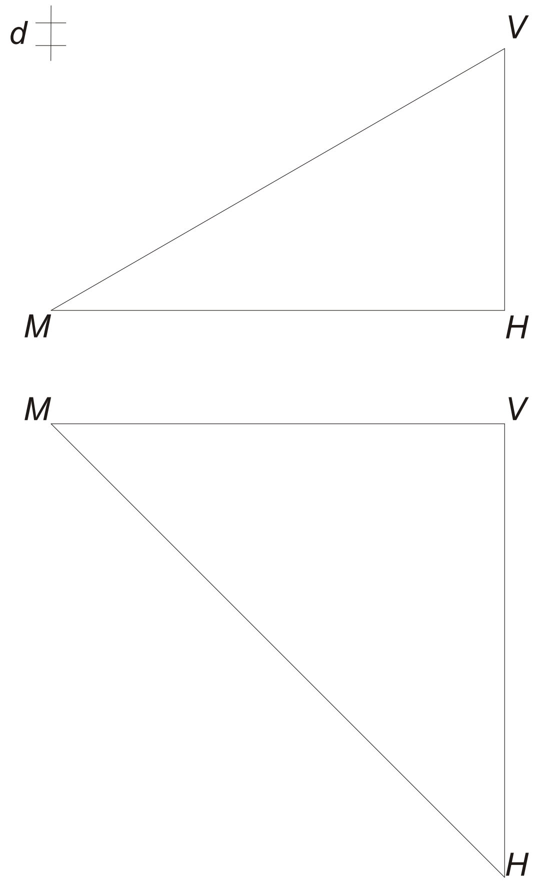
Construction of a circle

Construction of a circle contained in an oblique plane

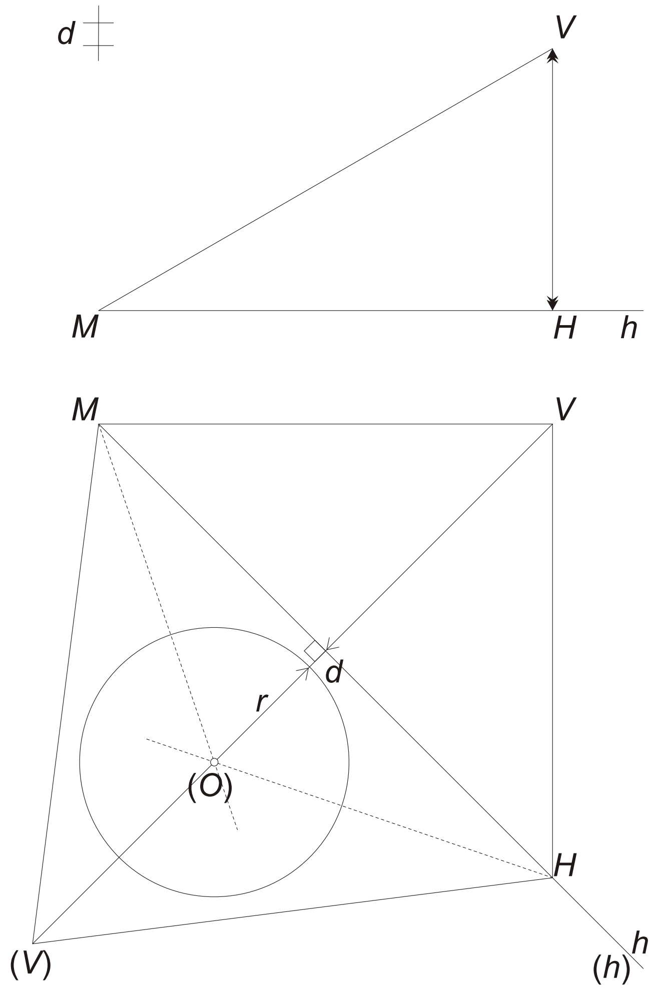


Exercise. *Given the triangle MVH such that its horizontal projection is an isosceles right triangle and a distance d. Construct the circle, contained in the plane of the triangle which is at the distance d from each sideline.*

To solve the problem we need to know

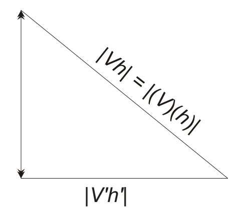
- the plane,
- the center,
- and the radius

of the circle. Among these, we know only the plane of the circle directly. The first step will be thus the construction of the center of the radius.
The circle can be obtained by constructing the incircle of the triangle, and then contracting it from the incenter in a suitable ratio. Hence, the center of the circle is the incenter, and its radius is the difference between the inradius and d.
To find the incenter, we rotate the plane of the triangle into horizontal position, where the bisectors of the triangle can be constructed directly. Then the center of the circle is the intersection of the three angular bisectors.



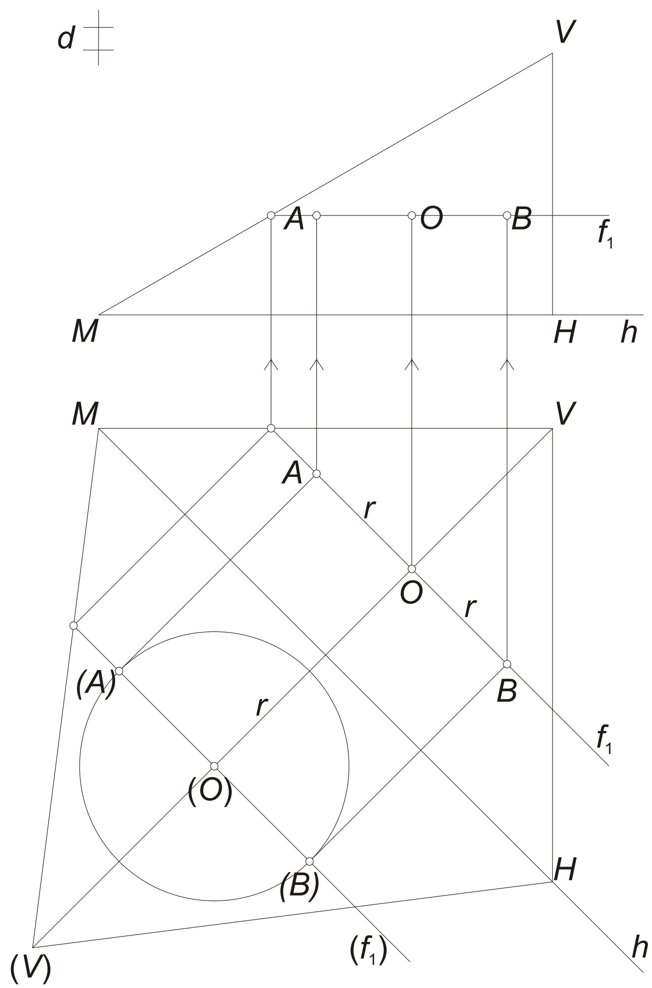
We choose the horizontal line $h = MH$ as the axis of the rotation. Then, during the rotation, the points of the axis, e.g. M remain at the same place, while V moves on a circle arc, perpendicular to h, into the rotated position (V) in such a way that the distance of (V) and $(h) = h'$ is equal to the real distance of V and h.

The latter one can be obtained from the corresponding difference triangle: the horizontal leg is the distance of V' and h', the vertical leg is the difference of the levels of V'' and h'', and the hypotenuse is the real distance of V and h.



In the rotated triangle, the center (O) of our circle is the intersection point of the angular bisectors. Then, measuring back d on the radii from the tangent points of the incircle we construct the radius of our circle, which now we can draw in the rotated plane.

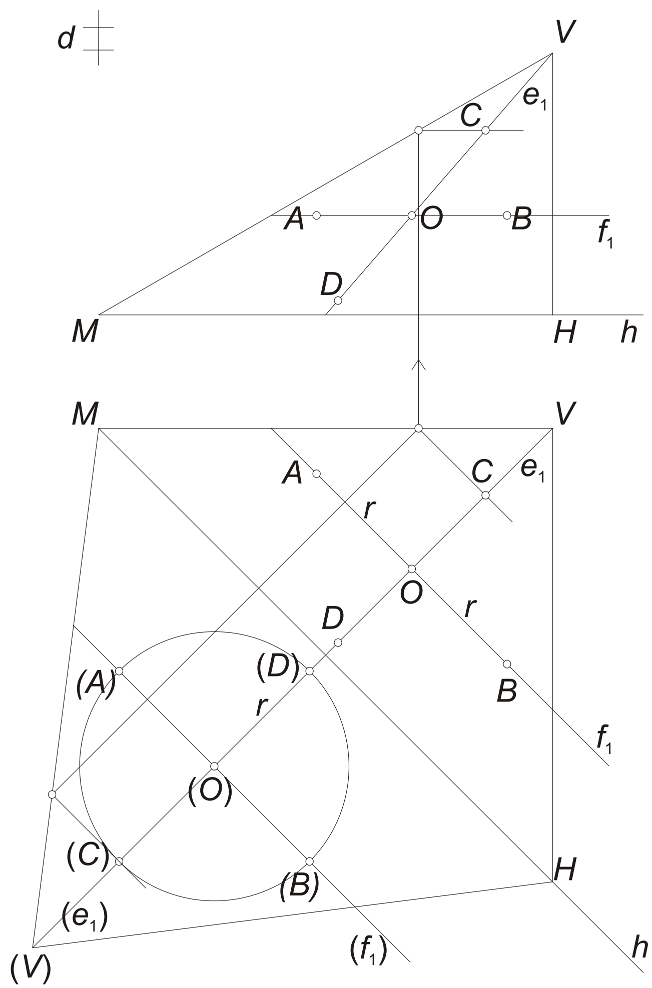
Since in our case the horizontal projection of MHV is an isosceles triangle and MH is a horizontal line, the MVH triagle itself is isosceles: $MV = VH$. Thus the angular bisector at V is an altitude and a median as well.



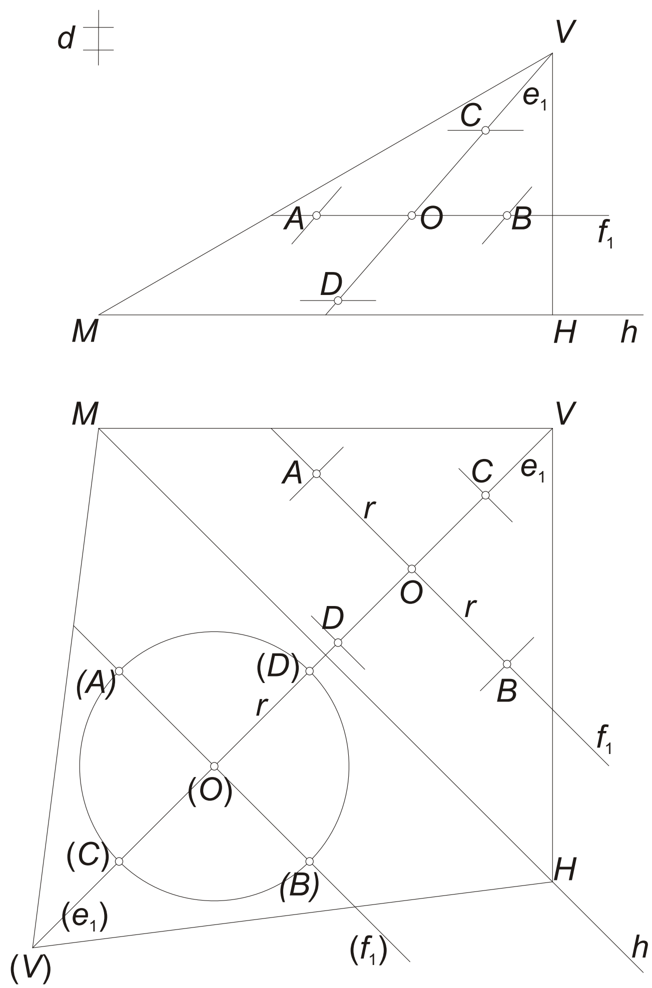
We construct the diameters on the axes of the ellipse-shaped projections.

1.a The major axis of the horizontal projection is the projection of the diameter on a horizontal line. This is *not* shortened during the projection, and its length is equal to the diameter or the circle.

In a plane, the horizontal lines are parallel. Thus, in the rotated plane we can draw the horizontal line (f_1) through (O) parallel to h', which contains the endpoints (A) and (B) of the diameter we are constructing. Now we can draw the horizontal and the vertical projections of these points.



1.b The minor axis of the ellipse in the horizontal projection is the projection of a horizontal slope line of the plane of the circle. This diameter is shortened in the largest degree: this has the shortest projection.
The horizontal slope lines of a plane are perpendicular to the horizontal lines of the plane. In our case the horizontal slope line passing through O is $e_1 = VO$. In the rotated plane, we can draw the rotated copies (C) and (D) of the endpoints of this diameter.
To obtain the two projections of C and D, consider, for example, the tangent line of the circle at C. Similarly to h and f_1, this is a horizontal line in the plane. First, we rotate back the intersection point of this line with MV, to construct the two projections of this point. Drawing lines Through these projections parallel to the projections of h, we obtain the projections of the tangent line. Then we can construct the projections of C.
To construct the projections of D, we can repeat this method, or observe that O is the midpoint of the segment CD. This latter property is preserved under projections.

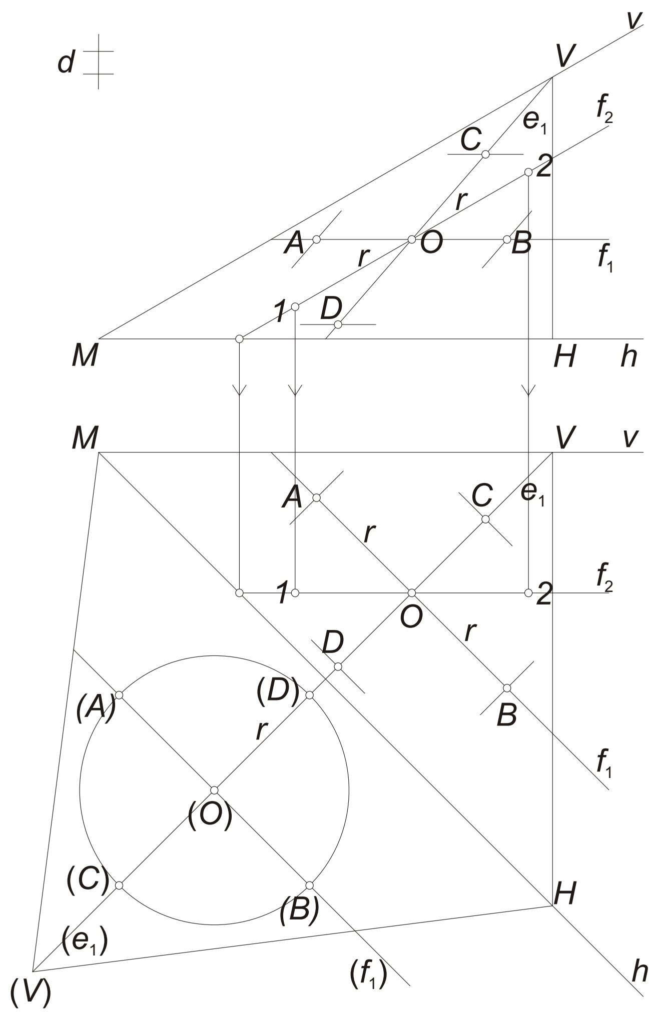


To draw the tangent lines at A, B, C and D, we can use the fact that AB and CD are orthogonal diameters of the circle. Thus, their projections are conjugate diameters of the ellipses. This means that the tangent lines at the endpoints of one of the diameters is parallel to the other one, and vica versa.

Hence, the tangent lines at C and D are parallel to AB, and the ones at A and B are parallel to CD.

This, naturally, is in connection with the fact that in the horizontal projection these points are endpoints of the axes, and thus, the lines through A' and B', parallel to $C'D'$ are perpendicular to $A'B'$, and vica versa.

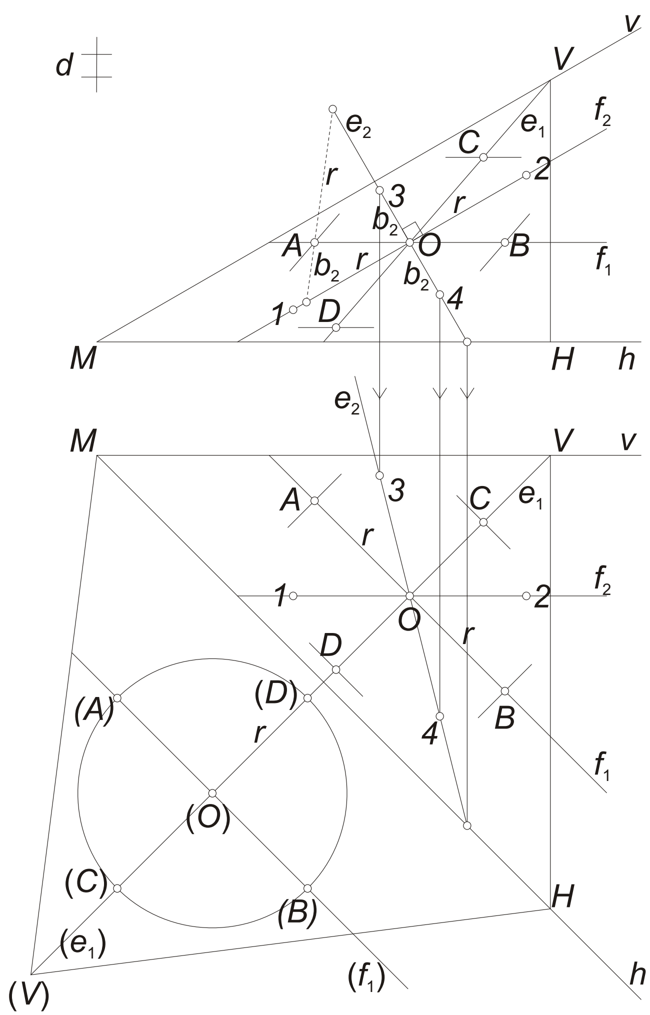
Finally, we remark that we have already constructed the tangent line at C, when constructing C. This line clearly coincides with the one obtained using the above principle.



2.a. The major axis of the vertical ellipse-shaped projection is the projection of the vertical line of the plane through O. Similarly like in the horizontal projection, the length of the major axis is equal to the diameter of the circle.

We construct the vertical line f_2 of the plane through O. Its horizontal projection is perpendicular to the lines of recall. We take the intersection of this line with h'. We construct the vertical projection of this point using its line of recall. Connecting it with O w obtain the required vertical line. (We can also notice that MV is also a vertical line of the plane, and thus, f_2 is parallel to MV.)

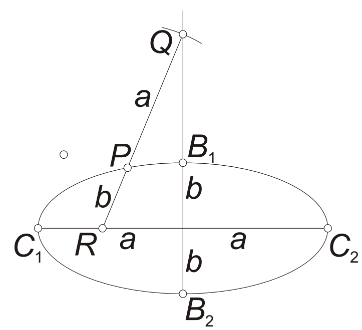
Measuring the real length of the radius on f_2 from O'' we obtain the vertical projections of the endpoints 1 and 2 of the required diameter. To construct their horizontal projections, we use their lines of recall.



2.b. The minor axis of the vertical projection of the circle is the vertical slope line e_2 of the plane of the circle, passing through the center O. In the vertical projection, this is the direction in which the diameter of the circle is most shortened.

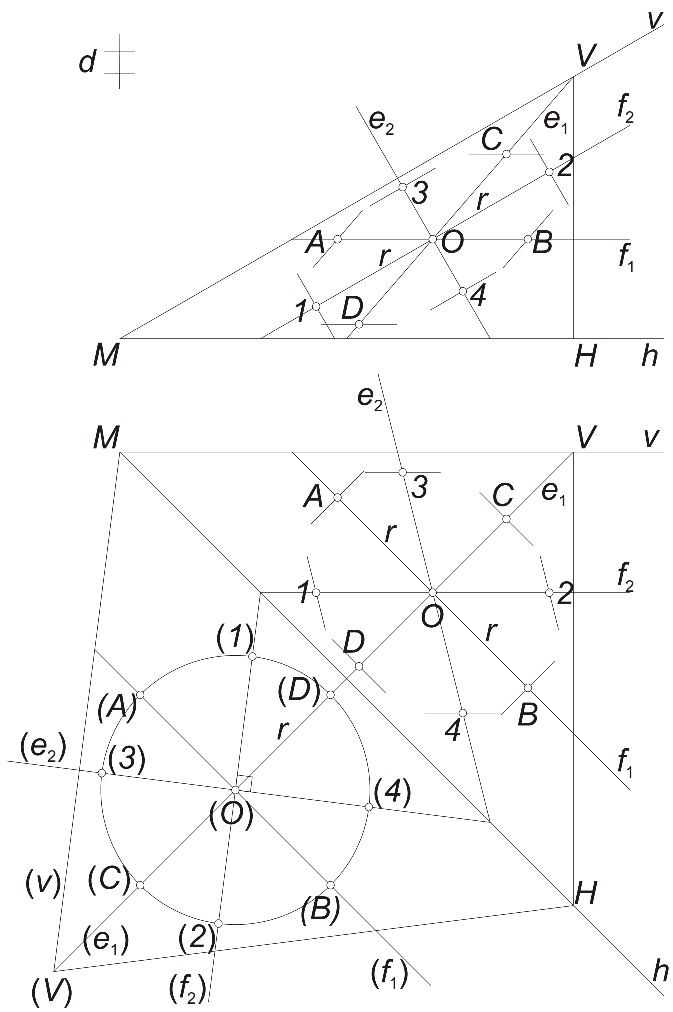
We construct this vertical slope line. Its vertical projection is perpendicular to the vertical projection of f_2. To construct its horizontal projection, we find one of its points, for example, its intersection point with h.

To construct the endpoints of this diameter of the ellipse, we can apply directly the so-called *reversed ellipsograph principle*. According to this, if a point P and the major axis C_1C_2 of an ellipse are given, then its minor axis B_1B_2 can be constructed in the following way.



The line of B_1B_2 is perpendicular to C_1C_2. Drawing a circle around P with radius $a = \frac{C_1C_2}{2}$ we obtain an intersection point Q with this line, on the side of P. The line of PQ intersects the line of C_1C_2 at the point Q. Then $PR = b = \frac{B_1B_2}{2}$ is half the length of the minor axis.

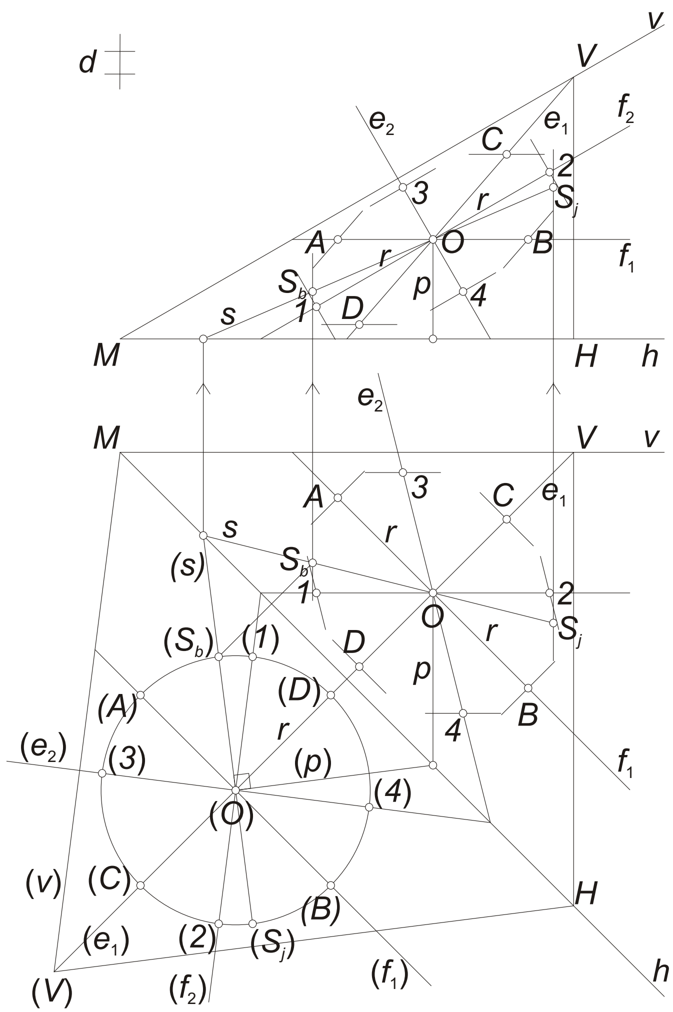
Now we know, for instance, the point A'' of the ellipse, and its major axis $1''2''$. Half the length of the major axis is the radius r of the circle. Applying the above construction, we obtain half the length of the minor axis, denoted by b_2. Measuring it on e_2'' we have the endpoints $3''$, $3''$ of the minor axis. We can construct the horizontal projections of these two points by using their lines of recall.



We construct the tangent lines at the endpoints of the points $1, 2, 3, 4$. Again, we apply the observation that these two diameters of the circle are perpendicular, and hence, their projections are conjugate to each other. Accordingly, at 1 and 2, the tangent lines are parallel to 34, and at 3 and 4, they are parallel to 12.

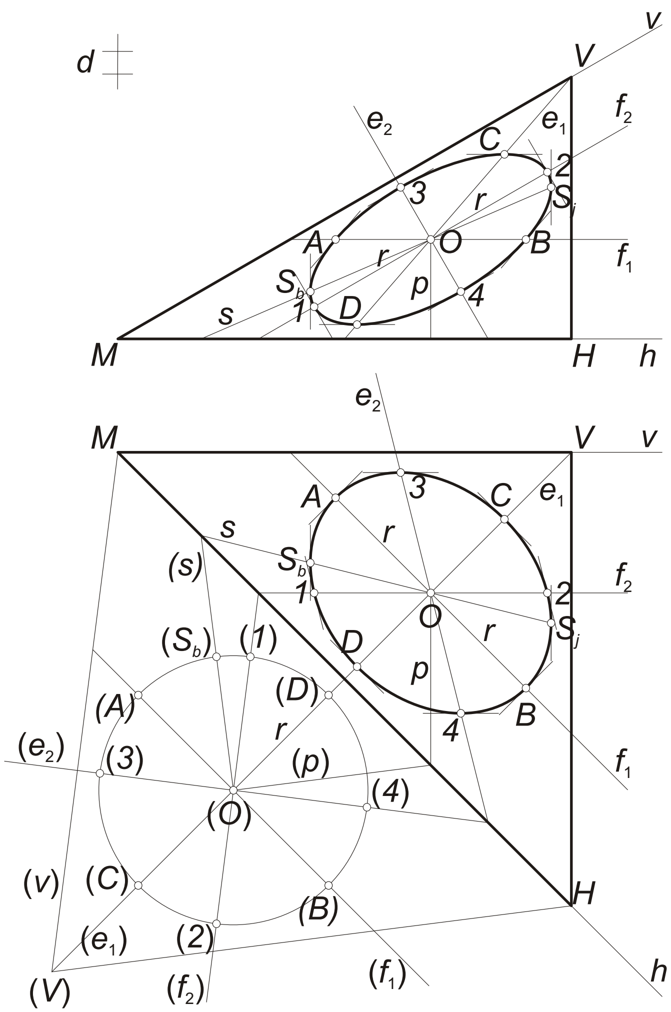
We remark that the axes of the vertical projection can be constructed in the rotated plane as well, which also enables us to find their horizontal and vertical projections. The reason is since MV and f_2 are parallel, we can draw (f_2) in the rotated plane parallel to $M(V)$. Then, (e_2) is perpendicular to f_2), which determines their intersection points $(1), (2), (3), (4)$ with the circle. Rotating these points back yields the horizontal and the vertical projections of the points.

Nevertheless, in general the first method provides a more precise construction.



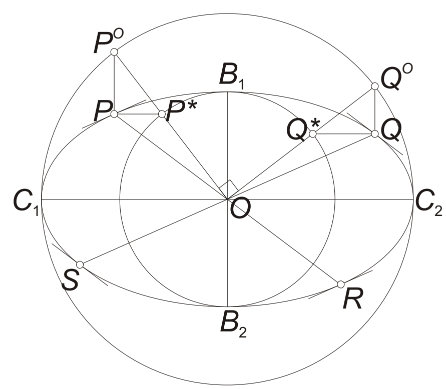
The last essential diameter is the one containing the left-most and right-most point of the circle. As the last step, we construct this diameter. In these points the tangent lines of the ellipses are lines of recall; that is, they are profile lines of the plane of the circle. Thus, the diameter is perpendicular to the profile lines of the plane.

Let us consider now a profile line in the plane, say the line p passing through O (another choice can be VM, for example). Its intersection point with h does not move during the rotation, and thus, connecting it to (O) we obtain the rotated copy (p) of the line. In the rotated plane we can draw the required line (s) perpendicular to (p). One of the two intersection points of (s) and the circle is the rotated copy (S_b) of the left-most point. The common point of (s) and h does not move during the rotation. Using this, we can draw s_1 and s_2, which enables us to find S_b' and S_b'' as well. The point S_j is the reflection of S_b about O. The tangent lines of S_b and S_j are the lines of recall of the points.



Finally, we draw the ellipses using the constructed points and tangent lines.

In some cases, for a more precise drawing we need to construct additional points and tangent lines of the ellipses. This, based on the knowledge of the two axes of the ellipses, can be done using the so-called *two-circle construction*. Applying this, it is useful to construct the four endpoints of two conjugate diameters at the same time in the following way.



We draw around O two circles, using the axes B_1B_2 and C_1C_2 as diameters. These are called the *minor* and the *major* circles of the ellipse, respectively. We choose an arbitrary point P° on the major circle, and draw also the point Q° such that OP° and OQ° are perpendicular. The radii OP° and OQ° intersect the minor circle at the points P^* and Q^*, respectively.

Drawing parallel lines at P° and Q° to the minor axis, and at P^* and Q^* to the major axis, their intersection points will be the points P, Q of the ellipse. Their reflections about O will be the ellipse points R and S. Then PR and QS are conjugate diameters, and thus, the tangent lines at P and R are parallel to QS, and the ones at Q and S are parallel to PR.