Gentle Statistical Mechanics — Second HW problem set

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▷ **Exercise 1.** Recall that we have stated the **Harris-FKG inequality** says that if $X_1, \dots, X_n$ are independent $\mathbb{R}$-valued random variables, and $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ are increasing functions, then

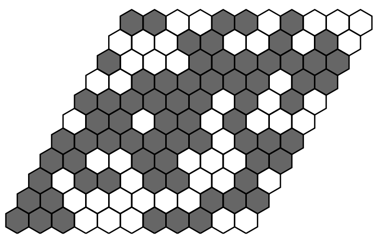
$$\mathbf{E}\big[f(X_1, \dots, X_n)\, g(X_1, \dots, X_n) \big] \geq \mathbf{E}\big[f(X_1, \dots, X_n) \big]\, \mathbf{E}\big[g(X_1, \dots, X_n) \big] .$$

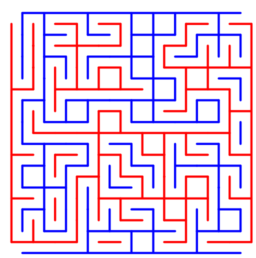
Let's prove it!

(a)$^\bullet$ For $n = 1$, show that if X and Y are i.i.d., then $\mathbf{E}\big[(f(X) - f(Y))(g(X) - g(Y)) \big] \geq 0$, and conclude from this.

(b)$^\bullet$ For general n, use induction.

Recall the self-duality of bond percolation on $\mathbb{Z}^2$ and site percolation on the triangular grid TG (same as colouring the faces of the hexagonal lattice) at density $p = 1/2$.





Given this, we have accepted the **Russo-Seymour-Welsh box-crossing estimates**: if $\mathsf{LR}(r, s)$ is the event that the $r \times s$ rectangle has a left-to-right crossing, then, in either model, for any aspect ratio $0 < \rho < \infty$ there exists a constant $c_\rho > 0$ such that, for every n,

$$c_\rho \leq \mathbf{P}_{1/2}\big[\mathsf{LR}(n, \rho n) \big] \leq 1 - c_\rho .$$

▷ **Exercise 2.** Show that for $p = 1/2$ site percolation on TG or bond percolation on $\mathbb{Z}^2$, for the **one-arm probability** $\alpha_1(r, R) := \mathbf{P}[\, \partial B_r(o) \longleftrightarrow \partial B_R(o) \,]$ and $\alpha_1(n) := \mathbf{P}[\, 0 \longleftrightarrow \partial B_n(o) \,]$, we have the following **quasi-multiplicativity** and **polynomial decay**:

(a)$^\bullet$ there exists $c > 0$ such that, for any radii $\rho \leq r/2 < 2r < R$,

$$c\, \alpha_1(\rho, r)\, \alpha_1(r, R) \leq \alpha_1(\rho, R) \leq \alpha_1(\rho, r)\, \alpha_1(r, R)\,; \tag{1}$$

(b)$^\bullet$ there exist constants c_i, γ_i such that

$$c_1\, n^{-\gamma_1} \leq \alpha_1(n) \leq c_2\, n^{-\gamma_2} . \tag{2}$$

The remaining exercises are basic theoretical material, kind of Probability 2. If you get lost, it might be OK to ignore them for now.

▷ **Exercise 3** (A useful calculus lemma).$^\bullet$ Prove that $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$, and $e^{-x} \leq 1 - x/2$ for all $0 < x < \epsilon$, if $\epsilon > 0$ is small enough. Conclude that, for any sequence $\epsilon_n \in (0, 1)$, we have:

$$\sum_n \epsilon_n = \infty \iff \prod_n (1 - \epsilon_n) = 0.$$

▷ **Exercise 4.** Consider the probability space $[0,1]$ with the uniform measure.

(a)$^\bullet$ Give a sequence of events (intervals) $\{A_n\}_{n\geq 1}$ with $\mathbf{P}[\, A_n \,] \to 0$ but $\mathbf{P}\big[\, A_n$ occurs for infinitely many $n \,\big] = 1$. Conclude that convergence in probability does not imply almost sure convergence.

(b)$^\bullet$ **Borel-Cantelli** cannot be reversed without independence: give a sequence of events $\{A_n\}_{n\geq 1}$ with $\sum_n \mathbf{P}[\, A_n \,] = \infty$ but $\mathbf{P}\big[\, A_n$ occurs for infinitely many $n \,\big] = 0$.

(c)$^\bullet$ Give a sequence of random variables $\{X_n\}_{n\geq 1}$ such that $X_n \to 0$ almost surely, but $\mathbf{E}[\, X_n \,] \to \infty$.

▷ **Exercise 5.**

(a)$^\bullet$ Use Cauchy-Schwarz or Jensen's inequality to show that if $\mathbf{E}[\, X^4 \,] < \infty$, then $\mathbf{E}[\, X^2 \,] < \infty$.

(b)$^{\bullet\bullet}$ Assuming that $X_1, X_2, \dots$ are i.i.d. with $\mathbf{E}[\, X_i \,] = \mu$ and $\mathbf{E}[\, X_i^4 \,] < \infty$, give a good bound on $\mathbf{E}\big[\, |X_1 + \cdots + X_n - n\mu|^4 \,\big]$, then use "turbo-Markov" to show that, for any $\epsilon > 0$ fixed,

$$\sum_{n=1}^{\infty} \mathbf{P}\Big[\, \Big| \frac{X_1 + \cdots + X_n}{n} - \mu \Big| > \epsilon \,\Big] < \infty \,,$$

and then we can use Borel-Cantelli to conclude the SLLN.

For random variables $X_1, X_2, \dots$ and X, with values in some metric space (Ω, d), **weak convergence** $X_n \Rightarrow X$ means that $\mathbf{E}f(X_n) \to \mathbf{E}f(X)$ for every continuous bounded function $f : \Omega \longrightarrow \mathbb{R}$. When Ω is $\mathbb{R}$, with the usual metric, then this is equivalent to the convergence of the Cumulative Distribution Functions at every continuity point, for which you actually saw some examples in Probability 1 (e.g., the deMoivre-Laplace CLT). One can also talk about the weak convergence of probability measures on Ω, also denoted by $\mu_n \Rightarrow \mu$, via the correspondence $\mu_n(A) = \mathbf{P}[\, X_n \in A \,]$, and then $\int_\Omega f(x)\, d\mu_n(x) \to \int_\Omega f(x)\, d\mu(x)$ for every bounded continuous f.

▷ **Exercise 6.** What is the weak limit of the following distributions on $\mathbb{R}$?

(a)$^\bullet$ Uniform distribution on $\{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$.

(b)$^\bullet$ For each i/n, $i = 0, 1, 2, \dots$ flip a coin with success probability λ/n. Let $X_n(t)$ be the number of successes in the interval $[0, t)$, and T_n is the position of the first success; that is $T_n \sim \frac{1}{n}\mathsf{Geom}(1/n)$. What is the weak limit of $X_n(t)$ and of T_n, as $n \to \infty$?

(c)$^{\bullet\bullet}$ Let $X_1, X_2, \dots$ be i.i.d. $\mathsf{Unif}[0,1]$, then let μ_n be the uniform distribution on $\{X_1, \dots, X_n\}$. Easier: if you don't know (average over) the realization $X_1, X_2, \dots$, what is μ_n? Harder: if you condition on $X_1, X_2, \dots$, so that μ_n becomes a *random* probability measure, where does it converge to, for almost every $X_1, X_2, \dots$?

For people with an abstract mindset (a statement easy to believe, tedious to prove):

▷ **Exercise 7.**$^{\bullet\bullet}$ Consider the metric space $\Omega = \{0,1\}^{\mathbb{Z}^d}$ with metric $d(\omega, \eta)$ being the 2^{-R} for the largest R such that ω and η agree on the ball $B_R(o)$. Show that (μ_n) on Ω converges weakly to some μ if and only if for any finite collection of vertices $v_1, \dots v_k \in \mathbb{Z}^d$ and possible values $(x_1, \dots, x_k) \in \{0,1\}^k$, the probabilities converge:

$$\lim_{n\to\infty} \mu_n\big(\omega(v_i) = x_i \text{ for all } i = 1, \dots, k\big) = \mu\big(\omega(v_i) = x_i \text{ for all } i = 1, \dots, k\big).$$

Another natural notion of convergence (notion of closeness) of probability measures is via the **total variation distance**. For two probability measures on a general probability space Ω,

$$d_{\mathrm{TV}}(\mu, \nu) := \sup\big\{|\mu(A) - \nu(A)| : \text{events } A \subseteq \Omega\big\}\,.$$

▷ **Exercise 8.**

(a)$^\bullet$ Show that $d_{\mathrm{TV}}(\mu_n, \mu) \to 0$ on any Ω implies $\mu_n \Rightarrow \mu$.

(b)$^\bullet$ Show that for $\Omega = \mathbb{R}$ the converse is false.

(c)$^{\bullet\bullet}$ If Ω is countable, then

$$d_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2} \sum_{x\in\Omega} \big|\mu(x) - \nu(x)\big|\,.$$

(d)$^\bullet$ If Ω is a finite set, then the converse of (a) is true. (In fact, over finite sets, any two reasonable notions of closeness of probability measures are equivalent. This is similar to the statement that, on any finite dimensional vector space, any two norms are equivalent.)