Projection pencils of quadrics and Ivory's theorem

Á.G. Horváth

Abstract. Using selfadjoint regular endomorphisms, the authors of [7] defined, for an indefinite inner product, a variant of the notion of confocality for the Euclidean space. Our aim is to give a definition that is a common generalization of the usual confocality, and the variant in [7]. We use this definition to prove a more general form of Ivory's theorem.

Mathematics Subject Classification (2010). 51M10; 15A21.

Keywords. Indefinite inner product, Ivory's theorem, projection pencils of quadrics.

1. Introduction

Two hundred years ago J.Ivory published a geometric theorem ([5]) whose importance can be measured by its influence on physic. Ivory's theorem states that the level surfaces of the gravitational potential in the exterior domain of an elliptic layer (which is an infinitely thin layer between two similar concentric ellipsoids) are confocal ellipsoids. In mechanics, the counterpart of this theorem is the so-called Newton's theorem (generalizing the theorem on the attraction of spheres to elliptic layers), which states that the gravitational potential inside an elliptic layer is constant.

Some important moments of the long history of this theorem can be found in [7]. We would like to mention here two papers were omitted there: the first one is a characterization of confocal conics in a pseudo-Euclidean space, due to G.Birkhoff and R.Morris [2], and the second one, written by V.V.Kozlov, giving analogues of the mechanical variations of Newton's and Ivory's theorems in spaces of constant curvature of dimension 3 (cf. [6]).

In this paper we deal with the property of confocality in spaces with indefinite inner products. The origin of this investigation is the concept of confocality introduced in [7], where the authors proved Ivory's theorem with respect to this concept. Nevertheless, this definition is a variant, and not a generalization of the original confocality of Euclidean conics. We remark also that H. Stachel, one of the authors of [7], proved Ivory's theorem for standard confocal conics in the physical Minkowski plane, without using projective tools. The aim of this paper is to introduce a definition of confocality which includes both the standard version and the one in [7] as a special cases, and to prove Ivory's theorem using this definition. In our investigation, we focus on singular selfadjoint endomorphisms, as they were not examined in [7].

Our paper basically follows the build-up of [7], uses its terminology, statements and proofs. We prove only the results which are not immediate consequences of the previous ones.

1.1. Notation and Terminology

In the paper, we present our results in a form that assumes that the reader is familiar with the fundamental notions, linear algebra and real projective geometry. In particular, we do not define the concepts of a real vector space, a direct sum of subspaces, and so on.

In our considerations, we use the following notations:

- V, $V^\star$, $L(V)$: A vector space, its dual space and the space of linear transformations of V, respectively.
- $\dim(V)$: The dimension of the vector space V. In this paper it is equal to $(n+1)$.
- $P(V)$: The projective space defined on V. In our paper $\dim(P(V)) = n$.
- $Q(V)$: The vector space of all the quadratic forms of V.
- $\operatorname{Ker} L$, $\operatorname{Im} L$: The kernel and the image space of the linear transformation L, respectively.
- $\mathbb{C}$, $\mathbb{R}$: The field of complex and real numbers, respectively.
- $\langle\cdot,\cdot\rangle$: The indefinite inner (scalar) product defined on the space.
- $\Phi(x,y)$,Φ, Ω: A symmetric bilinear function, its zero set defining the corresponding quadric, and the absolute quadric associated to a fixed projection, respectively.
- l, g, id, L, G, E: Linear mappings $l, g, id : V \longrightarrow V$ (and also their projective classes in $P(L(V))$), and their $(n+1)\times(n+1)$ matrices with respect to a homogeneous coordinate system, respectively.

2. Quadrics in a finite dimensional projective space

There is a well-known theory of projective quadrics in an n-dimensional projective space $P(V)$ over a commutative field K of characteristic different from 2 (cf. [1]). By definition, a quadratic form $q : V\times V \longrightarrow K$ is associated with a symmetric, bilinear form $\Phi : V \times V \longrightarrow K$ called the polar form of q that satisfies the equality:

$$\Phi(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y))$$

for all $x, y \in V$. Let $P(V)$ denote the n-dimensional projective space associated to the $(n+1)$-dimensional vector space V and let $p : V \setminus 0 \longrightarrow P(V)$ denote the canonical projection. The *isotropic cone* of q defined by $p(q^{-1}(0) \setminus 0)$ does not change when we replace q by kq for some $k \in K^*$. This means that such a projection is actually associated to a point of the projective space $P(Q(V))$. The *projective quadrics* (in algebraic sense) are the elements of $P(Q(V))$. A quadric is a *conic* when $n = 2$. It is proper if it has a non-degenerate equation; otherwise it is called degenerate. The classification of quadrics in real and complex cases are known, we have to determine the orbits of $P(Q(V))$ under the action of the isometry group $GP(V)$. If the field K is $\mathbb{C}$, then we have exactly $(n + 1)$ orbits, classified by the rank k, where $1 \leq k \leq (n + 1)$, and in the real case the orbits are classified by pairs (r, s) such that $1 \leq s \leq r \leq n + 1$. In particular, there exist $\frac{1}{2}n(n + 1) + 1$ types of proper quadrics. (See 14.1.5.1. Theorem in [1].) In present paper we consider only real vector spaces.

From geometric point of view, we can represent a quadric as the zero set of a quadratic form (or the zero set of its symmetric bilinear form). Fixing a regular symmetric bilinear form as an indefinite inner product $\langle \cdot, \cdot \rangle$ any quadric can be regarded as the zero set of a symmetric bilinear function $\langle x, l(y) \rangle$, where l is a selfadjoint transformation with respect to the product $\langle \cdot, \cdot \rangle$. Every such transformation represents a quadric, but the elements of the projective class of l belong to the same quadric. It is also possible that the same quadric is associated to two non-equivalent selfadjoint transformations. Every linear transformation l of V determines an endomorphism of the corresponding projective space $P(V)$. We do not use another notation for the endomorphisms and the linear transformations corresponding to them. In a numerical calculation, we can use matrix representations of the transformations in suitable bases. There is a fundamental result on the characterization of a fixed symmetric bilinear form $\langle \cdot, \cdot \rangle$ and a selfadjoint linear mapping l by simultaneous normal form - meaning the choice of a basis with respect to which both of them have simple coordinate elements (cf. Th. 5.3 in [4] or [3]). The matrix representations of linear transformations are denoted by capital letters.

Now we can introduce the *dual of a quadric*, which is a quadric of the dual space. Two points x and y of $P(V)$ are *conjugate* with respect to the symmetric bilinear form Φ if $\Phi(x, y) = 0$. The set of points conjugate to x is a subspace of $P(V)$ of dimension at least $(n - 1)$. Since the $(n - 1)$-dimensional subspaces in $P(V)$ are the zero sets of the linear forms $a^* \in P(V^*)$, the set of conjugate points is a point x^* of $P(V^*)$. The bilinear form

$$\widehat{\Phi}(x^*, y^*) := \Phi(x, y)$$

is called the dual of the form Φ. The corresponding quadric of $P(V^*)$ is the dual of the quadric defined by Φ. If there is a correspondence (distinct for the duality) between the dual form and a bilinear symmetric quadratic form of $P(V)$ then we say that we gave *a representation* of the dual form in $P(V)$. We denote by $\widetilde{\Phi}$ this representation of $\widehat{\Phi}$. To see this formally assume that the bilinear form is $\Phi(x,y) := x^TTy$ for a symmetric matrix T, and its dual quadric is the zero set of $\widehat{\Phi}(x^*,y^*)$. To determine $\widehat{\Phi}(x^*,y^*)$ we must describe the duality map. Let $(\cdot)^* : V \longrightarrow V^*$ be defined by

$$x \mapsto x^* = \Phi_x(\cdot) := \Phi(x,\cdot).$$

For a linear transformation L of V, let L^T denote the transposed of L with respect to the fixed product. Then

$$\Phi(L(x),L(y)) = (L(x))^TTL(y) = x^T(L^TTL)y$$

and

$$L(x)^* = \Phi_{L(x)}(\cdot) = \Phi(L(x),\cdot).$$

If $u \in \operatorname{Im} T$, then there is an $x \in V$ for which $T(x) = u$, and hence $T^{-1}u := x + \operatorname{Ker} T$ is well-defined. Thus for $u, v \in \operatorname{Im} T$, we can define a representation $\widetilde{\Phi}$ of the dual quadric $\widehat{\Phi}$. More specifically,

$$\widehat{\Phi}(x^*,y^*) = \widehat{\Phi}((T^{-1}(u))^*,(T^{-1}(v))^*) =$$
$$= \widehat{\Phi}(\Phi_{T^{-1}u}(\cdot),\Phi_{T^{-1}v}(\cdot)) = u^T((T^{-1})^TT(T^{-1}))v,$$

thus the definition

$$\widetilde{\Phi}(u,v) := u^T((T^{-1})^TT(T^{-1}))v$$

yields a representation of the dual form. The set valued mapping T^{-1} is an isomorphism on $\operatorname{Im} T \leq V$ to the factor space $V/\operatorname{Ker} T$, thus the mapping $T(T^{-1})$ is the identity on $\operatorname{Im} T$. Consequently T^{-1} can be considered as a symmetric linear transformation on $\operatorname{Im} T$ to a subspace of V, we denote it T^{-1}, too. Using the symmetry of T^{-1} the representation of the dual quadratic form can be defined on $\operatorname{Im} T$ by

$$\widetilde{\Phi}(u,v) := u^T(T^{-1})^Tv = u^TT^{-1}v.$$

We extract now the linear transformation $T^{-1} : \operatorname{Im} T \longrightarrow V$ to V by the zero map of the complementary subspace of $\operatorname{Im} T$ giving a singular transformation which defines the singular symmetric bilinear form $\widetilde{\Phi}(u,v)$ of V.

3. The confocality of conics

In this section we assume that $n = 2$. To define confocality in the general case, we shortly recall the description of Euclidean confocal conics. Working in Euclidean homogeneous coordinates, finite points and asymptotic directions (``points at infinity´´) are and given by column vectors. A row vector specify as a line with normal the mentioned vector. The line at infinity contains the infinite points. Change-of-basis transformations are represented by matrices. These transformations act on a point by left multiplication with the corresponding matrix, and on planes by right multiplication with the inverse matrix. In this way, point-line products are preserved. Euclidean transformations take the form

$$T = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

where A is a rotation matrix and b a translation vector. A becomes a rescaled rotation for a scaled Euclidean or similarity transformations, and an arbitrary nonsingular 3×3 matrix for an affine one.

Consider first the *absolute quadric* corresponds to the symmetric rank 2 matrix

$$\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The coordinates of a point x of the absolute holds the equality $x_1^2 + x_2^2 = 0$ thus it has only one projective point $(0,0,1)^T$. The representation of the dual form is

$$\widetilde{\Phi}(u,v) = (u_1, u_2, 0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix},$$

where $u = (u_1, u_2, 0)^T$ and $v = (v_1, v_2, 0)^T$ are elements of $\operatorname{Im} \Omega$. The representation of the dual quadric is also the absolute and hence it has only one element, which is associated to the projective point $(0,0,1)^T$ (note that the 3×3 identity matrix as a selfadjoint transformation identical with its inverse thus it defines a self-dual conic which is the empty set). It is easy to see that the euclidean confocal conics with foci $(0, \pm c)$ can be written in a suitable homogeneous coordinate system in the form:

$$0 = (x_1, x_2, x_3) \begin{pmatrix} \frac{1}{c^2+\lambda} & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Since this quadratic form is defined by a regular matrix, the representation of its dual is:

$$0 = (x_1, x_2, x_3) \begin{pmatrix} c^2+\lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

showing that the representations of the dual forms of confocal conics form a linear subset of all forms, containing the representation of the dual of the absolute. More precisely,

$$\alpha \begin{pmatrix} c^2+\lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} c^2+\lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} (\alpha+\beta)c^2+(\alpha\lambda_1+\beta\lambda_2) & 0 & 0 \\ 0 & (\alpha\lambda_1+\beta\lambda_2) & 0 \\ 0 & 0 & -(\alpha+\beta) \end{pmatrix} =$$

$$= \begin{cases} \begin{pmatrix} c^2+\lambda_3 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \text{if} \quad (\alpha+\beta)\neq 0 \\ \begin{pmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{if} \quad (\alpha+\beta)=0. \end{cases}$$

On the other hand, this linear hull does not contain the 3×3 identity transformation. This implies that the definition of [7] is not a generalization of the usual confocality of conics. As it can be seen easily, for a non-zero c the connection between the distances $2c_\alpha$ and $2c$ of the foci of the conics Φ_α and Φ for which $\Phi_\alpha^{-1}=\alpha\Phi+\beta I$, where I is the 3×3 identity matrix, is

$$c_\alpha^2=\frac{\alpha}{\alpha-\beta}c^2$$

showing that with respect to the pencil of the conics in [7], the distances of the foci are not constant.

Our second example is the confocal family of conics with two axes of symmetry, belonging to the pseudo-Euclidean (Minkowski) plane. In [2] these conics were called *relativistic confocal conics* in space time, and it was shown that they are geometrically tangent to the null lines (isotropic lines) through the foci. In [8], this configuration was called type B, and was illustrated in two nice figures. In our setting, we have the following:

The regular bilinear function in a Cartesian homogeneous coordinate system of the embedding Euclidean plane is:

$$\langle x,y\rangle=(x_1,x_2,x_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

The absolute Ω can be considered as the zero set of the bilinear function:

$$\left\langle (x_1,x_2,x_3), \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right\rangle = x_1y_1-x_2y_2,$$

defined by the projection

$$P=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can write the equation:

$$\frac{x_1^2}{\sigma}+\frac{x_2^2}{\tau}=1 \text{ with } \sigma\tau(\sigma+\tau)\neq 0,$$

using the selfadjoint transformation

$$G = \begin{pmatrix} \frac{1}{\sigma} & 0 & 0 \\ 0 & -\frac{1}{\tau} & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

into the form

$$\langle x, Gx \rangle = 0.$$

Thus the normal form of the pencil of the corresponding confocal conics can be obtained from the equality

$$0 = \langle x, (G^{-1} - tP)^{-1} x \rangle = (x_1, x_2, x_3) \begin{pmatrix} \frac{1}{\sigma - t} & 0 & 0 \\ 0 & \frac{1}{-(-\tau - t)} & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Hence, with respect to the original inhomogeneous Cartesian coordinates, we have:

$$\frac{x_1^2}{\sigma - t} + \frac{x_2^2}{\tau + t} = 1 \text{ for } t \in \mathbb{R} \setminus \{\sigma, \tau\}.$$

The elliptic and hyperbolic cases can be considered in a similar way. We consider special ovals of the projective space as intersections of a family of quadratic cones and the model planes, respectively. In the first case, the point of the model of the elliptic plane is a pair of antipodal points the unit sphere of V, where the lengths of the vectors are calculated using the bilinear function

$$\langle x, y \rangle = (x_1, x_2, x_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

In the second case the bilinear function is

$$\langle x, y \rangle = (x_1, x_2, x_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

and a point of the model is a pair of antipodal points of the hyperboloid containing the vectors with imaginary unit lengths. The family of cones is defined by the equality:

$$\frac{x_1^2}{c^2} + \frac{x_2^2}{c^2 - \beta^2} \pm \frac{x_3^2}{c^2 + \gamma^2} = 0,$$

where c is a parameter and $\beta^2 \pm \gamma^2 = \pm 1$ in the two respective cases. (The definition in the elliptic case is also motivated by physical argument of a gravitating arc, as we can see in [6].)

Using the selfadjoint transformation

$$G = \begin{pmatrix} \frac{1}{c^2} & 0 & 0 \\ 0 & \frac{1}{c^2 - \beta^2} & 0 \\ 0 & 0 & \frac{1}{c^2 + \gamma^2} \end{pmatrix},$$

we can rewrite the last equality in the form

$$\langle x, Gx\rangle = 0.$$

The normal form of the pencil of the corresponding confocal conics can be obtained from the equality

$$0 = \langle x, (G^{-1} - tI)^{-1}x\rangle =$$

$$= (x_1, x_2, x_3)\begin{pmatrix} \frac{1}{c^2-t} & 0 & 0 \\ 0 & \frac{1}{(c^2-t)-\beta^2} & 0 \\ 0 & 0 & \frac{1}{(c^2-t)+\gamma^2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the matrix of the identity. Thus the normal form of this pencil of conics is

$$\frac{x_1^2}{c^2-t} + \frac{x_2^2}{(c^2-t)-\beta^2} \pm \frac{x_3^2}{(c^2-t)+\gamma^2} = 0 \text{ where } x_1^2 + x_2^2 \pm x_3^2 = \pm 1.$$

We can conclude that for the description of confocality we must use both the singular and the nonsingular projections of the space V. This motivates our further examination.

On the Euclidean and the hyperbolic plane the different types of conics are the ellipses (with no ideal points) and hyperbolas (with two ideal points). Within the family of all conics, the singular quadrics (determining the common line of the two foci) separate the family into these two types. This situation can be observed in the elliptic case, too, but there is no other (metric or affine) possibility to distinguish the ovals of the two class to each other.

In the previously investigated case of the pseudo-Euclidean plane there are three types of conics as it can be seen either in [8] or in [2].

4. The Ivory property, projection pencils of quadrics and p-quadrics

The planar Euclidean version of Ivory's theorem states that the two diagonals of a certain curvilinear quadrangle formed by four confocal conics have the same length. With respect to the scalar product this equality means:

$$\rho^2(x, y') = \langle x - y', x - y'\rangle = \langle y - x', y - x'\rangle = \rho^2(x', y),$$

where the pairs of points $\{x, y\}, \{x, x'\}, \{y, y'\}, \{x', y'\}$ are on four confocal quadrics which intersect each other at the examined points. Also in the Euclidean space, there is an equivalent reformulation of this theorem, using the language of affine mappings, since if we have two confocal conics of the same type (e.g. ellipses) then there exists an affine mapping l, with the property that whenever a conic of the other type (hyperbola) intersects the first ellipse at a point, then it intersects the other one in at the point which is the image of the first one under this affinity. Both intersections are orthogonal, and now Ivory's theorem states that:

$$\rho(x, l(y)) = \rho(y, l(x)),$$

where ρ is the Euclidean distance. In a projective pseudo-Euclidean space the hyperbolic and the elliptic metrics (based on the inner product) are functions of the quantity

$$\rho(x, y) := \frac{\langle x, y\rangle}{\sqrt{|\langle x, x\rangle\langle y, y\rangle|}},$$

and thus, it was a natural idea to compute the lengths of the diagonals by substituting its endpoints into the function ρ (cf. [7]).

A linear transformation $p : V \longrightarrow V$ is a projection if the equality $p^2 = p$ holds. The vector space V can be regarded as the direct sum of its subspaces $\mathrm{Ker}\, p$ and $\mathrm{Im}\, p$. The restriction of the projection to its image spaces and to its kernel is the identity mapping and the zero map, respectively.

Note that the property of a linear transformation defined in the previous paragraph is not a ``projective property´´: in the projective class of the projection p the only projection is the transformation p itself. Indeed, if $\lambda \neq 0, 1$ then $(\lambda p)^2 = \lambda^2 p \neq \lambda p$, showing that the transformation λp is not a projection. Therefore we say that a point of $P(L(V))$ is a projection if there is a member of its class which is a projection in $L(V)$.

Definition 4.1. A maximal set of quadrics is called a *pencil of quadrics* if the set of the selfadjoint linear transformations corresponding to their duals belongs to a two-dimensional subspace of the vector space $L(V)$. We say that a pencil of quadrics is a *projection pencil associated to the projection* p, if the corresponding two-space contains p. Within a projection pencil of quadrics spanned by the invertible linear transformation l_0 and the projection p, the connected components of

$$\{(\lambda, \mu) \mid \lambda(l_0^{-1} + \mu p)\ \lambda, \mu \in \mathbb{R}\}$$

correspond to quadrics of different type.

Lemmas 9 and 10 in [7] imply a theorem stating that the special case investigated in Lemmas 6,7 and 8 is actually the general case of the proof of Ivory's theorem. Our goal is to describe a more general situation when this representation is possible. For this reason we introduce a new concept, the concept of p-quadric.

Definition 4.2. Let p be a projection. The quadric Φ generated by the selfadjoint transformation g is a *p-quadric* if for every $w = u + v \in V$ for which $u \in \mathrm{Im}\, p$ and $v \in \mathrm{Ker}\, p$ we have $g(u + v) = p\,(g(u)) - v$.

We remark that every quadric is an id-quadric. Furthermore, if Φ is a p-quadric, then $\mathrm{Im}\, p$ is an invariant subspace of g, because if $u \in \mathrm{Im}\, p$ then

$$g(u) = p(g(u)),$$

which yields that $g(u)$ is also an element of $\operatorname{Im} p$.

Using the method of H.Stachel and J.Wallner, we prove the following theorem.

Theorem 4.3 (A generalization of Ivory's theorem). *Let $P(V)$ be a projective space with the metric*

$$\delta(x,y) := \frac{\langle x,y\rangle}{\sqrt{|\langle x,x\rangle\langle y,y\rangle|}}$$

where $\langle\cdot,\cdot\rangle$ is a fixed indefinite inner product of V. Let p be a projection of V and denoted by $\Phi_0 = \{x \mid 0 = \langle x, l(x)\rangle\}$ and $\Phi_1 = l_1(\Phi_0)$ two regular p-quadrics of the same type which belong to the projection pencil $\mathcal{L}$ associated to l and p. Then there is a smooth family $\Phi_\lambda = l_\lambda(\Phi_0)$ $(0 \le \lambda \le 1)$ of p-quadrics of $\mathcal{L}$, such that l_λ is selfadjoint and has the Ivory property:

$$\delta(x, l_\lambda(y)) = \delta(l_\lambda(x), y) \text{ for all } x, y \in \Phi_0 \cap \operatorname{Im} p.$$

Any further p-quadric Ψ corresponding to the same projection pencil and containing a point $x \in \Phi_0 \cap \operatorname{Im} p$, also contains the entire path $l_\lambda(x)$, which intersects all quadrics Φ_λ orthogonally in $\operatorname{Im} p$.

5. The complete list of the cited definitions and statements

We now give the complete list of statements and definitions in [7]. This is necessary to understand the present modificated proof.

Definition 1. *A (nondegenerate) quadric Φ is the zero set of a (nondegenerate) symmetric bilinear form $\sigma(x,y) = \langle x, l(y)\rangle$, with a selfadjoint (nonsingular) linear endomorphism l. The endomorphism $l = id$ corresponds to the absolute quadric Ω, which is the set of absolute points.*

Definition 2. *The quadric $\widehat{\Phi} = \{v \mid \widehat{\sigma}(v,v) = \langle v, l^{-1}(v)\rangle = 0\}$ in the dual space is called the dual of the original quadric Φ defined by $\sigma(v,w) = \langle v, l(w)\rangle$.*

Definition 3. *If k is a linear endomorphism and the quadric Φ is given by the endomorphism l, then we define the dual k-image of Φ to have the equation*

$$\widehat{\sigma}(v,v) = 0, \text{ with } \widehat{\sigma}(v,v) = \langle v, kl^{-1}k^*(w)\rangle$$

$\widehat{\sigma}$ is understood to apply to gradients.

Definition 4. *Φ_0 and Φ_1 are said to be confocal (or homofocal), if one of the following equivalent conditions holds true:*

(i) the bilinear forms $\widehat{\sigma_0}$, $\widehat{\sigma}_1$, $\widehat{\langle\cdot,\cdot\rangle} = \langle\cdot,\cdot\rangle$ are linearly dependent,

(ii) the linear endomorphisms l_0^{-1}, l_1^{-1}, id are linearly dependent,

(iii) the coordinate matrices Q_1^{-1}, Q_2^{-1}, H^{-1} are linearly dependent.

The family of quadrics Φ confocal to Φ_0 is defined by the endomorphisms l which satisfy $l^{-1} = \lambda l_0^{-1} + \mu id$, $(\lambda,\mu) \in \mathbb{R}^2$, $\lambda \neq 0$.

Definition 5. *Within the family of confocal bilinear forms spanned by* l_0, *the connected components of* $\{(\lambda, \mu) \mid \lambda(l_0^{-1} + \mu id)$ *nonsingular*$\}$ *correspond to quadrics of different types.*

Lemma 3 *In the* n*-dimensional elliptic or hyperbolic space* $(n > 1)$ *all confocal families possess at least two types of quadrics.*

Lemma 4 *If the confocal quadrics* Φ_0 *and* Φ_λ *intersect, they do so orthogonally.*

Lemma 5 *Assume that* Φ *is a quadric, possibly singular but not contained in a hyperplane, and that there is a mapping* $x \longrightarrow x'$ *such that*

$$\langle x_1', x_2 \rangle = \langle x_1, x_2' \rangle \text{ for all } x_1, x_2 \in \Phi,$$

then there is a selfadjoint linear endomorphism l *of* $\mathbb{R}^{n+1}$ *such that* $x' = l(x)$ *for all* $x \in \Phi$.

Lemma 6 *If the linear endomorphism* l *is selfadjoint, then the quadric*

$$\Phi_0 : \sigma(x, x) := \langle x, x \rangle - \langle l(x), l(x) \rangle = 0$$

together with its l*-image* Φ_1 *has the Ivory property*

$$\delta(l(x), y) = \delta(x, l(y)) \text{ for all } x, y \in \Phi \text{ with } \langle x, x \rangle, \langle y, y \rangle \neq 0.$$

The restriction of l *to any linear subspace contained in* Φ_0 *is isometric in the sense of* δ.

Lemma 7 *Assume that* l *is selfadjoint and that the quadric* Φ_0 *given in Lemma 6 is regular. Then* Φ_0 *and* $\Phi_1 = l(\Phi_0)$ *are confocal. (The dual of* $l(\Phi_0)$ *defined by the endomorphism* $lg_0^{-1}l^*$ *if* Φ_0 *given by* g_0*.)*

Lemma 8 *If* l *is selfadjoint, then in most cases the quadric* Φ_0 *as defined in Lemma 6 is of the same type as* $l(\Phi_0)$ *provided both are regular. Different types are only possible when the normal form of* l *contains a block matrix* $R_2(0, b)$ *or* $R_{2k}(0, b, 1)$.

Lemma 9 *Consider two regular confocal quadrics* Φ_0, Φ_1 *which are of the same type. Then there is a selfadjoint endomorphism* l *such that* $\Phi_1 = l(\Phi_0)$ *and the equation of* Φ_0 *is given by* $\langle x, x \rangle - \langle l(x), l(x) \rangle = 0$.

Lemma 10 *We use the notation of the proof of Lemma 9. There is* $\delta > 0$ *such that* $id - \lambda g_0$ *has a square root which smoothly depends on* λ, *for* $-\delta < \lambda < 1 + \delta$.

Lemma 11 *Suppose that* $P, \Phi_0, \Phi_1, g_0, g_1, l$ *are as in Lemma 9 and its proof. Then there is a smooth family* l_λ *of transformations with* $l_0 = id$ *and* $l_1 = l$, *such that the quadric* $\Phi_\lambda = l_\lambda(\Phi_0)$ *is defined by the endomorphism* g_λ *with*

$$g_\lambda^{-1} = g_0^{-1} - \lambda id.$$

All quadrics Φ_λ *are confocal with* Φ_0. *They orthogonally intersect the path* $l_\lambda(x)$ *of a point* $x \in \Phi_0$.

Lemma 12 *We use the notations of Lemma 11 and consider the quadrics* Φ_λ*, defined by endomorphisms* g_λ*. If* $\Psi \neq \Phi_0$ *is confocal with* Φ_0*, and* $x \in \Phi_0 \cap \Psi$*, then also* $l_\lambda(x) \in \Psi$*.*

Theorem 2 *Let* $P(V)$ *be a projective space with the metric*

$$\delta(x,y) := \frac{\langle x,y\rangle}{\sqrt{|\langle x,x\rangle\langle y,y\rangle|}}$$

where $\langle\cdot,\cdot\rangle$ *is a fixed indefinite inner product of* V*. Let* Φ_0 *and* $\Phi_1 = l_1(\Phi_0)$ *denote two regular confocal quadrics of the same type. Then there is a smooth family* $\Phi_\lambda = l_\lambda(\Phi_0)$ $(0 \leq \lambda \leq 1)$ *of quadrics confocal with* Φ_0 *and* Φ_1*, such that* l_λ *is selfadjoint and has the Ivory property:*

$$\delta(x, l_\lambda(y)) = \delta(l_\lambda(x), y) \text{ for all } x,y \in \Phi_0 \cap \operatorname{Im} p.$$

Any further quadric Ψ *confocal with* Φ_0 *which contains a point* $x \in \Phi_0$ *contains the entire path* $l_\lambda(x)$*, which intersects all quadrics* Φ_λ *orthogonally.*

6. The proof of Theorem 4.3

Now we modify the statements of the previous section where necessary. By the definition of a projection p, the proof of Lemma 3 can be applied in our case, too. Lemma 4 for our projection pencil of quadrics can be formulated in the following way:

Proposition 6.1 (Lemma 4'). *If quadrics* Φ_0 *and* Φ_λ *corresponding to a projection pencil intersect, they do so orthogonally with respect to the quadratic form of* p*. Thus, if* x *is a common point, then we have* $0 = \langle p(g_0(x)), g_\lambda(x)\rangle$*.*

Proof. We have

$$0 = \langle x, g_0(x)\rangle = \langle x, v\rangle = \langle g_\lambda^{-1}(w), v\rangle = \langle (g_0^{-1} + \mu p)(w), v\rangle =$$
$$= \langle g_0^{-1}(w), v\rangle + \mu\langle p(w), v\rangle = \langle w, g_0^{-1}v\rangle + \mu\langle p(w), v\rangle =$$
$$= \langle g_\lambda(x), x\rangle + \mu\langle p(w), v\rangle = \mu\langle p(w), v\rangle,$$

as we stated.

□

Lemma 5 on the Ivory property is also holds in our setting. In Lemma 6-8 we change the selfadjoint transformation l to the selfadjoint transformation $l' = lp + (\mathrm{id} - p)$ and consider the quadric Φ_0' with the equation:

$$\langle p(x), p(x)\rangle - \langle l'(x), l'(x)\rangle = 0.$$

We remark that $\operatorname{Im} p \cap \Phi_0' = \operatorname{Im} p \cap \Phi_0$, where Φ_0 is defined by the equality

$$\langle x, x\rangle - \langle l(x), l(x)\rangle = 0.$$

It is also true that $\operatorname{Ker} p \cap \Phi_0' = \{x \in \operatorname{Ker} p \mid \langle x, x\rangle = 0\}$.

As it can be seen easily, the following variation of Lemma 6 is true for every projection pencil of quadrics:

Proposition 6.2 (Lemma 6'). *If the linear endomorphism l is selfadjoint and invariant on the subspace* $\operatorname{Im} p$ *then the quadric*

$$\langle p(x), p(x)\rangle - \langle l'(x), l'(x)\rangle = 0$$

together with its l' image $\Phi_1 = l'(\Phi_0)$ has the Ivory property

$$\delta(l'(x), y) = \delta(x, l'(y)) \text{ for all } x, y \in \Phi \text{ with } \langle x, x\rangle, \langle y, y\rangle \neq 0.$$

The restriction of l' to any linear subspace contained in Φ_0 is isometric in the sense of δ.

The following modification is more interesting:

Proposition 6.3 (Lemma 7'). *Assume that* $\operatorname{Im} p$ *is an invariant subspace of l, and that the quadric Φ_0 given by the equality:*

$$\langle p(x), p(x)\rangle - \langle l'(x), l'(x)\rangle = 0, \text{ where } l' = lp + (\mathrm{id} - p)$$

is regular. Then $\Phi_1 = l'(\Phi_0)$ is in the projection pencil of Φ_0 and p.

Proof. Rewriting the equation of Φ_0, we obtain:

$$0 = \langle x, p(x)\rangle - \langle l'(x), l'(x)\rangle = \langle x, (p - (l')^2)x\rangle.$$

V is a direct sum of $\operatorname{Ker} p$ and $\operatorname{Im} p$ for arbitrary p. Furthermore $l' = l$ on $\operatorname{Im} p$ and $l' = id$ on $\operatorname{Ker} p$. The dual of Φ_0 is represented by

$$0 = \langle x, (p - (l')^2)^{-1}x\rangle,$$

and the dual l'-image of Φ_0, according to Def. 3 in [7], is defined by

$$0 = \langle x, l'(p - (l')^2)^{-1}l'x\rangle.$$

Consider now the transformation

$$(p - (l')^2)^{-1} - l'(p - (l')^2)^{-1}l'.$$

Observe that $\operatorname{Im} p$ is an invariant subspace of $(p - (l')^2)$, as for any $u \in \operatorname{Im} p$, we also have $(p - (l')^2)(u) = u - l^2(u) \in \operatorname{Im} p$. Thus $\operatorname{Im} p$ is an invariant subspace of its inverse, and for a vector $u \in \operatorname{Im} p$, applying the argument of Lemma 7 to the invariant subspace $\operatorname{Im} p$, we have

$$(p - (l')^2)^{-1} - l'(p - (l')^2)^{-1}l'(u) = ((\mathrm{id} - l^2)^{-1} - l(\mathrm{id} - l^2)^{-1}l)(u) = u.$$

On the other hand for a vector $v \in \operatorname{Ker} p$,

$$(p - (l')^2)(v) = -v$$

showing that $v \in \operatorname{Ker} p$ and that $(p - (l')^2)$ is a reflection on $\operatorname{Ker} p$. Thus

$$((p - (l')^2)^{-1} - l'(p - (l')^2)^{-1}l')(v) = (p - (l')^2)^{-1}(v) - (p - (l')^2)^{-1}(v) = 0,$$

implying the required equality:

$$(p - (l')^2)^{-1} - l'(p - (l')^2)^{-1}l' = p.$$

□

In the remaining part, we consider only those selfadjoint transformations that leave $\operatorname{Im} p$ invariant.

Proposition 6.4 (Lemma 8'). *If l is selfadjoint with the invariant subspace* $\operatorname{Im} p$*, then the quadric Φ_0, defined by $g_0 = (p-(l')^2)$, is of the same type as $\Phi_1 = l'(\Phi_0)$ provided both are regular and unless the normal form of $l'|_{\operatorname{Im} p}$ contains a block matrix $R_2(0,b)$ or $R_{2k}(0,b,1)$ (see Th.1 in* [7] *or Th.5.3. in* [4]*).*

Proof. The convex combination of the selfadjoint transformations $g_0^{-1} = (p-(l')^2)^{-1}$ and $g_1^{-1} = l'(p-(l')^2)^{-1}l'$ can be investigated in the same way as in Lemma 8, using the result of our Prop. 6.2:

$$g_\lambda^{-1} := (1-\lambda)g_0^{-1} + \lambda g_1^{-1} = g_0^{-1} - \lambda p = (p-(l')^2)^{-1} - \lambda p,$$

if $0 \le \lambda \le 1$. For $u \in \operatorname{Im} p$ we have

$$\left((p-(l')^2)^{-1} - \lambda p\right)(u) = \left((\mathrm{id}-l^2)^{-1} - \lambda \mathrm{id}\right)(u)$$

and the proof of Lemma 8 can be applied. For $v \in \operatorname{Ker} p$, we obtain that

$$\left((p-(l')^2)^{-1} - \lambda p\right)(v) = -\mathrm{id}(v)$$

which is always non-singular. □

Since the quadric g_0 used in Lemmas 6'-8' is a p-quadric, we can give a representation theorem only for p-quadrics.

Proposition 6.5 (Lemma 9'). *Consider two regular p-quadrics of a projection pencil, say Φ_0 and Φ_1. Assume that they are of the same type with respect to the projection p. Then there is a selfadjoint transformation l invariant on the subspace* $\operatorname{Im} p$ *such that $\Phi_1 = l'(\Phi_0)$ where $l' = lp + (\mathrm{id}-p)$, and the equation of Φ_0 is given by*

$$\langle p(x), p(x)\rangle - \langle l'(x), l'(x)\rangle = 0.$$

Proof. Without changing the quadrics we can consider regular representing selfadjoint transformation g_0 and g_1 for which $\operatorname{Im} p$ is an invariant subspace and

$$g_0^{-1} - g_1^{-1} = p.$$

We have to show that there exists l such that it is invariant on the subspace $\operatorname{Im} p$ and $g_0 = p-(l')^2$. From the equality containing g_i^{-1}, we can see that on $\operatorname{Im} p$ we have $g_0^{-1} - g_1^{-1} = id$, and on $\operatorname{Ker} p$ $g_0^{-1} = g_1^{-1} = -id$. Now g_0 is a regular transformation of $\operatorname{Im} p$ thus the proof of Lemma 9 shows that there exist an invertible selfadjoint transformation $\widetilde{l} : \operatorname{Im} p \longrightarrow \operatorname{Im} p$ for which $g_0 = \mathrm{id}|_{\operatorname{Im} p} - \widetilde{l}^2$. Extract this transformation to an $l : V \longrightarrow V$ transformation by the equalities:

$$l(u) = \begin{cases} \widetilde{l}(u) & \text{if } u \in \operatorname{Im} p \\ u & \text{if } u \in \operatorname{Ker} p \end{cases}$$

Now for an element u of $\operatorname{Im} p$, we have

$$g_0(u) = u - \widetilde{l}^2(u) = u - l^2(u) = (p-(l')^2)(u),$$

and for $v \in \operatorname{Ker} p$,

$$g_0(v) = -v = (p - (lp + (\mathrm{id} - p))^2)(v) = (p - (l')^2(v),$$

showing that $g_0 = p - (l')^2$ and that Φ_0 is defined by the equality

$$\langle p(x), p(x) \rangle - \langle l'(x), l'(x) \rangle = 0.$$

It remains to show that, indeed, $l'(\Phi_0) = \Phi_1$. But by Lemma 7', $l'(\Phi_0)$ is defined by a transformation $\overline{g}_1$ with the property that

$$g_0^{-1} - (\overline{g}_1)^{-1} = p$$

and thus, $(\overline{g}_1)^{-1} = g_1^{-1}$. □

Proposition 6.6 (Lemma 10'). *Using the notation of Lemma 9', there is a value $\delta > 0$ such that $p - \lambda g_0 = (l')^2_\lambda$, which smoothly depends on λ for $-\delta < \lambda < 1 + \delta$.*

The proof of this proposition is a straightforward modification of the proof of Lemma 10, which we omit. Lemma 11 states the existence of a smooth family of regular transformations corresponding to two "confocal quadrics which are of the same type". Our method of generalization leads to the following proposition:

Proposition 6.7 (Lemma 11'). *Suppose that $p, \Phi_0, \Phi_1, g_0, g_1, l$ are as in Lemma 9' and its proof. Then there is a smooth family l_λ of transformations with $l_0 = id$ and $l_1 = l$, such that the quadric $\Phi_\lambda = l'(\Phi_0)$ is defined by the transformation g_λ with*

$$g_\lambda^{-1} = g_0^{-1} - \lambda p.$$

All quadrics Φ_λ are p-quadric belonging to the projection pencil of p. Their restrictions to $\operatorname{Im} p$ *intersect the path $l_\lambda(u)$ of a point $u \in \Phi_0 \cap \operatorname{Im} p$ orthogonally.*

Proof. By definition, $g_0 = p - (l')^2$ and $g_1^{-1} = g_0^{-1} - p$. Consider λg_0 instead of g_0. Then $\lambda g_0 = -\mathrm{id}$ on $\operatorname{Ker} p$. We define l_λ by the equalities

$$\lambda g_0 = \mathrm{id} - (l_\lambda)^2 \text{ on } \operatorname{Im} p,$$

$$l_\lambda = \mathrm{id}|_{\operatorname{Ker} p} \text{ on } \operatorname{Ker} p.$$

By Lemma 10', $(l'_\lambda) = lp + (\mathrm{id} - p)$ exists and depends smoothly on λ. Now $(l'_\lambda)p = p(l'_\lambda)$, since on Im$p$ it is the identity and on $\operatorname{Ker} p$ both sides are zero (we note that $l'_\lambda = \mathrm{id}|_{\operatorname{Ker} p}$ is invariant on $\operatorname{Ker} p$ by its definition). Thus for a non-zero λ on $\operatorname{Im} p$, we have

$$(l_\lambda)g_0 = (l_\lambda)\lambda^{-1}(\mathrm{id}|_{\mathrm{Im} p} - (l_\lambda)^2) = \lambda^{-1}(\mathrm{id}|_{\mathrm{Im} p} - (l_\lambda)^2)(l_\lambda) = g_0(l_\lambda).$$

Hence on $\operatorname{Im} p$,

$$g_\lambda^{-1} = l_\lambda g_0^{-1} l_\lambda = (l_\lambda)^2 g_0^{-1} = (\mathrm{id}|_{\mathrm{Im} p} - \lambda g_0)g_0^{-1} = g_0^{-1} - \lambda \mathrm{id}.$$

On the other hand, for an element of $\operatorname{Ker} p$, by definition,

$$g_\lambda^{-1} = g_0^{-1}$$

showing that on V we have

$$g_\lambda^{-1} = l'_\lambda g_0^{-1} l'_\lambda = g_0^{-1} - \lambda p$$

as we stated. From Lemma 7', we can see that Φ_0 and $l'_\lambda(\Phi_0)$ generates a projection pencil corresponding to the projection p. Finally, we must prove the statement on orthogonality. We can compute the derivative of the mapping $l_\lambda(x) : \mathbb{R} \longrightarrow V$ if we use the direct product structure of V. Let $w = u+v \in V$ where $u \in \operatorname{Im} p$ and $v \in \operatorname{Ker} p$. Then we have:

$$\lambda g_0(u+v) = (u - l_\lambda(u) l_\lambda(u)) - v.$$

Differentiating both sides with respect to λ, we obtain

$$g_0(u+v) = -2\dot{l_\lambda}(u) l_\lambda(u),$$

implying

$$\dot{l_\lambda}(u) = -\frac{1}{2} g_0(u+v)(l_\lambda(u))^{-1} = -\frac{1}{2} g_\lambda(u+v) l_\lambda(u),$$

as in [7]. Thus if $v = 0$, then the tangent hyperplane of Φ_λ in $l_\lambda(u)$ has the gradient vector $g_\lambda l_\lambda(u)$, which yields that in $P(V)$ the corresponding point is conjugate to the tangent hyperplane with respect to the identity quadric of $\operatorname{Im}P$, namely to p. □

We can modify Lemma 12 as well, in a natural way.

Proposition 6.8 (Lemma 12'). *Using the notation of the previous lemmas, if $\Psi \neq \Phi_0$ are p-quadrics belonging to the same projection pencil (of p and ${g_0}^{-1}$, and if $u \in \Phi_0 \cap \Psi \cap \operatorname{Im} p$, then $l_\lambda(u) \in \Psi$.*

Proof. We have

$$x \in \Phi_0 \Longleftrightarrow \langle x, g_0(x)\rangle = 0 \text{ and } x \in \Psi \Longleftrightarrow \langle x, g_\mu(x)\rangle = 0.$$

By definition, $g_\mu^{-1} = g_0^{-1} - \mu p$ with $\mu \neq 0$. Consider the following expression:

$$\lambda g_0 g_\mu^{-1} - \mu l_\lambda g_\mu l_\lambda g_\mu^{-1} - (\lambda - \mu) g_\mu g_\mu^{-1} =$$

$$= \lambda g_0(g_0^{-1} - \mu p) - \mu l_\lambda g_\mu l_\lambda g_\mu^{-1} - (\lambda - \mu)\mathrm{id} =$$

$$= \lambda \mathrm{id} - \lambda\mu g_0 p - \mu l_\lambda^2 - (\lambda - \mu)\mathrm{id} =$$

$$= \lambda \mathrm{id} - \lambda\mu g_0 p - \mu(p - \lambda g_0) - (\lambda - \mu)\mathrm{id} =$$

$$\mu(-p + \mathrm{id})(\lambda g_0 + \mathrm{id}).$$

For a point of $\operatorname{Im} p$, this expression is zero, since $p = \mathrm{id}$. Thus we also have that

$$\lambda g_0 - \mu l_\lambda g_\mu l_\lambda - (\lambda - \mu) g_\mu = 0$$

on $\operatorname{Im} p$. Hence on $\operatorname{Im} p$ we obtain

$$\mu\langle l_\lambda(u), g_\mu l_\lambda(u)\rangle = \mu\langle u, l_\lambda g_\mu l_\lambda(u)\rangle = \lambda\langle u, g_0(u)\rangle - (\lambda - \mu)\langle u, g_\mu(u)\rangle = 0,$$

showing that $l_\lambda(u) \subset \Psi \cap \operatorname{Im} p$. □

Now to prove Theorem 4.3, we need only replace the lemmas in the proof of Theorem 2 of [7] by our modified versions. More specifically, by Lemma 9' (Prop. 6.5), there exists l such that $\Phi_1 = l'(\Phi_0)$ with the transformation $l' = lp + (\mathrm{id} - p)$ and the equation of Φ_0 is given by

$$\langle p(x), p(x)\rangle - \langle l'(x), l'(x)\rangle = 0.$$

Lemma 11' (Prop. 6.7) shows the existence of Φ_λ and l_λ. By Lemma 6' (Prop. 6.2), l_λ has the Ivory property. Finally, Lemma 12' (Prop 6.8) shows the statement about the quadric Ψ, if it exists.

References

[1] Berger, M.: *Geometry I-II.* Springer, Berlin Heidelberg (1987)

[2] Birkhoff G., Morris R.: *Confocal conics in Space-Time.* Amer. Math. Monthly **69(1)**, 1–4 (1962)

[3] Gohberg, I., Lancester, P., Rodman, L.: *Indefinite Linear Algebra and Applications.* Birkhauser, Basel-Boston-Berlin (2005)

[4] Gohberg, I., Lancester, P., Rodman, L.: *Matrices and indefinite scalar products.* Birkhauser, Basel-Boston-Berlin (1983)

[5] Ivory, J.: *On the Attractions of homogeneous Ellipsoids.* Phil. Trans. Royal Society of London, 345–372 (1809)

[6] Kozlov V.V.: *The Newton and Ivory theorems of attraction in spaces of constant curvature.* Moscov University, Mechanics Bulletin **55(5)**, 16–20 (2000)

[7] Stachel, H., Wallner, J.: *Ivory's theorem in hyperbolic spaces.* Siberian Math. J. **45/4**, 785–794 (2004)

[8] Stachel, H.: *Ivory's theorem in the Minkowski plane.* Math. Pannon. **13**, 11-22 (2002)

Á.G. Horváth
Department of Geometry,
Budapest University of Technology and Economics,
H-1521 Budapest,
Hungary

e-mail: ghorvath@math.bme.hu

Received: 26 May 2009.

Revised: 8 March 2011.