Non-amenable Poisson Zoo

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Plan of talk

1. Definitions and results
2. Two pieces of motivation from measurable group theory
3. Proof ideas
4. What is missing for the main open question

Please interrupt me with questions at any time.

The Poisson Zoo model

Infinite transitive graph G, such as a Cayley graph $G = \mathrm{Cay}(\Gamma, S)$.

$\nu = \nu_o$ is a probability measure on finite rooted lattice animals $o \in H \subset V(G)$, invariant under the automorphisms of G that fix o. **E.g.1.** Ball $B_\rho(o)$ of random radius. **E.g.2.** SRW trajectory $(X_i)_{i=0}^{\ell}$ of random length.

Can map by any graph automorphism $\varphi : o \mapsto x$ to get $\nu_x := \varphi_*(\nu_o)$.

Fix intensity $\lambda > 0$, then $N_x \sim \mathrm{Poi}(\lambda)$ iid variables for each $x \in V(G)$, then N_x iid samples from ν_x. What are the clusters in the union?

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Assume that G is unimodular: $|\mathrm{Aut}_x y| = |\mathrm{Aut}_y x|$ for every x, y, or the Mass Transport Principle: $\sum_y f(x,y) = \sum_x f(x,y)$ for invariant functions.

Fact 1. If $m_1 := \mathbf{E}_\nu |H| = \infty$, then the union a.s. covers all of G.

Fact 2. If $m_2 := \mathbf{E}_\nu |H|^2 < \infty$, then for λ small enough there are only finite clusters, while for λ large enough there are infinite clusters, a.s.

The moment conditions: size-biasing

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Proof. From MTP, the expected number of animals covering o is $\lambda m_1 = \infty$. But this is a Poisson random variable, so, it is infinite a.s.

Fact 2. If $m_2 := \mathbf{E}_\nu|H|^2 < \infty$, then $\lambda_c \in (0, \infty)$ for percolation.

Proof. The expected total size of animals covering o is λm_2.

For λ small, domination by subcritical branching process.

For λ large, already singletons percolate.

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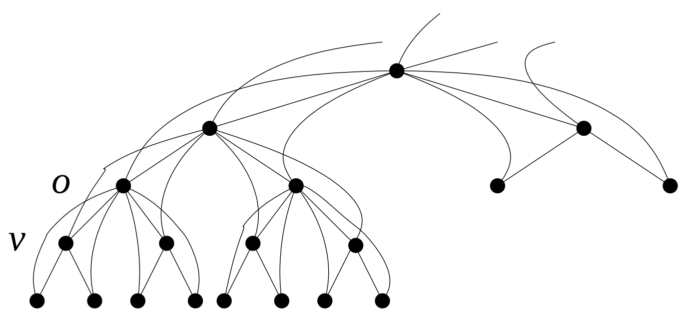
Importance of unimodularity:

Downwards random path
with random length $\mathbf{P}[\,\mathcal{L} = \ell\,] \asymp \ell^{-3}$.
$m_1 < \infty$, $m_2 = \infty$, but expected total length of paths covering a vertex is finite.



In Euclidean spaces

Gouéré '08: balls in $\mathbb{R}^d$ with random radii: $m_1 < \infty \implies \lambda_c \in (0, \infty)$

Ráth-Rokob '22:

1. The above implies the same for balls in $\mathbb{Z}^d$ by a simple coupling.

2. In $\mathbb{Z}^d$, $d \geqslant 5$, random length SRW trajectories (worms):

$m_{2-\epsilon} = \infty \implies \lambda_c = 0$, i.e., percolation at arbitrarily low density.

Much more precisely, $\mathbf{P}[\,\mathcal{L} = \ell\,] \asymp \frac{(\log\log\ell)^\epsilon}{\ell^3 \log\ell}$ for any $\epsilon > 0$ is enough.

But maybe $m_2 = \infty$ is enough?

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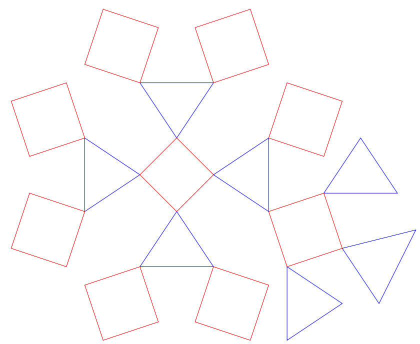
So, balls and worms are pretty much on the two extremes.

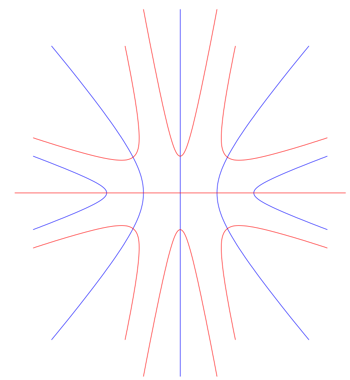
How about **non-amenable** ($\frac{|\partial S|}{|S|} > c\ \forall S$) groups, to make things simpler?
Should we care?

On non-amenable graphs

Theorem 1. If Γ is a non-amenable free product of unimodular transitive graphs, then ANY ν with $m_2 = \infty$ satisfies $\lambda_c = 0$.

$\mathbb{Z}_3 * \mathbb{Z}_4$





$\mathbb{Z} * \mathbb{Z}$

Theorem 2. If Γ is ANY nonamenable unimodular transitive graph, and the lattice animals are worms (simple random walk trajectories of random length) with $m_2 = \infty$, then $\lambda_c = 0$.

In particular, if $m_1 < \infty$ and λ is small, then we have infinite clusters, despite having arbitrarily low total density.

Question. If Γ is ANY nonamenable unimodular transitive graph, then ANY ν with $m_2 = \infty$ satisfies $\lambda_c = 0$?

Other low density FIID percolations with ∞ clusters

A site percolation σ on a group Γ is a **factor of IID (FIID) process** if there is a measurable **coding map** $\psi : [0,1]^\Gamma \longrightarrow \{0,1\}$, or $[0,1]^{E(\Gamma,S)} \longrightarrow \{0,1\}$ for some generating system S, such that for ω iid $\mathrm{Unif}[0,1]$ input,

$$\sigma(x) = \psi\big(\omega(x\cdot)\big)\,, \quad x \in \Gamma\,.$$

E.g.1. Consider Bernoulli percolation on a Cayley graph of Γ at $p = p_c + \epsilon$. Assuming $\theta(p_c) = 0$, true for non-amenable groups (BLPS '99), density of infinite clusters is small. Delete all finite clusters.

E.g.2. Take iid Unif$[0,1]$ labels on the edges $E(\Gamma, S)$. The Wired Minimal Spanning Forest is by deleting the edge with the largest label in every cycle, possible through infinity. This is infinitely many 1-ended trees for any non-amenable group Γ. Prune the leaves, repeatedly, a 1000 times.

E.g.3. Random interlacements (PPP of bi-infinite SRW trajectories) at low intensity. FIID by Borbényi-Ráth-Rokob '21.

These constructions inherently give ∞ many ∞ clusters.

One motivation: measurable cost

$\text{cost}(\Gamma, S) := \frac{1}{2} \inf \Big\{ \mathbf{E}_\mu[\deg(o)]$: μ is an invariant probability measure
on connected spanning subgraphs of $\text{Cay}(\Gamma, S) \Big\}$.

$\text{cost}^*(\Gamma, S) := \frac{1}{2} \inf \Big\{ \mathbf{E}_\mu[\deg(o)]$: μ is a FIID measure
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Obvious: $\text{cost}^*(\Gamma, S) = 1 - \frac{1}{|\Gamma|}$ for finite Γ, while $\text{cost}(\Gamma, S) \geqslant 1$ for $|\Gamma| = \infty$.

Ornstein-Weiss '87, BLPS '99: $\text{cost}^*(\infty \text{ amenable}, S) = 1$.
Can construct an FIID spanning tree with one or two ends.

Gaboriau '00: $\text{cost}(F_d, S) = \text{cost}^*(F_d, S) = d$ for free groups.

If Γ has infinitely many ends, then $\text{cost}(\Gamma, S) > 1$.

Gaboriau '98: $\text{cost}^*(\Gamma_1 \times \Gamma_2, S) = 1$ if either group is infinite.

The fixed price 1 question

Question (Gaboriau '02). Take infinite $\mathrm{Cay}(\Gamma, S)$ with no non-constant harmonic functions with finite Dirichlet energy, $\sum_{x \sim y} |h(x) - h(y)|^2 < \infty$. (This does not depend on the generating set S. Equiv to WUSF $=$ FUSF.)

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Does $\mathrm{cost}(\Gamma, S) = 1$ hold? Or even $\mathrm{cost}^*(\Gamma, S) = 1$?

A famous test case was Kazhdan (T) groups. Hutchcroft-P. '20 proved $\mathrm{cost}(\Gamma, S) = 1$, but not a FIID construction.

HP'20 constructed a Sparse Unique Cluster, a unique infinite cluster with arbitrarily small density, which implies cost $= 1$. Is there a FIID SUC?

Frączyk-Mellick-Wilkens '23+: $\mathrm{cost}^* = 1$ for higher rank Lie groups and products of trees. But no FIID SUC yet.

I am pretty sure that $\mathbb{F}_d \times \mathbb{Z}$ has an FIID SUC, as a Poisson Zoo, but even this has not been checked. Other examples?

Another motivation: indistinguishability

An invariant percolation has indistinguishable infinite clusters if they agree on every translation invariant property (e.g., one-ended, transient, $p_c < 1$).

Lyons-Schramm '99: insertion tolerant percolations on unimodular transitive graphs have indistinguishability.

Chifan-Ioana '10: for any FIID percolation, there are at most countably many distinguishable non-hyperfinite cluster types.

An invariant percolation cluster is δ-non-hyperfinite if any invariant deletion of at most δ density of the edges still gives infinite clusters.

Csóka-Mester-P.: For any FIID percolation on trees, there are at most $K(\delta) < \infty$ many distinguishable δ-non-hyperfinite cluster types. Equiv, any FIID percolation with δ-non-hyperfinite clusters has density $\geqslant \epsilon(\delta)$.
Proof uses entropy inequalities, from counting in random regular graphs.

Every non-amenable group? Idea from Hutchcroft: in Gromov-Osajda monsters, low density Poisson Zoo of expanders could be a counterexample.

Proof ideas for free products

Idea: use the free product structure, virgin territories, lots of independence, to find a branching exploration process $(E_n)_{n\geqslant 0}$ with large mean offspring inside the cluster.

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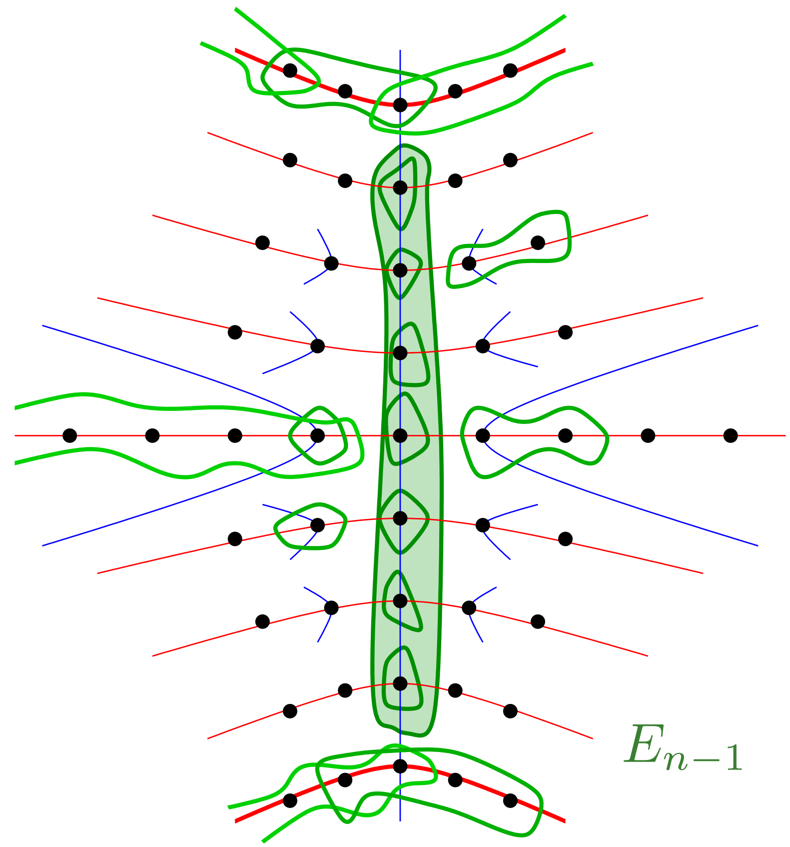
However, **an issue:** the two factors in $\Gamma_1 * \Gamma_2$ may behave differently.

E.g., in $\mathbb{Z} * \mathbb{Z}$, the animal measure ν extends only in the red direction.

Depending on $\mathbf{E}_{n-1}$, maybe only a small part of its boundary is virgin territory for growth.



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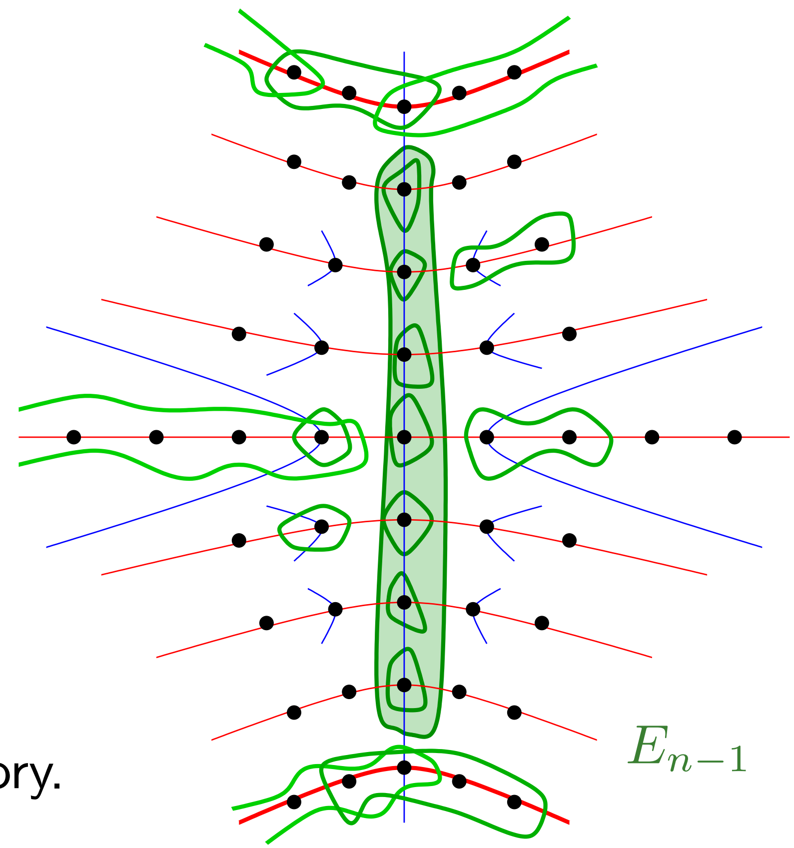

Solution: use sprinkling to reach virgin territory. This has a probability cost, but compared to the infinite mean, it is OK.

Proof ideas for worms

Random walk capacity: cap$(K) := \sum_{x \in S} \deg(x)\, \mathbf{P}_x[\, \tau_K^+ = \infty \,]$

$$:= \inf \left\{ \frac{1}{2} \sum_{x \sim y} |f(x) - f(y)|^2 : f \geqslant \mathbf{1}_K \right\}.$$

Lemma. If G is non-amenable, then cap$(K) \asymp |K|$ for any finite K.

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Exploration process: Animals of size at most R.
E_0 is the union of animals covering o.
$B_{n-1} := \partial E_{n-1} \setminus \partial E_{n-2}$
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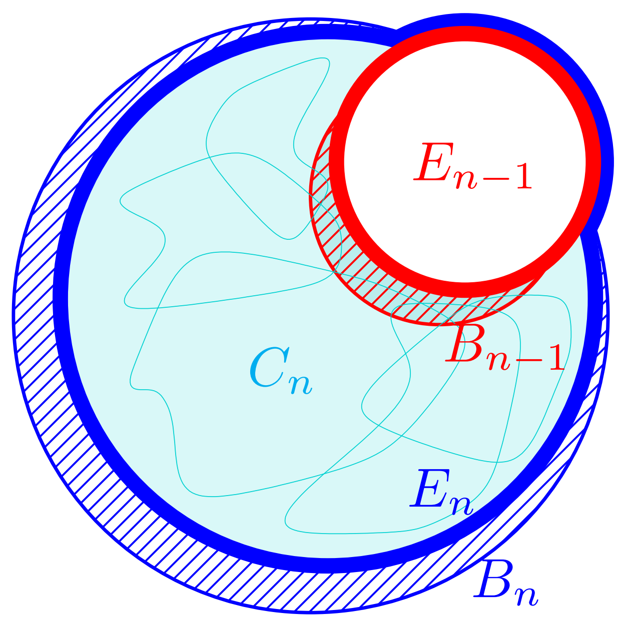

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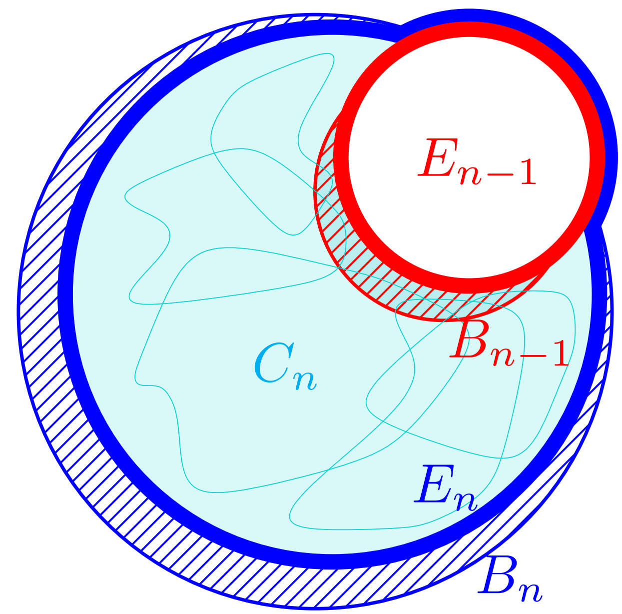

If $|E_{n-1}| > \vartheta_R |E_{n-2}|$ with ϑ_R large,
then $|B_{n-1}|$ is large, and choosing R large,
get large 1st and small 2nd moment for $|C_n|$, so
also $\mathbf{P}\big[\, |E_n| > \vartheta_R |E_{n-1}| \,\big|\, \mathcal{F}_{n-1} \,\big] > 1 - \frac{A_R}{|E_{n-1}|}$.

Altogether, $\mathbf{P}\big[\, |E_n| > \vartheta_R |E_{n-1}|$ for $n = 1, 2, \ldots \,\big] > 0$, with ϑ_R large.

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for some absolute constant $c > 0$, then the second proof works.

This $(*)$ does not hold in $\mathbb{T}_k \times \mathbb{T}_k$, for instance.

It holds in hyperbolic space $\mathbb{H}^d$, $d \geqslant 2$, hence the 2nd moment version of Gouéré's theorem seems to hold (Elias-P-Rokob, in preparation).

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Even if $(*)$ does not hold, could it hold for "typical" $K = E_{n-1}$?

Thank you for your attention!