\ \forall C$ implies that, for every $r \geqslant 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{P}\Big(B(X_0, r) \longleftrightarrow B(X_n, r)\Big) = 0,$$

and thus $\inf_{x,y\in\Gamma} \mu\big(B(x,r) \longleftrightarrow B(y,r)\big) = 0$, which implies that the q-thinning in $B(x,r)$ and $B(y,r)$ are getting independent. □

Step 3 (ergodicity)

We are not done yet: ergodicity of μ^q does not imply ergodicity of $\mu^q \otimes \mu^q$.

(E.g., $\ldots 1010 \ldots$ or $\ldots 0101 \ldots$ with probability $1/2$ each. In two independent copies, "agreeing" is an invariant event of probability $1/2$.)

However, μ_1 is Bernoulli percolation, not just ergodic, but weakly mixing $\Longleftrightarrow$ $\mu_1 \otimes \mu_1$ is ergodic $\Longleftrightarrow$ $\mu_1^{\otimes k}$ is ergodic for any $k \geqslant 2$.

Version of Proposition: if $\mu_i \otimes \cdots \otimes \mu_i$ is ergodic for μ_i that has zero frequencies, then $\mu_i^q \otimes \cdots \otimes \mu_i^q$ is also ergodic.

$\mu_i^{\otimes 2}$ ergodic $\Longrightarrow$ $\mu_i^{\otimes 4}$ ergodic $\Longrightarrow$ $(\mu_i^q)^{\otimes 4}$ ergodic $\Longrightarrow$ $\mu_{i+1}^{\otimes 2}$ ergodic.

So, on a Kazhdan group, there is μ_{n-1} that has a cluster with positive frequency. $\square$

Open questions

Problem 1. Can we get a small density unique infinite cluster in a factor of iid way?

A random coloring $\sigma : \Gamma \longrightarrow \{0,1\}$ is a **factor of iid** if there is a measurable map $\psi : [0,1]^\Gamma \longrightarrow \{0,1\}$ s.t., for $\omega \sim \mathrm{Unif}[0,1]^\Gamma$, $\sigma(x) = \psi\big(\omega(x+\cdot)\big)$.

If yes, then the group has fixed price 1, because Abért-Weiss '13 says these have the highest cost.

Problem 2. Is the cost of any group Γ realized inside any Cayley graph?

Problem 3. Our iterative process seems to condensate into a unique infinite cluster also on $\mathbb{Z}^3$, but not on $\mathbb{Z}^2$. Why?

Wild guess: transient graphs without non-constant HD functions are exactly those where free effective resistances between vertices remain bounded.
If this implied condensation, then $\beta_1^{(2)} = 0$ would imply cost $= 1$.

Open questions

P-Timár '21 proved that, surprisingly, FUSF is disconnected in some Cayley graph of the virtually free group $F_k \times \mathbb{Z}_{k^9}$ (and connected in some other).

Problem 4. Here, is the independent union of FUSF and Bernoulli(ϵ) bond percolation connected, for any $\epsilon > 0$? If not, is there a general invariant way?

This would kill only Gaboriau's proposed strategy to prove cost $= 1 + \beta_1^{(2)}$, not the statement, since cost$(F_k \times H) = 1 + \frac{k-1}{|H|}$ and $\beta_1^{(2)}(F_k \times H) = \frac{k-1}{|H|}$.