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NORMALLY DISTRIBUTED PROBABILITY MEASURE ON THE METRIC SPACE OF NORMS*

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Abstract In this paper we propose a method to construct probability measures on the space of convex bodies. For this purpose, first, we introduce the notion of thinness of a body. Then we show the existence of a measure with the property that its pushforward by the thinness function is a probability measure of truncated normal distribution. Finally, we improve this method to find a measure satisfying some important properties in geometric measure theory.

Key words Hausdorff metric; Borel; Dirac; Haar and Lebesgue-measure; space of convex bodies; metric space of norms

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1 Introduction

In this paper we shall investigate the probability space of norms defined on a real, n-dimensional Euclidean space V. A norm function on V defined by its unit ball K, which is (in a fixed, cartesian coordinate system of the Euclidean vector space $(V,\langle\cdot,\cdot\rangle)$ with origin O is a centrally symmetric in O) convex body. Such bodies give a closed proper subset $\mathcal{K}_0$[1] of the space of convex bodies $\mathcal{K}$ of $(V,\langle\cdot,\cdot\rangle)$. It is known that the Hausdorff distance δ^h (which we define in the next section) is a metric on $\mathcal{K}$ and with this metric $(\mathcal{K},\delta^h)$ is a locally compact space (see [5 ,6]). Thus there should be many measures available on these space. Unfortunatelly this is not so. Bandt and Baraki in [1] proved answering to a problem of McMullen [15] that there is no positive σ-finite Borel measure on it which is invariant with respect to all isometries of $(\mathcal{K},\delta^h)$ into itself. This result exclude the possibility of the existence of a natural volume-type measure. It was a natural question that can whether be found such a σ-finite Borel measure on $\mathcal{K}$ which holds the property that it is non-zero for any open set of $\mathcal{K}$ and invariant under rigid motions of the embedding vector space. This long standing question was answered in the last close by Hoffmann in [9]. His result can be summarized as follows. Each σ-finite rotation

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[1]We rather denote in this paper the space of O-symmetric convex bodies by $\mathcal{K}_0$ as the space of convex bodies with centroid O.

and translation invariant Borel measure on $(\mathcal{K}, \delta^h)$ is the vague limit of such measures and that each σ-finite Borel measure on $(\mathcal{K}, \delta^h)$ is the vague limit of measures of the form $\sum_{i=1}^{\infty} \alpha_n \delta_{K_n}$, where $\{K_n\ ,\ n \in \mathbb{N}\}$ is a countable, dense subset of $(\mathcal{K}, \delta^h)$, (α_n) is a sequence of positive real numbers for which $\sum_{i=1}^{\infty} \alpha_n < \infty$ and δ_{K_n} denote the Dirac measure concentrated at K_n.

Hoffmann also observed that a result of Bárány [2] "suggest that it might not be possible to define a "uniform" probability measure on the set of all polytopes which have rational vertices and are contained in the unit ball". The known concept of Gaussian random convex bodies [13] gives a poor class of Gaussian measures because of a random convex body is Gaussian if and only if there exists a deterministic body and a Gaussian random vector such that the random body is the sum of the deterministic one and the random vector almost surely. He asked whether there exists an alternative approach to "Gaussian" random convex bodies which yields a richer class of "Gaussian" measures on $(\mathcal{K}, \delta^h)$.

The investigations of the author on the generalized space-time models propose that define "Gaussian" (or other type) probability measure on $(\mathcal{K}, \delta^h)$ with respect to a given measurable function of the space (see in [10]). Our observation is that on certain probability space the uniformity or normality properties could be only "relative" one and thus we can require these properties in their impacts through a given function of the space. More precisely, we require the normality or uniformity on a pushforward measure by a given geometric function of the elements of the space (here on the space of convex bodies). To this purpose we will use the thinness function $\alpha_0(K)$ of K defined by the help of the concepts of diameter $d(K)$ and width $w(K)$. As a concrete construction we will give a probability measure on $(\mathcal{K}_0, \delta^h)$ which pushforward measure by the function $\alpha_0(K)$ has truncated normal distribution on the range interval $[\frac{1}{2}, 1)$ (Theorem 2). We note that a method which sends a convex body to a well-defined O-symmetric convex body by a continuous mapping, define a pullback measure of to the all space of convex bodies. The pushforward of this pullback measure by the composition of the mapping of the method and the function $\alpha_0(K)$ has the same properties as the measure of $(\mathcal{K}_0, \delta^h)$. To this purpose we can use the Minkowski symmetrization process sending a body K into the body $\frac{1}{2}(K + (-K))$ with the same diameter, width and thinness (Corollary 1). Our last statement (Theorem 3) that the previously construction can be modified such that the set of basic bodies will be dense and countable with smooth elements. Thus the set of polytopes has zero measure, the set of smooth bodies has measure 1, and every neighborhood has positive measure.

2 Definitions

For self-readable of this paper we recall some necessary definitions. Deeper understanding of the subject on convex geometry and geometric measure theory I suggest to read the nice books [6], [11] and [14] where all properties of the following concepts can be found. The

Definition 1 Let $\mathcal{K}$ be the set of convex bodies of an Euclidean vector space of dimension n. It is endowed with the topology induced by the Hausdorff metric δ^h, which is defined as follows:

$$\delta^h(C, D) = \max\left\{ \max_{x\in C} \min_{y\in D} \|x - y\|, \max_{y\in D} \min_{x\in C} \|x - y\| \right\} \text{ for } C, D \in \mathcal{K}.$$

If we consider a topology on $\mathcal{K}$ or on a subspace of it, such as the space of O-symmetric convex bodies $\mathcal{K}_0$, it is always assumed that it is the topology induced by δ^h.

From geometric measure theory we will use the concepts of Borel, Dirac, Haar and Lebesgue-measure. All of these concepts can be found in [3] or [7]. The definitions and properties of Dirac and Lebesgue measures investigated by a lot of book on modern analysis. We now give a short summary on Haar and Borel measures.

Let X be a locally compact Hausdorff space, and let $\mathfrak{B}(X)$ be the smallest σ-algebra that contains the open sets of it. This is the σ-algebra of Borel sets. Any measure defined on the σ-algebra of Borel sets is called a Borel measure. A measure μ on a measurable space is called inner regular if, for every measurable set A,

$$\mu(A) = \sup\{\mu(K) | K \subseteq A, \quad K \text{ is compact}\}.$$

Analogously if for a measurable set B we have

$$\mu(B) = \inf\{\mu(U) | U \supseteq A, \quad U \text{ is open}\},$$

we say that the measure is outer regular. If a Borel measure is both inner regular and outer regular, it is called a regular Borel measure. Note that a locally finite Borel measure automatically finite for every compact sets.

A topological group is a group G which at the same time is a topological space such that the mapping $(x, y) \mapsto xy^{-1}$ of $G \times G$ into G is continuous. For each $a \in G$ it is easy to check that the mappings $x \mapsto ax$, $x \mapsto xa$ and $x \mapsto x^{-1}$ are homeomorphisms of G onto G. A compact group is a topological group which is a compact space. If G is a compact group, then there is a unique positive regular Borel measure m on G such that

- $m(G) = 1$;
- If U is a nonempty open subset of G, then $m(U) > 0$;
- If B is any Borel set of G and $s \in G$, then

$$m(B) = m(sB) = m(Bs) = m(B^{-1}).$$

This measure m is called the Haar measure for G.

We also use some basic tools of probability theory, e.g. the concepts of truncated Gaussian and uniform distributions, and the concept of the pushforward and pullback of a measure. The reader can read on these concept on the internet or in basic works on probability theory e.g. in [4] or [8].

3 The Thinness Function

Let denote by $w(K)$ the infinum of the distances between parallel support hyperplanes of the convex body K. This is the width of K. The diameter of K is the supremum of the distances between two points of K. It can be regarded also as the supremum of the distances between parallel support hyperplanes of K. By these two quantities we define a new one.

Definition 2 Let denote by $\alpha_0(K)$ the number

$$\alpha_0(K) = \frac{d(K)}{w(K) + d(K)}.$$

We call it the thinness of the convex body K.

The thinness is $\frac{1}{2}$ in the case of the Euclidean ball only and it is equal to 1 if K has of dimension less or equal to $n-1$.

Let now B_E be the unit ball of the embedding Euclidean space and let define the unit sphere of $\mathcal{K}_0$ around B_E by the equality:

$$\mathcal{K}_0^1 := \{K \in \mathcal{K}_0 \mid \delta^h(K, B_E) = 1\}.$$

The following lemma shows the usable of the thinness function in our investigation.

Lemma 1 If $K \in \mathcal{K}_0^1$ and $\alpha_0 := \alpha_0(K)$ is the thinness of K then we have

$$\delta^h(\alpha K, B_E) = \begin{cases} 2\alpha - 1 & \text{if } \alpha_0 \le \alpha, \\ 2\alpha + 1 - 2\dfrac{\alpha}{\alpha_0} & \text{if } 0 \le \alpha < \alpha_0. \end{cases}$$

Proof Assume that $\delta^h(K, B_E)$ is the distance of the points $x \in \mathrm{bd}B_E$ and $y \in \mathrm{bd}K$. Then $\|y\|_E = \|x\|_E + 1 = 2$ and $0, x, y$ are collinear. (We note that the norm of the point y is also the half of the diameter $d(K)$ of K with respect to the Euclidean metric.) This implies that for $\alpha > 1$ the points $\frac{1}{\alpha}x$ and y give a segment with length $\delta^h\left(K, \frac{1}{\alpha}B_E\right)$ and thus

$$\delta^h\left(K, \frac{1}{\alpha}B_E\right) = \left\|y - \frac{1}{\alpha}x\right\|_E = \|y\|_E - \frac{1}{\alpha}\|x\|_E = 2 - \frac{1}{\alpha}$$

holds. If $\alpha < 1$ then the situation is a little bit more complicated. In this case there is a real number $\alpha_0 \in \left[\frac{1}{2}, 1\right)$ such that if $\alpha_0 \le \alpha < 1$ then again

$$\delta^h\left(K, \frac{1}{\alpha}B_E\right) = \left\|y - \frac{1}{\alpha}x\right\|_E = \|y\|_E - \frac{1}{\alpha}\|x\|_E = 2 - \frac{1}{\alpha}$$

but for $\alpha_0 \ge \alpha > 0$ we have a new pair of points $y' \in \mathrm{bdK}$ and $x' \in \mathrm{bdB_E}$ where the distance attained. The point y' is a point of bdK with minimal norm and we have the equality

$$\frac{1}{\alpha_0} - \|y'\| = 2 - \frac{1}{\alpha_0}.$$

Thus the norm of y' is equal to $2(\frac{1}{\alpha_0} - 1)$. In this case

$$\delta^h\left(K, \frac{1}{\alpha}B_E\right) = \left\|-y' + \frac{1}{\alpha}x'\right\|_E = \frac{1}{\alpha} - 2\left(\frac{1}{\alpha_0} - 1\right) = 2 + \frac{1}{\alpha} - \frac{2}{\alpha_0}.$$

We thus have the equality

$$\delta^h(\alpha K, B_E) = \alpha\delta^h\left(K, \frac{1}{\alpha}B_E\right) = \begin{cases} 2\alpha - 1 & \text{if } \alpha_0 \le \alpha \\ 2\alpha + 1 - 2\dfrac{\alpha}{\alpha_0} & \text{if } 0 \le \alpha < \alpha_0. \end{cases}$$

The constant α_0 depends only on the body K and it has the following geometric meaning. $\|y'\|_E = \frac{2}{\alpha_0} - 2$ is the half of the width $w(K)$ of the centrally symmetric body K, because it is a point on $\mathrm{bd}K$ with minimal norm. So we can see that

$$\frac{1}{2} \le \alpha_0 = \frac{2}{\|y'\|_E + 2} = \frac{d(K)}{w(K) + d(K)} < 1$$

as we stated. □

4 Measure on $\mathcal{K}_0^1$ with Uniform Pushforward

We now construct a measure on $\mathcal{K}_0^1$ which pushforward by the thinness function has uniform distribution. To this (following Hoffmann's paper) we introduce the orbits of a body K about the special orthogonal group $SO(n)$ by $[K]$. These are compact subsets of $\mathcal{K}_0^1$, and if we consider an open subset of $\mathcal{K}_0^1$ then the union of the corresponding orbits is also open. Hence there exists a measurable mapping $s : \mathcal{K}_0^1 \longrightarrow \mathcal{K}_0^1$ such that $s(K) = s(K')$ if and only if K and K' are on the same orbit. Let $\widetilde{\mathcal{K}_0^1} := \{K \in \mathcal{K}_0^1 \ , \ s(K) = K\}$ which is measurable subset of $\mathcal{K}_0^1$. We equip it with the induced topology of $\mathcal{K}_0^1$. Finally let $\Phi_{2a}^1 : \widetilde{\mathcal{K}_0^1} \times SO(n) \longrightarrow \mathcal{K}_0^1$ is the mapping defined by the equality:

$$\Phi_{2a}^1(K, \Theta) = \Theta K.$$

Our notation is analogous with the notation of [9]. It was proved in [9] (Lemma 2) that a non-trivial σ-finite measure μ_0 on $\mathcal{K}_0$ is invariant under rotations (meaning that for $\Theta \in SO(n)$ we have $\mu_0(\mathcal{A}) = \mu(\Theta\mathcal{A})$ for all Borel sets $\mathcal{A}$ of $\mathcal{K}_0$) if and only if there exists a σ-finite measure $\widetilde{\mu_0}$ on $\widetilde{\mathcal{K}_0}$ such that $\mu_0 = \Phi_{2a}(\widetilde{\mu_0} \otimes \nu_n)$, where ν_n is the Haar measure on $SO(n)$. It is obvious that in the case of $\mathcal{K}_0^1$ there is a similar result by our mapping $\Phi_{2a}^1(K, \Theta)$ which is the restriction of Hoffmann's map $\Phi_{2a}(K, \Theta)$ onto the set $\mathcal{K}_0^1$.

First choose a countable system of bodies K_m to define a probability measure on $\widetilde{\mathcal{K}_0^1}$. Without loss of generality we may assume that each of the bodies of $\widetilde{\mathcal{K}_0^1}$ has a common diameter of length 4 denoted by d, which lies on the n-th axe of coordinates (hence it is the convex hull of the points $\{2e_n, -2e_n\}$). Consider the set of diadic rational numbers in $(0, 2]$. We can write them as follows:

$$\left\{ m(n,k) := \frac{k}{2^n} \text{ where } n = 0, \cdots, \infty \text{ and for a fixed } n, \ 0 < k \leq 2^{n+1} \right\}.$$

Define the body $K_{m(n,k)}$ as the convex hull of the union of the segment d and the ball around the origin with radius $m(n,k)$. For each n we have 2^{n+1} such bodies, thus the definition

$$\widetilde{\mu_0^1} := \lim_{n \to \infty} \sum_{k=1}^{2^{n+1}} \frac{1}{2^{n+1}} \delta_{K_{m(n,k)}}$$

define a probability measure on $\widetilde{\mathcal{K}_0^1}$. (The limit is the vague limit (or limit with respect to weak convergence) of measures.) In fact,

$$\widetilde{\mu_0^1}\left(\widetilde{\mathcal{K}_0^1}\right) = \lim_{n \to \infty} \sum_{k=0}^{2^{n+1}} \frac{1}{2^{n+1}} \delta_{K_{m(n,k)}} \left(\widetilde{\mathcal{K}_0^1}\right) = 1$$

Lemma 2 The pushforward measure $w(K)^{-1}(\widetilde{\mu_0^1})$ has uniform distribution on the interval $(0, 4]$. ($w(K)$ means the width of the body K.)

Proof Let $B' = (0, x]$ be a level set of $(0, 4]$. By definition

$$\begin{aligned} w(K)^{-1}(\widetilde{\mu_0^1})(B') &= \widetilde{\mu_0^1}\left(\left\{K \in \widetilde{\mathcal{K}_0^1} \mid w(K) \in B'\right\}\right) \\ &= \lim_{n \to \infty} \sum_{\substack{K_{m(n,k)} \in w(K)^{-1}(B') \\ 0 < k \leq 2^{n+1}}} \frac{1}{2^{n+1}} = \lim_{n \to \infty} \sum_{2m(n,k) \in B'} \frac{1}{2^{n+1}} \end{aligned}$$

$$= \lim_{n\to\infty} \sum_{2m(n,k)<x} \frac{1}{2^{n+1}} = \lim_{n\to\infty} \sum_{k=1}^{2^{n-1}x} \frac{1}{2^{n+1}} = \frac{x}{4}$$

showing that $w(K)^{-1}(\widetilde{\mu_0^1})$ is the uniform distribution of the interval $(0,4]$. □

The Gaussian measure γ of the n^2-dimensional matrix space $\mathbb{R}^{n\times n}$ defined by the density function $G(X)$

$$G(X)\mathrm{d}\lambda^{n^2} := \frac{1}{\left(\sqrt{2\pi}\right)^{n^2}} \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}(X^TX)}\mathrm{d}\lambda^{n^2},$$

where $\mathrm{d}\lambda^{n^2}$ is the n^2-dimensional Lebesgue measure. The Haar measure ν_n of $\mathbb{R}^{n\times n}$ is the pushforward measure of the Gaussian measure by the mapping M defined by the Gram-Schmidt process (see [12]). In fact, if $GL(n,\mathbb{R})$ is the group of nonsingular matrices then M is a mapping from $GL(n,\mathbb{R})$ to $O(n)$ with the following properties:

1. surjective;
2. if B is a Borel set then we have $M(QB) = QM(B)$ for every $Q \in O(n)$;
3. $M^{-1}(QB) = QM^{-1}(B)$.

It can be proved that for a Borel set B of $O(n)$ we have

$$\nu_n(B) = \gamma\left(M^{-1}(B)\right).$$

Furthermore ν_n is a probability measure because a matrix is invertible (with respect to the Gaussian measure) with probability 1 and thus

$$\nu_n(O(n)) = \gamma(GL(n,\mathbb{R})) = 1.$$

Since Haar measure by definition invariant under orthogonal transformations it is the unique “uniform” (geometric volume) distribution on $O(n)$ and thus on $SO(n)$, too.

We now state the following:

Theorem 1 Let define the measure $\widetilde{\nu_0^1}$ by density function $\mathrm{d}\widetilde{\nu_0^1} = \frac{4}{(w+4)^2}\mathrm{d}\widetilde{\mu_0^1}$. Then

$$\alpha_0(K)^{-1}\left(\Phi_{2a}^1\left(\widetilde{\nu_0^1}\otimes\nu_n\right)\right)$$

is a probability measure with uniform distribution on $[\frac{1}{2},1)$.

Proof We are stating that the pushforward measure $\alpha_0(K)^{-1}\left(\Phi_{2a}^1\left(\left(\widetilde{\nu_0^1}\otimes\nu_n\right)\right)\right)$ has uniform distribution on $[\frac{1}{2},1)$ if and only if the pushforward measure $w(K)^{-1}\left(\widetilde{\mu_0^1}\right)$ has uniform distribution on $(0,4]$. To prove this consider a Borel set B of $[\frac{1}{2},1)$ and its image B' under the bijective transformation

$$\tau : t \mapsto \tau(t) := \frac{4}{t} - 4.$$

Of course B' is a Borel set of the interval $(0,4]$ which is the image of $[\frac{1}{2},1)$ with respect to τ. We now have that

$$\begin{aligned}
&\int_B \mathrm{d}\alpha_0(K)^{-1}\left(\Phi_{2a}^1\left(\widetilde{\nu_0^1}\otimes\nu_n\right)\right)\\
&= \alpha_0(K)^{-1}\left(\Phi_{2a}^1\left(\widetilde{\nu_0^1}\otimes\nu_n\right)\right)(B) = \Phi_{2a}^1\left(\widetilde{\nu_0^1}\otimes\nu_n\right)\left(\alpha_0(K)^{-1}(B)\right)\\
&= \widetilde{\nu_0^1}\left(\left(\Phi_{2a}^1\right)_1^{-1}\left(\left(\alpha_0(K)^{-1}(B)\right)\right)\right)\nu_n\left(\left(\Phi_{2a}^1\right)_2^{-1}\left(\alpha_0(K)^{-1}(B)\right)\right),
\end{aligned}$$

where $\left(\Phi^1_{2a}\right)^{-1}_1$ and $\left(\Phi^1_{2a}\right)^{-1}_2$ means the components of the set-valued inverse of the function Φ^1_{2a}, respectively. Since $\left(\Phi^1_{2a}\right)^{-1}_2\left(\alpha_0(K)^{-1}(B)\right)$ is the group $O(n)$ we have that

$$\int_B \mathrm{d}\alpha_0(K)^{-1}\left(\Phi^1_{2a}\left(\widetilde{\nu^1_0}\otimes\nu_n\right)\right)=\widetilde{\nu^1_0}\left(\left(\Phi^1_{2a}\right)^{-1}_1\left(\alpha_0(K)^{-1}(B)\right)\right)=\int_{\left(\Phi^1_{2a}\right)^{-1}_1\left(\alpha_0^{-1}(B)\right)}\mathrm{d}\widetilde{\nu^1_0}.$$

On the other hand

$$\begin{aligned}\left(\Phi^1_{2a}\right)^{-1}_1\left(\alpha_0^{-1}(B)\right)&=\left\{\widetilde{K}\in\widetilde{\mathcal{K}^1_0}\mid\alpha_0(\widetilde{K})=\frac{4}{w(\widetilde{K})+4}\in B\right\}\\&=\left\{\widetilde{K}\in\widetilde{\mathcal{K}^1_0}\mid w(\widetilde{K})\in B'=\frac{4}{B}-4\right\}\end{aligned}$$

implying that

$$\int_{\left(\Phi^1_{2a}\right)^{-1}_1\left(\alpha_0^{-1}(B)\right)}\mathrm{d}\widetilde{\nu^1_0}=\int_{\left\{\widetilde{K}\in\widetilde{\mathcal{K}^1_0}\mid w(\widetilde{K})\in B'\right\}}\frac{4}{(w+4)^2}\mathrm{d}\widetilde{\mu^1_0},$$

and it is equal to

$$\int_{\tau\in B'}\frac{4}{(4+\tau)^2}\mathrm{d}\tau=\int_{t\in B}\mathrm{d}t,$$

if and only if $w(K)^{-1}\left(\widetilde{\mu^1_0}\right)$ has uniform distribution on $(0,4]$ as we stated.

Since Lemma 2 says that $w(K)^{-1}\left(\widetilde{\mu^1_0}\right)$ has uniform distribution on the interval $[0,4]$ we also proved the theorem. □

Let denote by ν^1_0 the measure $\Phi^1_{2a}\left(\widetilde{\nu^1_0}\otimes\nu_n\right)$.

5 Measure on $\mathcal{K}_0$ with Normal Pushforward

Finally we can identify $\mathcal{K}_0$ with $\mathcal{K}^1_0\times[0,\infty)$. To this end let Φ_4 be the mapping

$$\Phi_4:(K,\alpha)\mapsto\alpha K.$$

Lemma 3 From the image $K'=\Phi_4(K)$ we can determine uniquely the body K and the constant α.

Proof $K'=\alpha K$ implies that $\alpha_0(K)=\alpha_0(K')=\frac{d(K')}{w(K')+d(K')}$ and thus $\alpha_0(K)$ is uniquely determined. We also know the value of

$$\alpha':=\delta^h(\alpha K,B_E).$$

We consider two cases. In the first case we assume that $\alpha\geq\alpha_0$ and hence by Lemma 1 we get that

$$\alpha'=2\alpha-1\text{ or }\alpha=\frac{\alpha'+1}{2},$$

and in the second one we assume $0\leq\alpha\leq\alpha_0$ then we have

$$\alpha'=2\alpha+1-2\frac{\alpha}{\alpha_0}\text{ or }\alpha=\frac{\alpha'-1}{2-\frac{2}{\alpha_0}}=\frac{\alpha_0(\alpha'-1)}{2(\alpha_0-1)}.$$

From these equalities we get that the first case implies

$$\alpha_0\leq\frac{\alpha'+1}{2}\text{ so }\alpha'\geq2\alpha_0-1,$$

and in the second one we have

$$\alpha_0 \geq \frac{\alpha_0(\alpha'-1)}{2(\alpha_0-1)} \geq 0.$$

Hence we have

$$2\alpha_0 - 1 \geq \alpha' \geq 0.$$

So first we determine α' and the value

$$2\alpha_0 - 1 = \frac{2d(K)}{w(K)+d(K)} - 1 = \frac{d(K)-w(K)}{d(K)+w(K)}.$$

Then using the above equalities we can calculate α which is uniquely determined. Now K is equal to $\frac{1}{\alpha}K'$. □

Denote by $\Phi_4^{-1}(K') := \left(\left(\Phi_4^{-1}\right)_1(K'), \left(\Phi_4^{-1}\right)_2(K')\right)$ the pair (K,α) determined by the method of Lemma 3. If we have a σ-finite measure ν_0^1 on $\mathcal{K}_0^1$ then we also have a σ-finite measure ν_0 on $\mathcal{K}_0$ by the definition $\nu_0 = \Phi_4(\nu_0^1 \otimes \nu)$, where ν is a σ-finite measure on $(0,\infty)$.

Let define now the set function $p(\mathcal{A})$ as follows. If $\mathcal{A} \subset \mathcal{K}_0$ ν_0 is a measurable set let be

$$p(\mathcal{A}) := \frac{1}{\sqrt{2\pi\sigma^2}} \int_{K'\in\mathcal{A}} \mathrm{e}^{-\frac{\left(\delta^h\left(B_E, \frac{\alpha_0(K')}{\Phi_4^{-1}(K')_2}K'\right)\right)^2}{2\sigma^2}} \mathrm{d}\nu_0.$$

The following theorem is our main result.

Theorem 2 If ν_0^1 is such a probability measure on $\mathcal{K}_0^1$ for which $\alpha_0(K)^{-1}(\nu_0^1)$ has uniform distribution, $\nu_0 = \Phi_4(\nu_0^1 \otimes \nu)$ where ν is a probability measure on $(0,\infty)$ and Φ is the probability function of the standard normal distribution then

$$P(\mathcal{A}) := \frac{4p(\mathcal{A})}{\left(\Phi\left(\frac{1}{\sigma}\right) - \Phi(0)\right)} = \frac{4}{\left(\Phi\left(\frac{1}{\sigma}\right) - \Phi(0)\right)\sqrt{2\pi\sigma^2}} \int_{K'\in\mathcal{A}} \mathrm{e}^{-\frac{\left(\delta^h\left(B_E, \frac{\alpha_0(K')}{\Phi_4^{-1}(K')_2}K'\right)\right)^2}{2\sigma^2}} \mathrm{d}\nu_0$$

is a probability measure on $\mathcal{K}_0$. Moreover $\alpha_0(K)^{-1}(P)$ has truncated normal distribution on the interval $[\frac{1}{2}, 1)$, (with mean $\frac{1}{2}$ and variance $\left(\frac{\sigma}{2}\right)^2$), so

$$\alpha_0(K)^{-1}(P)\left(\left\{\frac{1}{2} \leq t \leq c\right\}\right) = P\left(\{\mathcal{K} \in \mathcal{K}_0 \mid \alpha_0(K) \leq c\}\right) = \frac{\Phi\left(\frac{c-\frac{1}{2}}{\frac{\sigma}{2}}\right) - \Phi(0)}{\Phi\left(\frac{1}{\sigma}\right) - \Phi(0)}.$$

Proof

$$p(\mathcal{A}) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{K\in\left(\Phi_4^{-1}\right)_1(\mathcal{A})} \int_{\alpha\in\left(\Phi_4^{-1}\right)_2(\mathcal{A})} \mathrm{e}^{-\frac{\left(\delta^h\left(B_E, \frac{\alpha_0(K')}{\alpha}\alpha K\right)\right)^2}{2\sigma^2}} \mathrm{d}\nu \mathrm{d}\nu_0^1$$

however $\alpha_0(K') = \alpha_0(K)$ so it is equal to

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi\sigma^2}} \int_{K\in\left(\Phi_4^{-1}\right)_1(\mathcal{A})} \left(\int_{\alpha\in\left(\Phi_4^{-1}\right)_2(\mathcal{A})} \mathrm{e}^{-\frac{\alpha_0(K)'^2}{2\sigma^2}} \mathrm{d}\nu\right) \mathrm{d}\nu_0^1 \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{K\in\left(\Phi_4^{-1}\right)_1(\mathcal{A})} \left(\int_{\substack{\alpha K\in\mathcal{A},\\ \alpha\geq\alpha_0(K)}} \mathrm{e}^{-\frac{(2\alpha_0(K)-1)^2}{2\sigma^2}} \mathrm{d}\nu\right. \\
&\quad \left. + \int_{\substack{\alpha K\in\mathcal{A}\\ 0\leq\alpha\leq\alpha_0(K)}} \mathrm{e}^{-\frac{\left(2\alpha_0(K)+1-2\frac{\alpha_0(K)}{\alpha_0(K)}\right)^2}{2\sigma^2}} \mathrm{d}\nu\right) \mathrm{d}\nu_0^1
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{K\in\left(\Phi_4^{-1}\right)(\mathcal{A})_1} \left(\int_{\alpha\in\left(\Phi_4^{-1}\right)_2(\mathcal{A})} \mathrm{e}^{-\frac{(2\alpha_0(K)-1)^2}{2\sigma^2}} \mathrm{d}\nu \right) \mathrm{d}\nu_0^1$$
$$= \frac{\nu\left(\alpha\in\left(\Phi_4^{-1}\right)_2(\mathcal{A})\right)}{\sqrt{2\pi\sigma^2}} \int_{K\in\left(\Phi_4^{-1}\right)(\mathcal{A})_1} \mathrm{e}^{-\frac{(2\alpha_0(K)-1)^2}{2\sigma^2}} \mathrm{d}\nu_0^1.$$

For $\mathcal{A}=\mathcal{K}_0$ we have that it is equal to

$$= \frac{\nu((0,\infty))}{\sqrt{2\pi\sigma^2}} \int_{\frac{1}{2}}^{1} \mathrm{e}^{-\frac{1}{2}\left(\frac{t-\frac{1}{2}}{\frac{\sigma}{2}}\right)^2} \mathrm{d}\left(\alpha_0(K)^{-1}(\nu_0^1)(t)\right).$$

Since ν is a probability measure on $(0,\infty)$ and $\alpha_0(K)^{-1}(\nu_0^1)$ has uniform distribution on $[\frac{1}{2},1)$ so we have that

$$p(\mathcal{K}_0) = \frac{1}{2\sqrt{2\pi}\frac{\sigma}{2}}\frac{1}{2}\left(\int_{-\infty}^{1} \mathrm{e}^{-\frac{1}{2}\left(\frac{t-\frac{1}{2}}{\frac{\sigma}{2}}\right)^2}\mathrm{d}t - \int_{-\infty}^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}\left(\frac{t-\frac{1}{2}}{\frac{\sigma}{2}}\right)^2}\mathrm{d}t\right) = \frac{\Phi\left(\frac{1}{\sigma}\right)-\Phi(0)}{4},$$

where the function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{x} \mathrm{e}^{\left(-\frac{u^2}{2}\right)} \mathrm{d}u$$

is the standard normal distribution function.

Analogously, for the set $\mathcal{K}_0(c) := \{K'\in\mathcal{K}_0 \mid \alpha_0(K') = \alpha_0(K) \le c\}$ we have

$$p(\mathcal{K}_0(c)) = \frac{\nu((0,\infty))}{\sqrt{2\pi\sigma^2}} \int_{\frac{1}{2}}^{c} \mathrm{e}^{-\frac{1}{2}\left(\frac{t-\frac{1}{2}}{\frac{\sigma}{2}}\right)^2} \mathrm{d}\left(\alpha_0(K)^{-1}(\nu_0^1)(t)\right) = \frac{\Phi\left(\frac{c-\frac{1}{2}}{\frac{\sigma}{2}}\right)-\Phi(0)}{4},$$

thus the measure

$$P(\mathcal{A}) := \frac{4}{\Phi\left(\frac{1}{\sigma}\right)-\Phi(0)} p(\mathcal{A})$$

is such a probability measure on $\mathcal{K}_0$ which pushforward by the function $\alpha_0(K)$ has normal distribution. □

Corollary 1 From Theorem 2 follows the existence of a measure with similar properties on the space $\mathcal{K}$ of convex bodies. Let denote by $m(K) := \frac{1}{2}(K+(-K))$ where the addition means the Minkowski sum of convex bodies. The mapping

$$m : \mathcal{K} \longrightarrow \mathcal{K}_0$$

is a continuous function on $\mathcal{K}$ and thus it defines a pullback measure μ on $\mathcal{K}$ by the rule

$$\mu(H) = P(m(H)) \text{ where } H = m^{-1}(H') \text{ for a Borel set } H' \in \mathcal{K}_0.$$

Observe that m has the following properties:

1. surjective
2. for any set $S\subset\mathcal{K}$ and a vector $t\in\mathbb{R}^n$ we have $m(S+t) = m(S)$
3. for any $K\in\mathcal{K}$ holds that $d(K) = d(m(K))$, $w(K) = w(m(K)$ implying that $\alpha_0(K) = \alpha_0(m(K))$.

This implies that the function α_0 is well-defined on $\mathcal{K}$ and for any Borel set $B\in\left[\frac{1}{2},1\right)$

$$\mu\left(\alpha_0^{-1}(B)\right) = P(m\left(\alpha_0^{-1}(B)\right)) = P(\alpha_0^{-1}|_{\mathcal{K}_0}(B)),$$

showing that the pushforward of the measure μ has truncated normal distribution on the interval $\left[\frac{1}{2},1\right)$.

6 Geometric Measure with Normal Pushforward

In this section we reformulate the preceding construction such a way that it will be useful to stochastic-geometric examination. Note that our constructed measure is a geometric measure in the sense that invariant under rigid motions. The basic questions on such a measure are: "Do the convex polytopes have measure zero, do the smooth bodies have positive measure, or does a neighborhood always have positive measure?" The improved construction gives positive answer to these questions. There is no polytope among our bodies $K_{m(n,k)}$ thus already for this system holds that the set of convex polytopes has measure zero. More precisely we have:

Lemma 4 Denote by $\mathcal{P}_0$ the set of O-symmetric convex polytopes. Then we have

$$P\left(\mathcal{P}_0\right)=0.$$

Proof Introduce the sets $\mathcal{P}_0^1$ and $\widetilde{\mathcal{P}_0^1}$ as we did in the case of the O-symmetric bodies $\mathcal{K}_0$. By definition

$$\widetilde{\mu_0^1}\left(\widetilde{\mathcal{K}_0^1}\setminus\widetilde{\mathcal{P}_0^1}\right)=1,$$

showing that

$$\widetilde{\mu_0^1}\left(\widetilde{\mathcal{P}_0^1}\right)=0.$$

Thus

$$\widetilde{\nu_0^1}\left(\widetilde{\mathcal{P}_0^1}\right)=\int_{\widetilde{\mathcal{P}_0^1}}\mathrm{d}\widetilde{\nu_0^1}=\int_{\widetilde{\mathcal{P}_0^1}}\frac{4}{(w+4)^2}\mathrm{d}\widetilde{\mu_0^1}=0,$$

and so

$$\nu_0^1\left(\mathcal{P}_0^1\right)=\Phi_{2a}^1\left(\widetilde{\mathcal{P}_0^1}\otimes\nu_n\right)\left(\widetilde{\mathcal{P}_0^1},SO(n)\right)=0.$$

Finally, we have

$$\nu_0\left(\mathcal{P}_0\right)=\Phi_4\left(\nu_0^1\otimes\nu\right)\left(\mathcal{P}_0^1,[0,\infty)\right)=0,$$

hence $p\left(\mathcal{P}_0\right)=P\left(\mathcal{P}_0\right)=0$, as we stated. □

Changing in the constructions the bodies $K_{m(n,k)}$ to smooth ones the calculation of Lemma 4 is also valid for the set of polytopes and gives the value 1 for the set of smooth O-symmetric bodies. On the other hand the measure of a neighborhood will be positive if and only if the system of the bodies for which the measure concentrated will be dense in $\widetilde{\mathcal{K}_0^1}$. In the following subsection we give such a system.

The new system of bodies

We define the new system in two steps.

- Change the the body $K_{m(n,k)}$ to a smooth body $K_{m(n,k)}^l$ defined by the convex hull of the ball around the origin with radius $m(n,k)$ and the two balls of radius $\varepsilon_l=\frac{1}{2^l}m(n,k)$ with centers $\pm(2-\varepsilon_l)e_n$.

- Substitute each elements of the system of the bodies $K_{m(n,k)}^l$ with a new countable system of bodies. Consider a dense, countable and centrally symmetric point system $\{P_1,-P_1,P_2,-P_2\cdots\}$ in the closed ball of radius 2 with the additional property that there is no two distances between the pairs of points which are equals to each other. (Such a point system is exist.) We assume that the first point P_1 is the endpoint of $2e_n$ and denote by S_i a similarity of E^n which sends P_1 into P_i and the ball of radius 2 at the origin into the ball of radius OP_i centered at the origin O, too. Consider the countable set of bodies

$$S\left(K^l_{m(n,k)}\right) := \left\{S_i\left(K^l_{m(n,k)}\right)\ ,\ i = 1, 2, \cdots\right\},$$

and define the elements of the new set $\mathcal{H}^l_{m(n,k)}$ by induction as follows:

– The first element is itself the set $K^l_{m(n,k)} := S_1\left(K^l_{m(n,k)}\right)$.

– In the second step consider such pairs from the list $S\left(K^l_{m(n,k)}\right)$ one of which has diameter 4 and construct their convex hulls. Add these bodies also to the set $\mathcal{H}^l_{m(n,k)}$.

– In the third step construct the convex hull of the triplet from which one has diameter 4. Add these bodies to $\mathcal{H}^l_{m(n,k)}$, too.

– ... and so on.

Hence we have a countable system of centrally symmetric convex bodies with diameter 4. The getting set $\mathcal{H}^l_{m(n,k)}$ has a partition into countable subsets. So we have:

$$\begin{aligned}\mathcal{H}^l_{m(n,k)} = K^l_{m(n,k)} \dot\cup \left\{\operatorname{conv}\left\{S_i\left(K^l_{m(n,k)}\right), S_j\left(K^l_{m(n,k)}\right)\right\} \text{ for } i, j\right\} \\ \dot\cup \left\{\operatorname{conv}\left\{S_i\left(K^l_{m(n,k)}\right), S_j\left(K^l_{m(n,k)}\right), S_k\left(K^l_{m(n,k)}\right)\right\} \text{ for } i, j, k\right\} \dot\cup \cdots,\end{aligned}$$

where all of the elements are smooth bodies having diameter 4. The following technical lemma is important.

Lemma 5 The bodies of

$$\mathfrak{H} = \left\{\mathcal{H}^l_{m(n,k)} \quad m, n, k, l \in \mathbb{N}\right\}$$

are pairwise non-congruent. For an arbitrary polytope $Q \in P_0$ and for a given number ε we can choose an element $R \in \mathfrak{H}$ for which hold that $\delta^h(Q, R) < \varepsilon$.

Proof The first statement follows from the fact that each of the bodies of $\mathfrak{H}$ contains a maximal flat part which is the convex hull of the points P_i. By the choice of the point system $\{P_i\}$ these parts are pairwise non-congruent. The proof of the second statement based on the fact that for large l, $m(n,k)$ with a small k the bodies $S\left(K^l_{m(n,k)}\right)$ essentially are O-symmetric segments and thus their convex hull is close to a polytope in Hausdorff distance. We here omit the straightforward argument. □

The last subsection of the present section contains the definition of the new measure and the corresponding Theorem 3.

Definition of the measure

We distribute the part of the measure $\widetilde{\mu^1_0}$ which originally concentrated on $K^l_{m(n,k)}$ among the elements of $\mathcal{H}^l_{m(n,k)}$.

For a fixed $r \in \mathbb{N}$ consider a sequence (α^r_i) of positive numbers which holds the property $\sum\limits_{i=1}^{\infty} \alpha^r_i = 1$. Let $L^r_i(l)$ be the i-th element of the r-th subset of the above partition of $\mathcal{H}^l_{m(n,k)}$. Thus it is a convex hull of exactly r copies of bodies from $S\left(K^l_{m(n,k)}\right)$. We give it the weight $\frac{\alpha^r_i}{2^r}$.

Definition 3 Choose a sequence of positive numbers β_l with again the property $\sum\limits_{l=1}^{\infty}\beta_l = 1$. Define a measure $\widetilde{\widetilde{\mu_0^1}}$ by the equality:

$$\widetilde{\widetilde{\mu_0^1}} := \lim_{n\to\infty}\sum_{k=1}^{2^{n+1}}\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}\sum_{i=1}^{\infty}\frac{\beta_l\alpha_i^r}{2^{n+1+r}}\delta_{L_i^r(l)}.$$

We state the following:

Theorem 3 On the space of norms there is a probability measure P with the following properties:

- The neighborhoods has positive measure.
- The set of polytopes has zero measure.
- The set of smooth bodies has measure 1.
- The pushforward $\alpha_0(K)^{-1}(P)$ of P has truncated normal distribution on the interval $[\frac{1}{2}, 1)$.

Proof Consider the measure $\widetilde{\widetilde{\mu_0^1}}$ without the measure $\widetilde{\mu_0^1}$ and expand it for $\mathcal{K}_0$ on the way as we did it with $\widetilde{\mu_0^1}$. The final measure P by Lemma 4 on the set of polytopes has zero value. By the remark before the definition of the new system we know that the set of smooth bodies of $\mathcal{K}_0$ has measure 1 since the elements of $\mathfrak{H}$ are smooth. The required property on the approximation of polytopes follows from Lemma 5 since for each polytope we can find a body from $\mathcal{H}^l_{m(n,k)}$ close to them. The definition of $\widetilde{\widetilde{\mu_0^1}}$ guarantees that the distribution of $\widetilde{\widetilde{\mu_0^1}}$ and $\widetilde{\mu_0^1}$ are agree proving our last statement. $\square$

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