ON DIRICHLET-VORONOI CELL

Part I. Classical problems *

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Dedicated to Professor Julius Strommer on the ocassion of his 75^{th} birthday

Abstract

This paper is basicly a survey that discusses some actual questions on the so-called DIRICHLET-VORONOI cell of a lattice of dimension n. The first part is about the classical results and some principal problems of this subject while we plan a second part on some constructive and algorithmic questions concerned with our topic.

Keywords: lattice of dimension n, DIRICHLET-VORONOI cell

1 Introduction

The concept of DIRICHLET-VORONOI cell (also DIRICHLET cell or VORONOI region, briefly D-V cell) was introduced in two classical paper by DIRICHLET [10] and VORONOI [24], respectively. Let us given a discrete point set L in the n-dimensional Euclidean space E^n. The DIRICHLET-VORONOI tiling of L is a tiling with convex tiles

$$D(\mathbf{z}) = \{\mathbf{y} \in E^n \quad | \quad |\mathbf{y}-\mathbf{z}| \leq |\mathbf{y}-\mathbf{x}| \text{ for all } \mathbf{x} \in L\} \qquad \mathbf{z} \in L.$$

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That means, $D(\mathbf{z})$ consists of those points $\mathbf{y}$ of E^n whose distance from the point $\mathbf{z}$ (the origin of D) is not greater than its distance from any other points of the set L. The tile $D(\mathbf{z})$ is called the cell of $\mathbf{z}$ where the function $|\cdot|$ is the usual Euclidean norm of E^n. In the case when L is a lattice (i.e. the endpoint set of the integer linear combinations of a fixed linearly independent vector system of E^n) these cells are translated copies of the cell $D(0)$ of the origin. In this paper we investigate only cells of lattices thus in most cases we apply the notation D only. The concept of D-V cell is useful in solving a lot of various problem.

First of all, we say some words about applications in discrete geometry. For example, the classical ball packing problem is to find out how densely a large number of balls can be packed together. This problem is in general unsolved even today. For measuring the density of such a packing we naturally come to the idea of D-V cell. The precise definition of density Δ of a lattice packing is the following:

$$\Delta = \frac{\text{volume of one ball}}{\text{volume of a basic lattice parallelepiped}} = \frac{\text{volume of one ball}}{\text{volume of the cell } D}.$$

In the plane this problem is solved, the optimal arrangement is the regular triangle (simplicial) (or regular hexagonal) lattice defined by two edge vectors of a regular triangle. For this arrangement the density is:

$$\Delta = \frac{\pi}{\sqrt{12}}.$$

(See Figure 1.) A lot of papers deal with this problem and its analogous concerned with the optimal arrangements of convex bodies in a space of dimension n. The most important classical results is due to GAUSS [12], KORKINE-ZOLOTAREFF [19],[20], BLICHFELDT [4],[5],[6].

Second, we mention the so-called *quantization of data* problem. This is the following: Suppose that certain data (symbols) are uniformly distributed over a large set S in E^n. If we have a lattice L with basis vectors of determinant 1 and each point of S we substitute for vector $\mathbf{x}$ to the nearest lattice point, the average squared error per symbol related to L is defined by the integral

$$G(L) = \int_D |\mathbf{x}|^2 d\mathbf{x}$$

over the D-V cell D of L. The problem is to minimize this average over all lattices of determinant 1. CONWAY and SLOANE in [7] support the conjecture that for higher dimensions the solution is provided by the polar

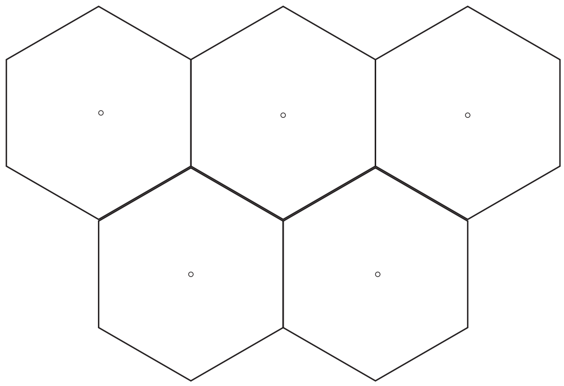

Figure 1: The regular triangle lattice and its D-V tiling.

(or dual) lattice of that lattice which gives the densest ball packing in E^n. The normalized form of this quantity called the *normalized second moment of* D, is a dimensionless real number defined

$$G^*(L) = \frac{1}{n}\mathrm{Vol}(D)^{-1-\frac{2}{n}} \int\limits_D |\mathbf{x}|^2 d\mathbf{x}.$$

(It is introduced by H.DAVENPORT in the paper [8].)

The last problem in this introduction is related to the *numerical integration.* Let G be a JORDAN measurable set of the plane E^2, and ω a continuous non-decreasing real function on $[0, \infty)$ with $\omega(0) = 0$. Let $\mathcal{H}^\omega(G)$ denote the set of real functions f on G such that

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq \omega(|\mathbf{x} - \mathbf{y}|) \text{ for } \mathbf{x}, \mathbf{y} \in G$$

Then we want to choose points $\mathbf{x}^1, \ldots, \mathbf{x}^k \in G$ and real numbers $\alpha_1, \ldots, \alpha_k$ for fixed $k = 1, 2, \ldots$ such that the maximal error

$$\max \left\{ \left| \int\limits_G f(\mathbf{x})d\mathbf{x} - \sum_{i=1}^{k} \alpha_i f(\mathbf{x}_i) \right| \qquad f \in \mathcal{H}^\omega(G) \right\}$$

let be minimal. Using the concept of D-V cells and some other results due to L.FEJES-TÓTH, BABENKO [1] found an optimal choice for $\mathbf{x}^i$ and α_i in the asymptotic sense as k tends to infinity. To extend this result to higher dimensions seems to be very difficult.

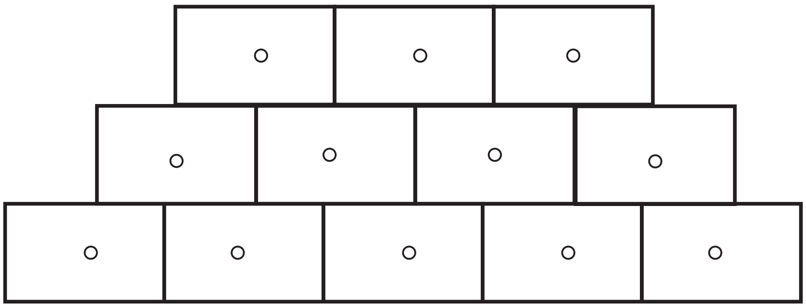

Figure 2: Non face-to-face lattice tiling.

From these introductory remarks we can see that the concept of D-V cells is important. In this paper we summarize some classical results on the geometric properties of D-V cell. The author will arise some problems solved and open as well. A new concept will also be introduced; the idea of *k-dimensional coveredness of a lattice parallelepiped.* We examine its connection with a lattice geometric problem discussed in the papers [3],[13].

2 Definitions, elementary properties of D

A *lattice* L of the Euclidean n-space E^n is defined by a linearly independent vector system of E^n as the set of all integral linear combinations of this system like a *basis*. We say that the lattice L' of dimension k ($1 \leq k \leq n$) is a *sublattice* of L if $L' \subset L$. The vector $\mathbf{m}$ is a *minimal* one if it is one of the shortest non-zero vector of L. (The length of a vector is regarded with respect to the usual Euclidean norm of E^n.) As in the introduction, the *D-V cell* $D(\mathbf{x})$ of a lattice poin $\mathbf{x}$ is the collection of those points of the space which are closer to $\mathbf{x}$ as any other points of the lattice. It is clear from the definition above that L is invariant under the translations by the lattice vectors and the reflections in a lattice point or in the midpoint of any lattice segment, respectively. From theese follows that any two D-V cell are translated copies of each other and, moreover, any cell and its $(n-1)$-dimensional faces (called *facets*) are centrally symmetric convex bodies, respectively. The definition of D implies (by virtue of the fact that a lattice is a discrete point system) that it is a polyhedron defined by finite intersections of certain half-spaces each of them contains the origin and is bounded by the midhyperplane of a lattice segment connecting the origin with a lattice points. The collection of the cells $D(\mathbf{x})$ for $\mathbf{x} \in L$ forms a so-called lattice tiling of the space E^n.

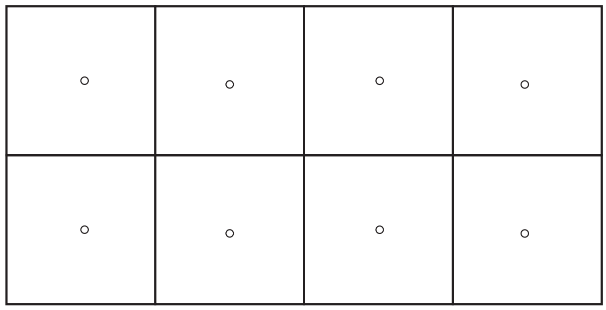

Figure 3: Non primitive D-V cell system.

This means that their union covers the space and their interiors are mutually disjoint. This tiling is *face-to-face*, so in particular, any facet of the tile $D(\mathbf{x})$ is also a facet of another tile. In Fig. 2 we pictured a non face-to-face lattice tiling formed by the translated copies of a rectangle. It is obvious that the D-V cell D is bounded and we assume in this paper that D also is closed, so this region is compact. The volume of D is equal to the volume of a *basic-parallelepiped* of L which is spanned by vectors of a basis of the lattice.

An important class of lattice tilings (whose convex tiles will be called parallelohedra as well) is the class of primitive tilings. An n-dimensional tiling is *primitive* if any vertex of a tile belongs to and is a vertex of precisely n other tiles. In the plane the D-V cells of the regular triangle lattice (see in Fig.1) form a primitive tiling, however, the D-V cells of the square lattice is not primitive (see fig.3).

3 Parallelohedron, extremal body

The concept of extremal body is due to H.MINKOWSKI, he proved the fundamental theorem of bounded centralsymmetric convex bodies. This is one of the most important theorem in the geometry of numbers and has a lot of consequences and applications in other parts of mathematics. (See in [21])

Theorem 1 ([21]) *A bounded central symmetric convex body K in E^n with the origin 0 in its centre and volume $v(K) > 2^n v(D)$ contains at least one lattice point different from 0. (D is the D-V cell of L, $v(\cdot)$ is the n-dimensional volume function.)*

An elegant proof of this theorem can be found e.g. in the book of P.GRUBER and G.LEKKERKERKER [18]. From this theorem immediately follows that such an 0-symmetric convex body which does not contain non-zero lattice points (so-called *empty body*) has a volume at most $2^n v(D)$. H.MINKOWSKI introduced the concept of *extremal body with respect to the lattice* L which is an empty $\mathbf{0}$-symmetric closed convex body with volume $2^n v(D)$. In his book he investigated this class of bodies and proved some interesting theorems on it. First of all he characterized the elements of this class:

Theorem 2 ([22]) *Let K be a (bounded) $\mathbf{0}$-symmetric convex body. Then K is extremal if and only if the following two properties hold:*
a, *The space E^n is covered by the bodies L-translates of $\frac{1}{2}K$.*
b, *Each point $\mathbf{x} \in E^n$ belongs to at most one body $\frac{1}{2} intK + \mathbf{u}$*
where $intK$ means the interior of the body K.

H.MINKOWSKI also proved that an extremal body is necessarily a closed polyhedron for which the following properties hold:

1. At most $2(2^n - 1)$ lattice points belong to the relative interiors of the faces of K,

2. K has at most $2(2^n - 1)$ faces,

3. On the boundary of K there lie at least $2(2^n - 1)$ lattice points.

In his famous works [24] and [25] VORONOI also studies this class of polyhedra. He introduced the concept of *parallelohedron* as a convex polyhedron P whose translates by a lattice L cover E^n and they have disjoint interiors. By a theorem of H.MINKOWSKI a polyhedron P parallelohedron if and only if $2P$ is extremal with respect to a lattice L. It is clear that the D-V cell D of the lattice L is a parallelohedron. Conversely Figure 4 shows a parallelohedron P which is not the cell of its lattice, but that prototile P is an affine image of the unit square D. Since this square is the D-V cell of the corresponding lattice it can be asked for the following question due to VORONOI:

Whether each parallelohedron is an affine image of a D-V cell?

This is one of the most famous open problems of this theme. For dimensions $n \leq 4$ this conjecture was proved by DELONE [9] while in the papers [24] and [25] VORONOI showed that in the space E^n each parallelohedron which is the prototile of a primitive lattice tiling is an affine image of a D-V cell. This result was refined later on by some authors. The history of this problem has been surveyed in the book [18].

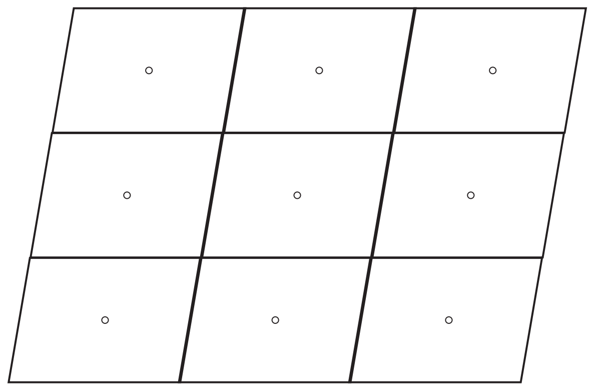

Figure 4: Non-primitive parallelohedron on the plane

We now revert to the problem on the number of lattice points on the boundary of an extremal body. In general we can not tell more than in the properties 1,2, and 3. We now investigate the special case when the extremal body is an enlarged copy $2D$ of the D-V cell of its lattice.We need to introduce the definition of relevant vector due to VORONOI. A lattice vector $\mathbf{x}$ is called *relevant vector* if it is actually needed to define the cell D. This means that the hyperplane containing the midpoint of $\mathbf{x}$ and perpendicular to $\mathbf{x}$, intersects the D-V cell D in a facet of dimension $(n-1)$. VORONOI proved the following characterization of relevant vectors:

Theorem 3 ([24]) *The lattice vector $\mathbf{x} \in L$ is a relevant one if and only if it is an unique minimal norm element of the coset $\mathbf{x} + 2L$. In this term "unique" means the property that if $\mathbf{y}$ is a minimal norm element of $\mathbf{x} + 2L$ then it is equal to $\mathbf{x}$ or $-\mathbf{x}$.*

This result says that $\pm\mathbf{x}$ is a unique pair of minima of its coset if and only if their endpoints are in the interiors of opposite facets of the extremal body $2D$. The author has generalized this result as follows :

Theorem 4 ([14]) *If a lattice vector $\mathbf{x}$ is in the relative interior of an $(n-k)$-dimensional face of the body 2D (for certain $k = 1, \dots, n-1$) then it is a minimum vector of its coset $\mathbf{x} + 2L$. Conversely if the rank of the set*

$$\mathcal{M}_{\mathbf{x}} := \{\mathbf{m} \in L | \mathbf{m} \textit{ is a minimum vector of the coset } \mathbf{x} + 2L\}$$

is equal to k then the elements of $\mathcal{M}_{\mathbf{x}}$ are in the relative interior of certain (pairwise distinct) $(n-k)$-dimensional faces of $2D$. Furthermore in the case *of $k = n$ the lattice vector $\mathbf{x}$ is a vertex of the body $2D$ if and only if $\mathbf{x}$ is a minimum vector of the coset $\mathbf{x} + 2L$ and the rank of $\mathcal{M}_{\mathbf{x}}$ is equal to n.*

Here the rank of a vector set means its dimension. Now we give some examples to illustrate this theorem. The number of elements of $\mathcal{M}_{\mathbf{x}}$ depends on the lattice. In the three-dimensional cubic lattice there are three types of lattice points belonging to the closed body $2D$. The vertex coordinates of $2D$ are congruent to $(1, 1, 1)$ componentwise mod2 (we take the coordinates with respect to the edge vectors of the basic cube of the lattice). The vertices are the minimal elements of the coset of $(1, 1, 1)$. So $|\mathcal{M}_{(1,1,1)}| = 8$ and $\text{rank } \mathcal{M}_{(1,1,1)} = 3$.

Regard now the so-called regular-simplex lattice of dimension 3. We can construct this lattice from the cubic lattice taking in addition the centers of the 2-dimensional faces of the basic cube also to lattice points. A basis $\{ \mathbf{e}_i \mid i = 1, 2, 3\}$ of this lattice points to centres of any three cube faces meeting in a cube vertex as origin. In Fig. 5 we see the body $2D$ which is a rhombic dodecahedron. We have two types of lattice vectors on the boundary of $2D$. E.g. the vertex $(-1, 1, 1)$ of $2D$ is a lattice point. The minimal elements of the coset of this point – denoted by double circles in Fig.5 – are the endpoints of the longer diagonals of the rhombic faces. This means that $|\mathcal{M}_{(-1,1,1)}| = 6$ and $\text{rank } \mathcal{M}_{(-1,1,1)} = 3$. By these two examples we see that the number of the minimal elements of a coset depends on the lattice and the combinatorial type of the corresponding face of $2D$.

These examples suggest the following theorem which gives an algebraic relation among the lattice points lying on the boundary of $2D$. The proof of this theorem can be found also in [14].

Theorem 5 ([14]) *Let $\mathbf{x}$ be a lattice point in the relative interior of an $(n - k)$-dimensional face Π of $2D$. $(1 \leq k \leq n)$. Then there are q facets of $2D$ (denoted by $\Pi_1, \cdots, \Pi_q$) each containing the face Π such that the sum of their relevants $\mathbf{y}_1, \cdots, \mathbf{y}_q$ is equal to $\mathbf{x}$:*

$$\mathbf{x} = \mathbf{y}_1 + \ldots + \mathbf{y}_q.$$

The number of these facets is not greater than k (for instance in the previous example $k = 3$ and $q = 2$). The relevants $\mathbf{y}_i$ above are orthogonal to each other and so

$$\mathbf{x}^2 = \mathbf{y}_1^2 + \ldots + \mathbf{y}_q^2.$$

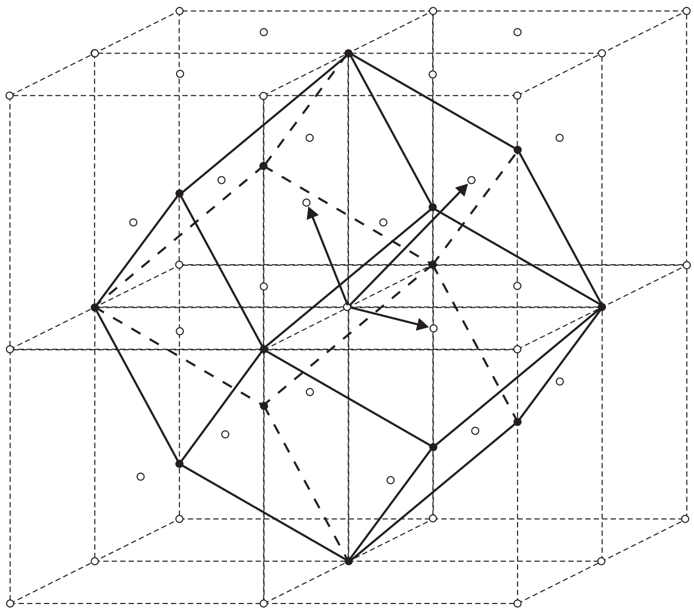

Figure 5: The regular simplex lattice and its extremal body $2D$.

4 On the nearest lattice point problem

The so-called *nearest lattice point problem* is an important problem of lattice geometry, convex optimization and other fields of mathematics. This is the following: How to find a lattice point of a fixed lattice L which is the nearest one to a given point of the space E^n? It is clear that the given point $\zeta \in E^n$ is in a D-V cell $D(\mathbf{x})$ of the D-V tiling of the lattice L, and the center $\mathbf{x}$ of this cell is the nearest lattice point to ζ. If we can choose a lattice hyperplane with the property that it is covered by all the D-V cells of L whose centers lie in this hyperplane, then the nearest lattice point problem for ζ and L can be simplified to the nearest lattice point problem for the orthogonal projection of ζ to this hyperplane and the $(n-1)$-dimensional sublattice of L lying the hyperplane considered. This principle would give the idea of a good algorithm to solve the original problem. Unfortunatelly, in general, there is no such a sublattice in higher dimensional spaces. Precisely the following theorem holds:

Theorem 6 ([13]) *For dimensions n=6,7,8 there exists an n-lattice L in* *which there is no sublattice L^* of dimension $(n-1)$ satisfying the assumption*

$$\cup(D + \mathbf{v} \mid \mathbf{v} \in L^*) \supset Lin[L^*]$$

where D and $Lin[L^*]$ denote the D-V cell of L and the subspace of dimension $(n-1)$ spanned by the sublattice L^*, respectively.

The proof of this theorem (see [13]) relies on the fact that if the sublattice L^* has the above-mentioned property then the difference between the number of minima of L and L^* is not greater than $4(n-1)$. On the other hand K.BEZDEK and T.ÓDOR proved in [3] that in the cases $n \leq 3$ we can choose such a sublattice L^* of the given lattice L. (See the problem survey [3].) By the papers [2] of BARNES and WALL and the other one [23] of Leech we proved that there is an infinite sequence of lattices in which the number of minima is not a polynomial function of the dimension n. (For details we refer to the papers [15],[16]). As a consequence there is an infinite series of lattices in which the required condition does not hold for its sublattices of dimension $(n-1)$.

The nearest lattice point problem motivates the following strict version of the previous one.

Whether a nearest lattice point can be found among the vertices of a given type of basic-parallelepiped containing this point?

In the paper [17] we have found a quantity which is characteristic for a lattice, if it is sufficiently small then the base has the desired property.

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be independent lattice vectors and denote by $G = (<\mathbf{e}_i, \mathbf{e}_j>)\; i,j = 1 \ldots n$ the Gram matrix of this system. The following matrix A is a modification of G:

$$A = \begin{pmatrix} 1 & \frac{<\mathbf{e}_1,\mathbf{e}_2>}{\mathbf{e}_1^2} & \cdots & \frac{<\mathbf{e}_1,\mathbf{e}_n>}{\mathbf{e}_1^2} \\ \frac{<\mathbf{e}_2,\mathbf{e}_1>}{\mathbf{e}_2^2} & 1 & \cdots & \frac{<\mathbf{e}_2,\mathbf{e}_n>}{\mathbf{e}_2^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{<\mathbf{e}_n,\mathbf{e}_1>}{\mathbf{e}_n^2} & \frac{<\mathbf{e}_n,\mathbf{e}_2>}{\mathbf{e}_n^2} & \cdots & 1 \end{pmatrix}.$$

Here $\mathbf{e}_i^2 = <\mathbf{e}_i, \mathbf{e}_i>$. The quantity mentioned above is the maximum norm of the inverse of this matrix A. We remark that the maximum norm of a vector $\mathbf{x}$ of E^n is defined as the maximal absolute value of its coordinates with respect to a fixed orthonormal basis of E^n and the maximum norm of the matrix A is defined by:

$$\|A\|_\infty = \max \left\{ \sum_{j=1}^{n} |a_{ij}| \quad \middle| \quad i = 1 \ldots n \right\}.$$

The theorem is the following:

Theorem 7 ([17]) *If $\|A^{-1}\|_\infty \le 2$ then for every point ζ of the parallelepiped spanned by the vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ holds the property: The nearest lattice point to ζ lies among the vertices of its parallelepiped.*

The following example shows that this condition is only sufficient but not necessary.

Consider the lattice that is spanned by the vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$ having the same length. Assume that the angle of these vectors is acute. Since the basic-parallelogram $P[\mathbf{e}_1, \mathbf{e}_2]$ of this lattice is covered by the D-V cells of the vertices of P, one of the lattice points nearest to the point ζ of P is a vertex of P. At the same time for the system $\{\mathbf{e}_1, \mathbf{e}_2\}$

$$A^{-1} = \frac{1}{1 - \frac{<\mathbf{e}_1,\mathbf{e}_2>^2}{\mathbf{e}_1^2\mathbf{e}_2^2}} \cdot \begin{pmatrix} 1 & -\frac{<\mathbf{e}_1,\mathbf{e}_2>}{\mathbf{e}_1^2} \\ -\frac{<\mathbf{e}_1,\mathbf{e}_2>}{\mathbf{e}_2^2} & 1 \end{pmatrix}$$

and so

$$\|A^{-1}\|_\infty = \|A\|_\infty \cdot \frac{1}{1-\cos^2\alpha} \ge \frac{1}{1-\cos^2\alpha},$$

where α is the angle of the examined vectors. Of cours this quantity is graeter than two if $0 < \alpha < \frac{\pi}{6}$.

A simple proof of Theorem 7 is the following:

Proof: By continuity, it sufficies to consider the case when the point ζ is an inner point of P. Let the lattice vector $\mathbf{x}^*$ be a solution of the problem for ζ. Then

$$(\mathbf{x}^* - \zeta)^2 \le (\mathbf{x}^* - \zeta \pm \mathbf{e}_k)^2 \text{ for } k = 1 \ldots n.$$

This means that

$$\pm 2 < \mathbf{x}^* - \zeta), \mathbf{e}_k > +(\mathbf{e}_k)^2 \ge 0 \text{ for} k = 1 \ldots n.$$

From this inequality we have

$$\left| \frac{< \mathbf{x}^* - \zeta), \mathbf{e}_k >}{(\mathbf{e}_k)^2} \right| \le \frac{1}{2}.$$

But the vector $(\mathbf{x}^* - \zeta)$ can be written as linear combination of the vectors $\mathbf{e}_i$ so

$$\mathbf{x}^* - \zeta = \sum_{i=1}^{n} (x_i - \zeta_i)\mathbf{e}_i$$

where by the assumptions $0 < \zeta_i < 1$ and $x_i's$ are integers. This means that

$$\left|\frac{< \mathbf{x}^* - \zeta), \mathbf{e}_k >}{(\mathbf{e}_k)^2}\right| = \left|\sum_{i=1}^{n}(x_i - \zeta_i)\frac{< \mathbf{e}_i, \mathbf{e}_k >}{(\mathbf{e}_k)^2}\right| \leq \frac{1}{2}.$$

In this formula the sum is a coordinate of the vector $A(\mathbf{x}^* - \zeta)$. Hence

$$\|A(\mathbf{x}^* - \zeta)\|_\infty \leq \frac{1}{2}$$

and so

$$\|(\mathbf{x}^* - \zeta)\|_\infty = \|A^{-1}A(\mathbf{x}^* - \zeta)\|_\infty \leq \|A^{-1}\|_\infty \cdot \|A(\mathbf{x}^* - \zeta)\|_\infty \leq 1.$$

This shows that $|x_i - \zeta_i| \leq 1$ for every i where x_i are integers and $0 < \zeta_i < 1$. Thus x_i equal to 0 or 1 which proves the statement. Q.E.D

Note that if the parallelepiped is spanned by vectors $\mathbf{e}_i$ which have the same length, say m, then $A = \frac{1}{m} \cdot G$ where G is the Gram matrix of the lattice. This means that $A^{-1} = m \cdot G^{-1}$ so the norm of A^{-1} is the product of the length of the edges of P and the maximum norm of the Gram matrix of the dual to the system $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. This is a basis of the dual lattice L^{-1} to L. The elements of the dual basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ are defined as normal vectors of the faces $P[\mathbf{e}_2, \ldots, \mathbf{e}_n]$,. . .,$P[\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}]$ furthermore, the lengthes of these vectors are equal to the reciprocal distances of the corresponding parallel opposite faces of P, say $\frac{1}{m_1}, \ldots, \frac{1}{m_n}$, respectively. This means that the maximum norm of G^{-1} is "small" if there are orthogonal walls of the parallelepiped P. Thus the condition can be used in that cases if the walls of the parallelepiped nearly perpendicular to each other.

On the other hand it is clear that a parallelepiped spanned by a regular simplex holds the desired property, the nearest lattice point to an inner point of the parallelepiped can be found among the vertices of it. This motivates the following discussion of this pharagraph. We first introduce the concept of the *k-dimensional coveredness* of a lattice parallelepiped. We now assume that L is generated by the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Denote by P the lattice parallelepiped spanned by these vectors, and let $D^n(Q)$ be the D-V cell of a vertex Q of P and denote D the cell of the origin.

Definition 1 *The k-dimensional skeleton of a parallelepiped P is the union of its k-dimensional faces. The parallelepiped P of the lattice L is k-dimensionally covered if its k-dimensional skeleton is covered by the union of n-dimensional D-V cells $D^n(Q)$ where Q runs over the vertices of P.*

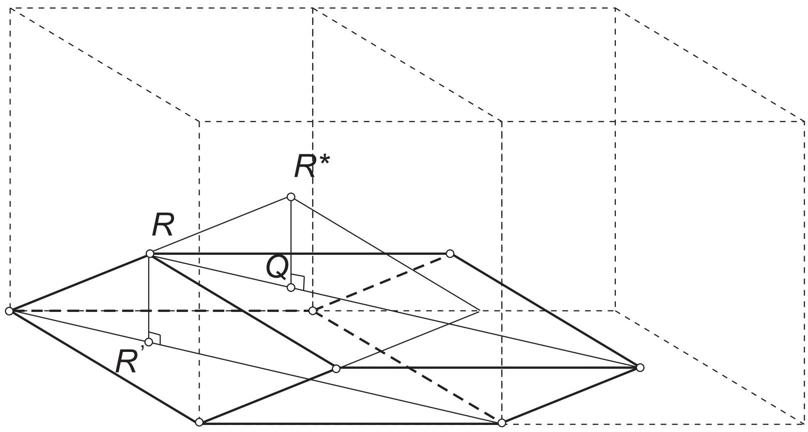

Figure 6:

It is clear that every lattice-parallelepiped P is 0-dimensionally covered, furthermore P is 1-dimensionally covered if and only if the endpoints of its edges $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ are on the boundary of the extremal body $2D$. It is not surprising that the k-dimensional coverdness implies the l-dimensional one if $l \leq k$. As a particular case the n-dimensional coveredness means that the nearest lattice point to a fixed point of P can be found among the vertices of P. The question is now how we can guarantee the n-dimensional coveredness of a lattice parallelepiped P? The following statement formulates the fact that the n and $(n-1)$-dimensional coverednesses are equivalent properties.

Theorem 8 *If the parallelepiped is $(n-1)$-dimensionally covered then it is n-dimensionally covered, too.*

Proof: Assume that the union of D-V cells

$$K = \cup\{D^n(Q) \quad | \quad Q \text{ is a vertex of } P\}$$

doesn't cover the parallelepiped P but it covers the $(n-1)$-skeleton $P \setminus \mathrm{int} P$. Then there exists a point $\mathbf{x}$ in the interior of P which is lying in the set $D^n(R) \setminus K$, where R is lattice point out of P. But $D^n(R)$ is convex (so connected), thus there exists such a point S on the $(n-1)$-dimensional skeleton of P which is lying in $D^n(R)$ and –by assumption– also covered by K. As our D-V cells have disjoint interiors hence we have a contradiction. Q.E.D.

The following example shows that the k-dimensional coveredness does not implies the n-dimensional one if $k \leq (n-2)$.

Assume that $n = 3$ and $k = 1$ and regard the lattice L spanned by the basis:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

As in Figure 6 we can readily verify that the orthogonal projection R' of the point R on the plane $[\mathbf{e}_1, \mathbf{e}_2]$ is in the cell $D^n(R)$. Hence the analogous projection Q of R^* on the plane $[\mathbf{e}_1, \mathbf{e}_2]$ in the region $D^n(R^*)$ and P is not 2 or 3-dimensionally covered. However, P is 1-dimensionally covered because the minimal distance of the midpoints of the vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$ to the lattice points is equal to $\frac{1}{2}$. (The closed cells around the endpoints of the edges of the parallelepiped contain the midpoint of the edges.)

The following theorem relates to the k and $(k-1)$-dimensional coveredness. Before the formulation of this theorem we introduce some notations.

Let $D_{i_1,\ldots,i_k}$ be the D-V cell of the origin with respect to the lattice $L_{i_1,\ldots,i_k}$ is spanned by the vectors $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}$. Denote by $P_{i_1,\ldots,i_k}$ the k-dimensional face of P containing the origin and being spanned by the vectors $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}$. Furthermore, let$V_{i_1,\ldots,i_k}$ be the k-dimensional subspace of E^n containing the face $P_{i_1,\ldots,i_k}$.

Theorem 9 *If the number k is less than n the following two statements are equivalent:*

1. *The parallelepiped P is k-dimensionally covered in the lattice L.*

2. *P is $(k-1)$-dimensionally covered and for each set of indices for which $1 \leq_1 < \ldots < i_k \leq n$ the following equality holds:*

$$D_{i_1,\ldots,i_k} = D \cap V_{i_1,\ldots,i_k}.$$

The staightforward proof is of technical character and omitted here.

Using this theorem it is not to hard to show that a parallelepiped which is spanned by a *super acute* simplex is n-dimensionally covered. We define by induction this type of simplices.

Definition 2 *A two dimensional simplex is super acute if it is acute triangle. A k-dimensional simplex super acute if its $(k-1)$-dimensional faces are super acute and the angles of its $(k-1)$-dimensional faces are acute.*

Our definition has been motivated by the fact that an l-dimensional face of such a simplex contains the centre of its circumscribed ball. From this observation we can see that for all k $(1 \leq k \leq n-1)$ the D-V cell $D_{i_1,\dots,i_k}$ of the sublattice $L_{i_1,\dots,i_k}$ is just equal to $D \cap V_{i_1,\dots,i_k}$ in the theorem above. In case $k = 2$ the k-dimensional coveredness follows from the assumption of super acuteness hence a parallelepiped spanned by a super acute simplex is always n-dimensionally covered.

We remark that the regular n-simplex is a super acute one, this implies that its well-known lattice A_n (see [7]) solves of nearest lattice point problem.

5 D-V cells in E^2 and E^3

Using Theorem 4 and Theorem 5 in the preceding section we derive the well-known combinatorial classification of D-V cells in dimensions two and three, respectively. For the classical derivation and other details we refer e.g. to the monograph of L.FEJES TÓTH [11].

In the plane E^2 we have only two combinatorial types of D-V cells like in Fig.1 and Fig.3, respectively. To prove this, we refer to our Theorem 4 about the relevant vectors. The double lattice $2L$ has 3 (non-zero) cosets in the original lattice L. From among these 3 cosets at least two contain an opposite pair of relevant vectors. If in the third coset thereis only one pair of shortest vectors then our D-V cell is a central symmetric hexagon, while being two pairs of minima then our D-V cell is a rectangle by Theorem 5. vectors mod2 from which at least two contain an opposite pair of relevant vectors. If in the third coset there is only one pair of shortest vectors then the D-V cell is a central symmetric convex hexagon and if in this coset being two distinct pair of minima then (by the Theorem 5) the D-V cell is a rectangle.

The situation in the space is more complicated. Before the discussion this classification we give an elementary lemma without proof.

Lemma 1 *Let P be a central symmetric convex hexagon in the plane. All the combinatorially possible decompositions of P into 2,3 or 4 central symmetric convex parts like a face-to-face tiling of P can be seen in Fig.7 .*

We need the concept of *zone* of a D-V cell of the space. Since the faces of such a polyhedron are central symmetric any edge e determines a zone of faces in which each face has two sides equal and parallel to the given edge e. The following lemma describes the types of the zones of a D-V cell.

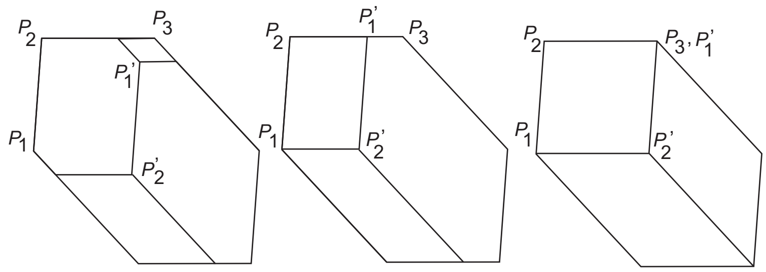

Figure 7: Decompositions of a central symmetric convex hexagon .

Lemma 2 *The number of the opposite pairs of relevant faces corresponding to a given zone is two or three. If it is two then the corresponding faces are orthogonal to each other.*

Proof: In fact, the D-V tiling decomposes into layers, each being uniquely determined by one of its cells, starting, for instance, with D at the origin. The considered zone of D determines a whole set of equal zones meeting one another along whole faces. The layer of D consists of the cells surrounded by these zones. The relevant vectors of the faces of this zone are orthogonal to the given edge e (which determines the zone) thus they are in a plane perpendicular to e at the origin. This means that the centres of D-V cells belonging to this layer lie on this plane. Furthermore the orthogonal projection of this layer on this plane is a lattice tiling where the tiles are D-V cells of the lattice of the considered centres. From the classification of the types of D-V cells of the plane we get the statements of the lemma. Q.E.D.

Now we can formulate the theorem.

Theorem 10 *There are only five different combinatorial types of D-V cells in the Euclidean 3-space E^3. The most symmetric representatives of them are the cub, the regular hexagonal prism, the elongated dodecahedron (bounded by a tetrahedral zone of regular hexagons and two caps each consisting of four rhombi), the rhombic dodecahedron and the truncated octahedron, respectively. These most important representatives are pictured in Fig.8 in combinatorially equivalent form.*

Proof: The basis of our discussion is the number σ of opposite pairs of relevants. Referring to the statements of Theorem 4, we have to discuss the respective cases of $\sigma = 3, 4, 5, 6$ and 7. We show that corresponding to the cases of $\sigma = 3, 4, 5$ there is only one type of D-V cells, in the case $\sigma = 6$

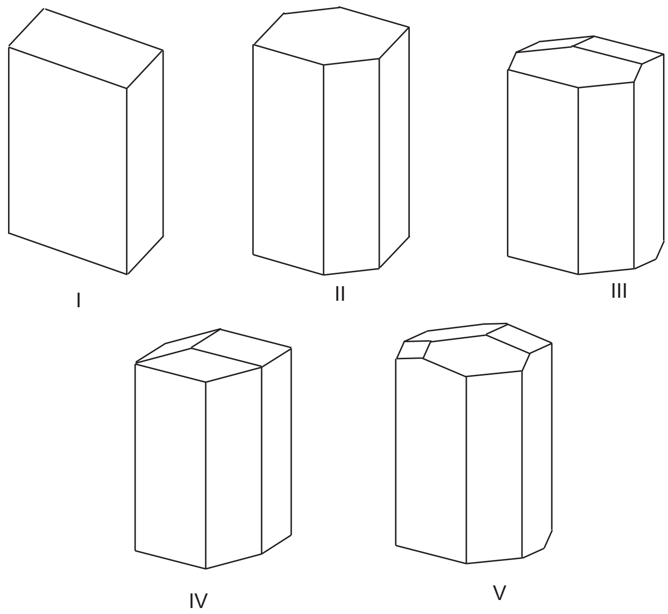

Figure 8: D-V cells of the space

there are two different types while if $\sigma = 5$ there is no such a topological face-edge-vertex complex which can be realized as a D-V cell. These types of polyhedra are illustrated in Fig.8 I,II,III, IV, and V, respectively, and we have Fig. 9 showing those D-V cells III,IV,V which are realizable as the cells of certain centered brick lattices. Let start our discussion with the case of

1. $\sigma = 3$. Since the number of faces of the polyhedron is six by the Lemma 2 the D-V cell is a brick. (See in Fig.8 I.)

2. When $\sigma = 4$ the D-V cell has eight faces. On the base of Lemma 2 again, there is a hexagonal zone of D that means that the polyhedron is a hexagonal prism. (See Fig.8 II.) (If there is no hexagonal zone of D then by Lemma 2 would exist four pairwise ortogonal vectors of the three-dimensional space which is a contradiction.)

3. In the third case $\sigma = 5$. This means that the polyhedron has ten faces and at least one hexagonal zone. The orthogonal projection of a cap to the plane of relevants of this zone decomposes the projection of the zone into two central symmetric convex components. (A cap would

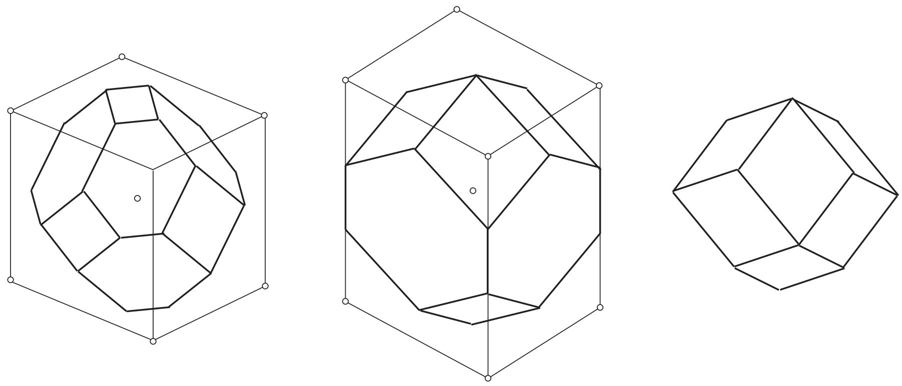

Figure 9: The possible D-V cells of a centered brick lattice

have two central symmetric faces by central symmetry of D.) But the projection of the zone is a central symmetric convex hexagon by Lemma 2 thus by virtue of Lemma 1 there is no D-V cell in this case.

4. Let now be $\sigma = 6$. Since the D-V cell has 12 faces, analogously to the previous case, we have to decompose a convex central symmetric hexagon into three parts, each of them is a central symmetric convex polygon. By Lemma 1 we get two different combinatorial types of decompositions, the first contains a hexagon and two parallelograms, the second consists of three parallelograms. Both possibilities can be realize as a D-V cell. See Fig.8 III,IV and Fig.9, respectively.

5. Finally, if $\sigma = 7$ then the D-V cell has 14 walls. By Lemma 1 a cap of the polyhedron consists of two hexagons and two parallelograms, respectively. The corresponding D-V cell is pictured in Fig.8 V and Fig.9, respectively.

Q.E.D.

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