Gentle Statistical Mechanics — First HW problem set

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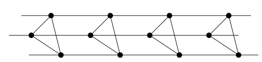
Two-point •• problems are a bit harder or more interesting or more important than one-point • problems.

▷ **Exercise 1.•** What is the critical bond percolation density for the infinite triangular ladder?



▷ **Exercise 2.•** Let $G(V, E)$ be any bounded degree infinite graph, and $S_n \nearrow V$ an exhaustion by finite connected subsets. Consider Bernoulli(p) bond percolation on G. Is it always true that, for $p > p_c(G)$, we have

$$\lim_{n\to\infty} \mathbf{P}_p[\,\text{largest cluster for percolation inside } S_n \text{ is the subset of an infinite cluster}\,] = 1\,?$$

▷ **Exercise 3.•** By noting that $\theta(p) = \lim_{n\to\infty} \mathbf{P}_p[\,o \longleftrightarrow \partial B_n(o)\,]$, show that $p \mapsto \theta(p)$ is right-continuous.

▷ **Exercise 4.•** Can it happen for some iid sequence $X_1, X_2, \dots$ that $(X_1 + \cdots + X_n)/a_n$ almost surely converges, for some deterministic sequence $a_n \to \infty$ to a random variable that is not an almost sure constant? (Hint: think of Kolmogorov's 0/1 law, discussed on March 6.)

▷ **Exercise 5.** Consider a Galton-Watson tree with offspring distribution ξ, with $\mathbf{E}\xi = \mu$. Let Z_n be the size of the nth level, with $Z_0 = 1$, the root. On March 6, we (will) have shown that $\mathbf{E}[\,Z_n\,] = \mu^n$, which easily implies that for $\mu < 1$ the GW tree is finite almost surely.

(a)•• If $\mu > 1$ and $\mathbf{E}[\,\xi^2\,] < \infty$, show that $\mathbf{E}\big[\,Z_n^2\,\big] \le C(\mathbf{E}Z_n)^2$ with a constant $C < \infty$ that does not depend on n. (Hint: use the conditional variance formula $\mathbf{D}^2[Z_n] = \mathbf{E}\big[\,\mathbf{D}^2[Z_n \mid Z_{n-1}]\,\big] + \mathbf{D}^2\big[\mathbf{E}[\,Z_n \mid Z_{n-1}\,]\big]$.) Using this and the **Second Moment Method**, namely, if $X \ge 0$ a.s., then Cauchy-Schwarz implies $\mathbf{P}[\,X > 0\,] \ge (\mathbf{E}X)^2/\mathbf{E}[\,X^2\,]$ (you can look this up, e.g., in PGG Section 12.3), deduce that the GW tree is infinite with positive probability.

(b)• Extend the previous part to the case $\mathbf{E}\xi = \infty$ or $\mathbf{D}\xi = \infty$ by a truncation $\xi \mathbf{1}_{\xi<K}$ for K large enough.

For Bernoulli bond percolation on any connected infinite graph G, any $o \in V(G)$, define

$$p_T := \inf\big\{p : \mathbf{E}_p[\,|\mathscr{C}_o|\,] = \infty\big\},$$

where $\mathscr{C}_o$ denotes the cluster of vertex o. T is for the honour of Temperley. As for the critical density p_c defined in class, one can show that this does not depend on o. Obviously, $p_T \le p_c$ for any graph.

▷ **Exercise 6.** Consider Bernoulli bond percolation on the canopy tree Λ: countably infinitely many leafs on level 0, joined pairwise by vertices on level 1, again joined pairwise by vertices on level 2, and so on. (The limit of large balls in the 3-regular tree $\mathbb{T}_3$ viewed from their boundaries).

(a)• Show that $p_c(\Lambda) = 1$.

(b)•• Find $p_T(\Lambda)$.

(A highly non-trivial theorem is that $p_T = p_c$ holds on every transitive graph. The simplest proof is due to Duminil-Copin and Tassion `https://arxiv.org/abs/1502.03050`.)

▷ **Exercise 7.**$^{\bullet}$ Prove using Fekete's subadditive lemma that, for Bernoulli percolation on any transitive graph, $\sigma(p) := \lim_{n\to\infty} \frac{-1}{n} \log \mathbf{P}_p[\, o \longleftrightarrow \partial B_n(o)\,]$ exists, and is in $[0,\infty)$.

▷ **Exercise 8.** Consider percolation on the d-regular tree $\mathbb{T}_d$, $d \geq 3$.

(a)$^{\bullet\bullet}$ By analyzing the location of the smaller solution of $f(s) = s$, where $f(s)$ is the generating function of the offspring distribution $\mathsf{Binom}(d-1, \frac{1}{d-1} + \epsilon)$, show that

$$\theta(p) := \mathbf{P}_p[\, |\mathscr{C}_o| = \infty \,] \asymp p - p_c \,,$$

meaning equality up to constant factors, as $p \searrow p_c$. (That is, the off-critical exponent on regular trees is $\beta = 1$.)

(b)$^{\bullet\bullet}$ Estimate the probability $\mathbf{P}[\, Z_n > 0 \,] = f^{\circ n}(0)$ to get that

$$\mathbf{P}[\mathscr{C}_o \text{ has height} \geq n] \asymp 1/n \,.$$

(That is, the one-arm critical exponent on regular trees is $1/\rho = 1$.)

▷ **Exercise 9.**$^{\bullet\bullet}$ Consider percolation on the 3-ary tree, and let

$$p_c^{(2)} := \inf \big\{ p : \mathbf{P}_p[\, \text{cluster of the root contains an infinite binary tree} \,] > 0 \big\}.$$

By writing a recursion similar to the simple percolation case, calculate $p_c^{(2)}$ (you will need to solve a cubic polynomial, but it has an obvious root, so it becomes quadratic). Show that the phase transition is discontinuous: at $p_c^{(2)}$, the probability of having the binary tree is already positive.

▷ **Exercise 10.**$^{\bullet\bullet}$ Consider site percolation on $\mathbb{Z}^2$. Show that $1/3 \leq p_c(\mathbb{Z}^2, \text{site}) \leq 5/6$.