On the Dirichlet-Voronoi cell of unimodular lattices *

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Abstract

This paper consists of two results concerning the DIRICHLET-VORONOI cell of a lattice. The first one is a geometric property of the cell of an integral unimodular lattice while the second one gives a characterization of all those lattice vectors of an arbitrary lattice whose multiples by $\frac{1}{2}$ are on the boundary of the cell containing the origin. This result is a generalization of a well-known theorem of VORONOI characterizing the so called relevants of the cell.

1 Introduction

The concept of DIRICHLET-VORONOI cell (also DIRICHLET cell or VORONOI region, briefly D-V cell), introduced in two classical papers by DIRICHLET [3] and VORONOI [8],[9] respectively, is useful in solving a lot of different problems.

In the first part of this paper we deal with some questions concerning the geometric properties of the D-V cell D of an integral, unimodular lattice of dimension n. We define a *lattice-section* (orthogonal to the direction $\mathbf{x}$) of D as the intersection of D and a lattice-hyperplane orthogonal to a fixed relevant vector $\mathbf{x}$. (The relevant vectors of D are the lattice vectors actually needed to define the region D.) We formulate some relations between the volumes of the lattice sections and the volumes of the cells $D, D_{\mathbf{x}}, \overline{D}_{\mathbf{x}}$ where the last two polyhedra are the D-V cells of the lattices $L \cap H_{\mathbf{x}}$, where $H_{\mathbf{x}}$ is the subspace of E^n orthogonal to $\mathbf{x}$ and the orthogonal projection of L to the hyperplane $H_{\mathbf{x}}$, respectively. We investigate the lattice sections from the point of view of symmetry, and raise some questions about it.

In the second part of this paper we prove an interesting result on a generalization of Voronoi's theorem on the relevants of the cell D. Using this

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theorem one can easily derive the well-known combinatorial classification of D-V cells in dimensions two and three, respectively [5]. For the classical derivation and other details we refer e.g. to the monographs of L.FEJES TÓTH [4] and B.N.DELONE [2]. We think that these results will play an important role in the solution of further classification problems on D-V cells.

2 Definitions

Let L be an n-dimensional integral, unimodular lattice. This means that L is an integral lattice (so the scalar product of any two vectors from L is an integer), with determinant $\det L = 1$. Here $\det L = \det G = \det\,(A^T \cdot A)$ where A is the coordinate matrix of a basis of L and G is the so-called Gram-matrix of the lattice L.

Let $\mathbf{x}$ be a vector of L. Then there is a sublattice $L_{\mathbf{x}} = L \cap aff L_{\mathbf{x}}$ of dimension $(n-1)$ containing the origin and orthogonal to $\mathbf{x}$. (See [1] or [6].) We denote by $H_{\mathbf{x}}$ the subspace of dimension $(n-1)$ spanned by $L_{\mathbf{x}}$. The lattice vectors corresponding to the hyperplanes which contain a proper facet ($(n-1)$-dimensional face) of the D-V cell D of L are called **relevants**. (D consists of those points $\mathbf{y}$ of E^n whose distance from the origin is not greater than its distance from any other points of the set L.) The height $k_{\mathbf{x}}$ of a vector $\mathbf{x} \in L$ is the number of those lattice-hyperplanes parallel to (and different from) $H_{\mathbf{x}}$ which intersect the segment $[0, \mathbf{x}]$.

Let $\mathbf{x}$ be a relevant. It is clear that the usual width of D (in the direction $\mathbf{x}$) is $|\mathbf{x}| = \sqrt{N(\mathbf{x})}$ while the lattice width of D is equal to $k_{\mathbf{x}}$. Let $D_{\mathbf{x}}$ and $\overline{D}_{\mathbf{x}}$ be the D-V-cells of the lattice $L_{\mathbf{x}} = L \cap H_{\mathbf{x}}$ and the orthogonal projection $\overline{L}_{\mathbf{x}}$ of L to the space $H_{\mathbf{x}}$, respectively. In fact, the projection $\overline{L}_{\mathbf{x}}$ is also a lattice [6]. We have the following relations for the regions $D_{\mathbf{x}}$, $\overline{D}_{\mathbf{x}}$ and $D \cap H_{\mathbf{x}}$:

$$D_{\mathbf{x}} \supset D \cap H_{\mathbf{x}} \supset \overline{D}_{\mathbf{x}}.$$

The first relation is trivial while the second one follows from the fact that if $\zeta \in \overline{D}_{\mathbf{x}}$ and there is a lattice vector in L for which $\mathbf{y} \neq \mathbf{0}, \mathbf{x}$, $N(\mathbf{y}-\zeta) < N(\zeta)$ then for the orthogonal projection $\mathbf{y}^*$ of $\mathbf{y}$ the inequalities $N(\mathbf{y}^* - \zeta) \leq N(\mathbf{y} - \zeta) < N(\zeta)$ hold. This is impossible because $\mathbf{y}^* \in \overline{L}_{\mathbf{x}}$ and ζ is in the cell $\overline{D}_{\mathbf{x}}$.

Throughout this paper the **lattice-sections** of D are written in the form

$$D \cap (H_{\mathbf{x}} + i \cdot \mathbf{e}_{\mathbf{x}}) \text{ where } i \text{ runs from } -\left[\frac{k_{\mathbf{x}}-1}{2}\right] \text{ to } \left[\frac{k_{\mathbf{x}}}{2}\right]$$

and $\mathbf{e_x}$ is a lattice vector with the property:

$$L = \cup\{(L_{\mathbf{x}} + i \cdot \mathbf{e_x}) \text{ where } i \text{ is integer}\}.$$

(The function $[\cdot]$ denotes the usual integer part function.)

3 On the volume of the lattice-sections

Let $\mathbf{x}$ be a relevant of the D-V-cell of the integral unimodular lattice L. The following theorem gives a relation between the volumes of the lattice sections and the volume of the D-V-cell of the lattice $L_{\mathbf{x}}$.

Theorem 1 *If L is a lattice then using the introduced notation we have the following equality:*

$$v(D_{\mathbf{x}}) = \sum_{i=-\left[\frac{k_{\mathbf{x}}-1}{2}\right]}^{\left[\frac{k_{\mathbf{x}}}{2}\right]} v(D \cap (H_{\mathbf{x}} + i \cdot \mathbf{e_x})),$$

where the function $v(\cdot)$ denotes the volume function of its argument. (In this case the argument is of dimension $n-1$ thus $v(\cdot)$ is the $(n-1)$-dimensional volume function.)

Proof: Let R be a large real number and consider the ball G of dimension $(n-1)$ with the radius R and center 0. Furthermore, let N_R^0 be the number of those lattice points which are in this ball. Similarly, let N_R^i be the number of those lattice points of $L_{\mathbf{x}} + i \cdot \mathbf{e_x}$ whose orthogonal projections to the subspace $H_{\mathbf{x}}$ are in the ball G. It is clear that the volume of the ball can be calculated asymptotically (for large R) as the product of the volume of $D_{\mathbf{x}}$ and the number N_R^0 and in another way as the sum of the numbers $N_R^i \cdot v(D \cap (H_{\mathbf{x}} + i \cdot \mathbf{e_x}))$ when i runs from $-\left[\frac{k_{\mathbf{x}}-1}{2}\right]$ to $\left[\frac{k_{\mathbf{x}}}{2}\right]$ since $v(D \cap (H_{\mathbf{x}} + i \cdot \mathbf{e})) = v(D + i \cdot \mathbf{e} \cap (H_{\mathbf{x}}))$. But the numbers N_R^i are asymptotically equal to each other so we have the required equality:

$$v(D_{\mathbf{x}}) = \sum_{i=-\left[\frac{k_{\mathbf{x}}-1}{2}\right]}^{\left[\frac{k_{\mathbf{x}}}{2}\right]} v(D \cap (H_{\mathbf{x}} + i \cdot \mathbf{e_x})). \qquad \text{Q.E.D.}$$

We remark that the above statement holds for general lattices and general "lattice hyperplane" H, taking all the sections of D with lattice hyperplanes parallel to H. The proof is word by word the same, only projecting along some lattice direction not parallel to H (or it follows as the union of the projections of the sections onto H is a (possible disconnected) tile for the projection of L). The volume of the D-V-cell $\overline{D}_{\mathbf{x}}$ of the orthogonal projection lattice $\overline{L}_{\mathbf{x}}$ can be given in the following way:

Lemma 1

$$v(\overline{D}_{\mathbf{x}}) = \frac{1}{k_{\mathbf{x}}} v(D_{\mathbf{x}}).$$

From the definition of the dual (or polar) lattice $L_{\mathbf{x}}^{-1}$ of $L_{\mathbf{x}}$ it is obvious that for an integral lattice L we have $L_{\mathbf{x}}^{-1} \supseteq L_{\mathbf{x}}$ and this relation is an equality if and only if the number $k_{\mathbf{x}}$ is equal to 1. Moreover, if $\mathbf{y}_1 \in \overline{L}_{\mathbf{x}}$ then there is an element $\mathbf{y} \in L$ for which $\mathbf{y} = \mathbf{y}_1 + \alpha\mathbf{x}$ where α is a real number. If now $\mathbf{z} \in L_{\mathbf{x}}$ is arbitrary then $\mathbf{z} \cdot \mathbf{y} = \mathbf{z} \cdot \mathbf{y}_1$ so $\mathbf{y}_1 \in L_{\mathbf{x}}^{-1}$. (Here the product " $\cdot$ " means the inner product of the vectors.) Therefore $\overline{L}_{\mathbf{x}} \subset L_{\mathbf{x}}^{-1}$ hence we have:

$$L_{\mathbf{x}} \subset \overline{L}_{\mathbf{x}} \subset L_{\mathbf{x}}^{-1}.$$

This relation holds for every integral lattice. If the examined lattice is unimodular and $\mathbf{x}$ is one of its relevants we can say a little bit more.

Lemma 2 *Let L be an integral unimodular lattice with a relevant $\mathbf{x}$. Then*

$$\overline{L}_{\mathbf{x}} = L_{\mathbf{x}}^{-1}.$$

These two lemmas are simple consequences of Lemma 1 and Theorem 1 of the paper [7]. In fact, from the second part of Lemma 1 of [7] we get that $\det(\overline{L}_{\mathbf{x}}) = \frac{1}{|\mathbf{x}|}$. where $\det(\textit{lattice}) = v(\text{ DV-cell})$ and from Theorem 1 (a) and Theorem 1 (b) of [7] we have the equalities $\det(L_{\mathbf{x}}) = |\mathbf{x}|$, and $k_{\mathbf{x}} = |\mathbf{x}|^2$, respectively. Now Lemma 1 is obvious and also Lemma 2:

$$\det(\overline{L}_{\mathbf{x}}) = \frac{1}{|\mathbf{x}|} = \frac{1}{\det(L_{\mathbf{x}})} = \det L_{\mathbf{x}}^{-1}.$$

We remark that Lemma 2 holds for any $\mathbf{x}$ not only for relevants.

The following theorem is an interesting consequence of the above lemma.

Theorem 2 *Let D be the D-V cell of an integral, unimodular lattice L, and $\mathbf{x}$ be one of its relevant. Then*

$$v(D) = \frac{|\mathbf{x}|}{k_{\mathbf{x}}} \cdot \sum_{i=-\left[\frac{k_{\mathbf{x}}-1}{2}\right]}^{\left[\frac{k_{\mathbf{x}}}{2}\right]} v(D \cap (H_{\mathbf{x}} + i \cdot \mathbf{e}_{\mathbf{x}})) = |\mathbf{x}| \cdot M_{\mathbf{x}}$$

where $M_{\mathbf{x}}$ is the average value of the volume of the lattice sections of D perpendicular to the direction $\mathbf{x}$.

Proof: From the equations

$$\frac{k_{\mathbf{x}}^2}{|\mathbf{x}|^2} = \det(L_{\mathbf{x}}) = |\mathbf{x}|^2$$

(see in the proof of Lemma 2) we have the equality $|\mathbf{x}|^2 = k_{\mathbf{x}}$. So from Theorem 1 and Lemma 1 we get

$$v(\overline{D}_{\mathbf{x}}) = \frac{1}{k_{\mathbf{x}}} \sum_{i=-\left[\frac{k_{\mathbf{x}}-1}{2}\right]}^{\left[\frac{k_{\mathbf{x}}}{2}\right]} v(D \cap (H_{\mathbf{x}} + i \cdot \mathbf{e}_{\mathbf{x}})) = \frac{|\mathbf{x}|}{k_{\mathbf{x}}}.$$

This gives the required formula:

$$v(D) = 1 = \frac{|\mathbf{x}|}{k_{\mathbf{x}}} \cdot \sum_{i=-\left[\frac{k_{\mathbf{x}}-1}{2}\right]}^{\left[\frac{k_{\mathbf{x}}}{2}\right]} v(D \cap (H_{\mathbf{x}} + i \cdot \mathbf{e}_{\mathbf{x}})). \qquad \text{Q.E.D.}$$

It is well-known that the D-V cell and its $(n-1)$-dimensional faces are centrally symmetric. This means that we have one or two lattice-sections which are centrally symmetric corresponding to the cases $k_{\mathbf{x}}$ is odd $(i=0)$ or $k_{\mathbf{x}}$ is even $(i=0,\ i=[\frac{k_{\mathbf{x}}}{2}])$, respectively. The following natural question is unsolved: *What is the necessary and sufficient condition that all the lattice-sections should be centrally symmetric?* If the norm of a relevant is one or two the value $k_{\mathbf{x}}$ is the same. This means that the corresponding lattice sections are centrally symmetric. We think that the converse statement is also true, if each lattice section perpendicular to the relevant $\mathbf{x}$ is centrally symmetric then the norm of $\mathbf{x}$ is one or two. Since the lattice vectors with norms one or two are precisely the roots of the lattice this conjecture says that all the lattice sections of $\mathbf{x}$ are centrally symmetric if and only if $\mathbf{x}$ is a root of the lattice.

4 Lattice vectors in $2D$.

G.F.VORONOI proved in [9] that the lattice vector $\mathbf{x}$ is a relevant of the cell D if and only if it is a minimum vector of the coset $\mathbf{x} + 2L$ and if the vector $\mathbf{y}$ is another minimal element of this coset then $\mathbf{y} = \pm\mathbf{x}$. In this paragraph we generalize this result. The geometric meaning of this theorem is the following: The lattice vectors $\pm\mathbf{x}$ are in the relative interior of an $(n-1)$-dimensional face of the body $2D$ if and only if they are the unique pair of minima of the coset $\mathbf{x} + 2L$ if and only if the hyperplane orthogonal to $\mathbf{x}$ through $\frac{1}{2}\mathbf{x}$ intersects D. We show the following theorem:

Theorem 3 *If a lattice vector* $\mathbf{x}$ *is on the boundary of the body 2D then it is a minimum vector of its coset* $\mathbf{x}+2L$*. If moreover the rank of the set*

$$\mathcal{M}_{\mathbf{x}} := \{\mathbf{m} \in L | \mathbf{m} \text{ is a minimum vector of the coset } \mathbf{x}+2L\} \ni \mathbf{x}$$

is equal to k $(k = 1, \ldots, n)$ *then the elements of* $\mathcal{M}_{\mathbf{x}}$ *are in the relative interiors of certain* $(n-k)$*-dimensional parallel faces of* $2D$*, these faces being distinct for distinct elements of* $\mathcal{M}_{\mathbf{x}}$*.*

Proof: If $\mathbf{x}$ is in the relative interior of an $(n-k)$-dimensional face of $2D$ then the point $\frac{1}{2}\mathbf{x}$ is in the corresponding $(n-k)$-dimensional face of the D-V cell D. Let $\mathbf{y}$ be an arbitrary element of $L \setminus 0$ then the closed halfspace

$$H_{\mathbf{y}} := \{\alpha \in E^n \quad | \quad \mathbf{y} \cdot \alpha \leq \frac{1}{2}\mathbf{y} \cdot \mathbf{y}\}$$

contains the point $\frac{1}{2}\mathbf{x}$ which means that $\mathbf{x} \cdot \mathbf{y} \leq \mathbf{y} \cdot \mathbf{y}$. Regarding now the norm of the element $\mathbf{x} - 2\mathbf{y}$ we get that

$$N(\mathbf{x} - 2\mathbf{y}) = N(\mathbf{x}) + 4N(\mathbf{y}) - 4\mathbf{x} \cdot \mathbf{y} \geq N(\mathbf{x})$$

which proves the first part of the statement.

Let $\mathbf{x}$ be a shortest element of its coset. Then for every lattice vector $\mathbf{y} \in L \setminus 0$ the inequality

$$N(\mathbf{x} - 2\mathbf{y}) = N(\mathbf{x}) + 4N(\mathbf{y}) - 4\mathbf{x} \cdot \mathbf{y} \geq N(\mathbf{x})$$

implies that the vector $\frac{1}{2}\mathbf{x}$ is in the halfspaces

$$H_{\mathbf{y}} := \{\alpha \in E^n | \mathbf{y} \cdot \alpha \leq \frac{1}{2}\mathbf{y} \cdot \mathbf{y}\}, \text{ for every } \mathbf{y} \text{ of } L \setminus 0$$

which means that it is on the boundary of D. In this case there is a face Π of dimension $n-l$ of D the relative interior of which contains the point $\frac{1}{2}\mathbf{x}$ $(1 \leq l \leq n)$. Since those tac-hyperplanes of D which contain the point $\frac{1}{2}\mathbf{x}$ also contain the face Π, we can see that the intersection of the hyperplanes corresponding to the lattice vectors $\mathbf{y} \in L$ defined by the equality $N(\mathbf{x} - 2\mathbf{y}) = N(\mathbf{x})$ contains Π, too. So the flat $\cap\{\mathrm{bd}(H_{\mathbf{y}}) | \mathbf{y} \in \mathcal{M}_{\mathbf{x}}\}$ is of dimension not less than $(n-l)$. This means that the rank of $\mathcal{M}_{\mathbf{x}}$ (which is k by the assumption) is at most l therefore $k \leq l$. On the other hand we show $k \geq l$. Namely there are at least l independent facet hyperplanes $\mathrm{bd}(H_{\mathbf{y}_i})$, $i = 1, \ldots, \sigma$ of D which contain an $(n-l)$-dimensional face of D (see e.g. [6]). But for these relevants $\mathbf{y}_i$ we have the same property that $\mathbf{x} - 2\mathbf{y}_i \in \mathcal{M}_{\mathbf{x}}$

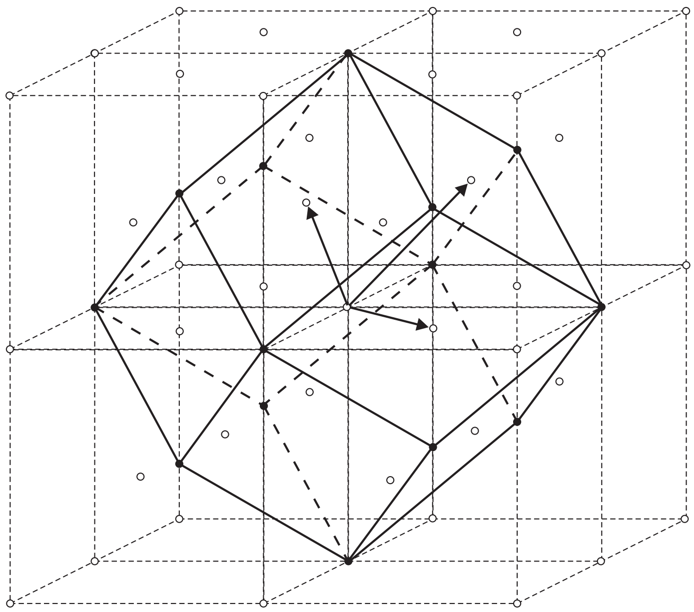

Figure 1: The regular simplex lattice and its body $2D$.

so $k = \operatorname{rank}(\mathcal{M}_{\mathbf{x}}) \geq l$. This means that $k = l$. If now $\mathbf{x} - 2\mathbf{y} \in \mathcal{M}_{\mathbf{x}}$, then $N(\mathbf{x} - 2\mathbf{y}) = N(\mathbf{x}) = \min\{N(\mathbf{x} - 2\mathbf{y} - 2\mathbf{z}) \mid \mathbf{z} \in L\}$, whence $\mathbf{x} - 2\mathbf{y} \in \operatorname{bd}(2D)$. Therefore we can repeat for $\mathbf{x} - 2\mathbf{y}$ the considerations made for $\mathbf{x}$, which proves the second part of the statement, except the distinctness of the $(n-k)$-faces whose relative interiors contain distinct elements of $\mathcal{M}_{\mathbf{x}}$. Let now e.g. $\mathbf{x} \neq \mathbf{x} - 2\mathbf{y} \in \mathcal{M}_{\mathbf{x}}$, $\frac{1}{2}\mathbf{x} \in \operatorname{relint}\Pi$, $\frac{1}{2}\mathbf{x} - \mathbf{y} \in \operatorname{relint}\Pi^*$, where Π, Π^* are $(n-k)$-faces of D. By the above proved facts we have that $\operatorname{aff}\Pi$ ($\operatorname{aff}\Pi^*$, respectively) is the translate of $\cap\{\operatorname{bd}(H_{\mathbf{y}} \mid \mathbf{y} \in \mathcal{M}_{\mathbf{x}}\}$ containing $\frac{1}{2}\mathbf{x}$ ($\frac{1}{2}\mathbf{x} - \mathbf{y}$ respectively). Moreover $\cap\{\operatorname{bd}(H_{\mathbf{y}} \mid \mathbf{y} \in \mathcal{M}_{\mathbf{x}}\}$ is orthogonal to $\mathbf{y}$, hence $\Pi \neq \Pi^*$. Q.E.D.

Remark 1: The number of elements of $\mathcal{M}_{\mathbf{x}}$ depends on the lattice. In the three-dimensional cubic lattice there are three types of lattice points belonging to the closed body $2D$. The vertex coordinates of $2D$ are congruent to $(1,1,1)$ componentwise mod2 (we take the coordinates with respect to the edge vectors of the basic cube of the lattice). The vertices are the minimal elements of the coset of $(1,1,1)$. So $|\mathcal{M}_{(1,1,1)}| = 8$ and $\operatorname{rank} \mathcal{M}_{(1,1,1)} = 3$.

Consider now the so-called regular-simplex lattice of dimension 3. We can construct this lattice from the cubic lattice taking in addition the centers of the 2-dimensional faces of the basic cube also to lattice points. A basis

$\{ \mathbf{e}_i \mid i = 1, 2, 3\}$ of this lattice points to centres of any three cube faces meeting in a cube vertex as origin. In Fig.1 we see the body $2D$ which is a rhombic dodecahedron. We have two types of lattice vectors on the boundary of $2D$. E.g. the vertex $(-1, 1, 1)$ of $2D$ is a lattice point. The minimal elements of the coset of this point – denoted by double circles in Fig.1 – are the endpoints of the longer diagonals of the rhombic faces. This means that $|\mathcal{M}_{(-1,1,1)}| = 6$ and rank $\mathcal{M}_{(-1,1,1)} = 3$. By these two examples we see that the number of the minimal elements of a coset depends on the lattice and the combinatorial type of $2D$.

These examples suggest the following theorem which gives an algebraic relation among the lattice points lying on the boundary of $2D$.

Theorem 4 *Let* $\mathbf{x}$ *be a lattice point in the relative interior of an* $(n-k)$*-dimensional face* Π *of* $2D$ *(*$1 \leq k \leq n$*). Then there are* q *facets of* $2D$ *(denoted by* $\Pi_1, \cdots, \Pi_q$*) each containing the face* Π *such that the sum of their relevants* $\mathbf{y}_1, \cdots, \mathbf{y}_q$ *is equal to* $\mathbf{x}$*:*

$$\mathbf{x} = \mathbf{y}_1 + \ldots + \mathbf{y}_q.$$

The number of these facets is not greater than k *(for instance in the previous example* $k = 3$ *and* $q = 2$*). The relevants* $\mathbf{y}_i$ *above are orthogonal to each other and so*

$$\mathbf{x}^2 = \mathbf{y}_1^2 + \ldots + \mathbf{y}_q^2.$$

Proof: We note that if $\mathbf{x}$ is a relative interior point of an $(n-k)$-dimensional face Π of $2D$ then, for each relevant $\mathbf{y} \neq \mathbf{x}$, that corresponds to a facet of $2D$ containing the face Π we have that the vector $\mathbf{x} - \mathbf{y}$ is an inner point of an $(n-l)$-dimensional face of $2D$, where $l < k$. In fact the vector $\mathbf{x} - \mathbf{y}$ is the midpoint of the segment with the respective endpoints $\mathbf{x}$ and $\mathbf{x} - 2\mathbf{y}$. Here we have $|\mathbf{x}| = |\mathbf{x} - 2\mathbf{y}|$ and thus $\mathbf{x} - 2\mathbf{y} \in \mathcal{M}_{\mathbf{x}}$. In fact, $|\mathbf{x}| = |\mathbf{y} + (\mathbf{x} - \mathbf{y})| = |\mathbf{y} - (\mathbf{x} - \mathbf{y})| = |\mathbf{x} - 2\mathbf{y}|$, since both $\mathbf{x}$ and $\mathbf{y}$ lie in a hyperplane with normal $\mathbf{y}$. From the the proof of the previous theorem we can see that these endpoints $\mathbf{x}$ and $\mathbf{x} - 2\mathbf{y}$ are relative inner points of one of two parallel faces of dimension $(n-k)$, Π and, say, Π^*, respectively. ($\mathrm{aff}\Pi^* = \mathrm{aff}(\Pi - 2\mathbf{y})$). But $\mathbf{x} - \mathbf{y}$ is not the zero vector thus it is not in the interior of $2D$ so it is in the relative interior of an $(n-l)$-dimensional face Π^1 of $2D$ where l is less than k. So for an arbitrary relevant $\mathbf{y}_1$ like above, $\mathbf{x}$ is the sum of the vectors $\mathbf{y}_1$ and $\mathbf{x} - \mathbf{y}_1$ where the second vector is in the relative interior of a face Π^1 of dimension greater than $(n-k)$. The original face Π belongs to this new face so facet containing Π^1 also contains Π. Let Π_2 be such a facet and Π_1 be the facet which corresponds to the relevant

$\mathbf{y}_1$. If $\mathbf{y}_2$ is the relevant of the new facet then the lattice vector $\mathbf{x}-\mathbf{y}_1$ can be decomposed to the vectors $\mathbf{y}_2$ and $(\mathbf{x}-\mathbf{y}_1)-\mathbf{y}_2$, respectively. Here the new vector $(\mathbf{x}-\mathbf{y}_1)-\mathbf{y}_2$ is in the relative interior of such a face Π^2 which contains the face Π^1 and has a dimension strictly greater than that of Π^1. This means that the statement of the theorem related to the decomposition is easy to prove by induction. The orthogonality of the relevants can be seen in the following way. The second facet Π_2 contains the second face Π^1 so also contains the segment parallel to $\mathbf{y}_1$ with the endpoints $\mathbf{x}$ and $\mathbf{x}-2\mathbf{y}_1$. Thus the normal vector $\mathbf{y}_2$ of Π_2 is orthogonal to $\mathbf{y}_1$. In the following step of the above construction we define a face Π^2 which contains the face Π^1 therefore also contains the above segment. The affin hull of this new face Π^2 contains the affine hull of the sets Π^1 and $\Pi^1-2\mathbf{y}_2$. From this definition we see that the facet Π_3 containing the face Π^2 (and thus Π^1) contains the segment parallel to $\mathbf{y}_2$ with the respective endpoints $\mathbf{x}-\mathbf{y}_1$ and $(\mathbf{x}-\mathbf{y}_1)-2\mathbf{y}_2$ (and the segment parallel to $\mathbf{y}_1$ with the endpoints $\mathbf{x}$ and $\mathbf{x}-2\mathbf{y}_1$). This means that the normal vector $\mathbf{y}_3$ of this facet is perpendicular to the vectors $\mathbf{y}_1$ and $\mathbf{y}_2$. From this follows the orthogonality of the vectors $\mathbf{y}_i$ by induction. The last statement of the theorem is obvious from the orthogonality. Q.E.D.
Corollary: For every n-dimensional lattice if the length of a lattice vector is greater than $n^{\frac{1}{2}}\cdot\max\{|\mathbf{y}_i| \mid \mathbf{y}_i \text{ is a relevant of the cell } D\}$ then it is not in the closed body $2D$.
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