Polygons with equal angles in the hyperbolic plane *

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Dec. 2, 2002

Abstract

In this paper we will show that if the long diagonals of a $2n$-polygon with equal angles meet at one point then the common perpendicular of its opposite sides also contains this point. Furthermore the polygon is centrally symmetric and regular if and only if the distance of the sides to this point are equal to each other. We give an analogous statement for $2n+1$-polygons in the hyperbolic plane.

In [4] and [5] the second author examined the following problem: "What is the condition for a 5-sided polygon or a hexagon in the hyperbolic plane with respective five or six right angles to be a regular one?"

To solve the problem on 5-sided polygons it is easy; it can be seen that the lines through a vertex perpendicular to the opposite side meet at one point and the polygon is regular iff the distances of this common point to the sides are equals.

For such an investigated hexagon is not obvious but thrue that there is a common perpendicular of a pair of its opposite sides and these three common perpendicular meet at one point, too. (See e.g. exercise 19.8.27 in [1].) By projectiv method I.Vermes proved that if the long diameters of the hexagon meet at one point then this point is the meeting point of the common perpendiculars, the hexagon is centrally symmetric and regular

*Partially supported by Hung.Nat.Found for Sci.Reseach (OTKA) grant no. T038397 (2002) and by the J.Bolyai fellowship (2000)

†2002

iff the distances of its center to the sides are equal. The proof was very complicated, concentrated to the orthogonality.

In this paper we considerably generalize this latter statement concentrated only to the equality of the angles. We prove for an analouge statement as the latter theorem on hexagons, for every n and α where α is the angle of the n-sided polygon holding the inequalities: $0 \leq \alpha \leq \frac{n-2}{n}\Pi$.

Theorem 1 *Let n be an integer, $\mathcal{P} = \{A_1, \dots, A_n\}$ an n-sided polygon with equal angles in the hyperbolic plane.*

1. *If $n = 2k+1$ and the lines through a vertex A_i perpendicular to the opposite sides meet at one point O, then the vertices of P on the rays $r_i = OA_i$ are uniquely determined.*

2. *If $n = 2k$ and the long diameters A_iA_{k+i} of the polygon meet at one point O, $\mathcal{P}$ is centrally symmetric and the common perpendiculars of the opposite sides meet at the point O, too.*

Proof: First we note (see e.g. [3], [2]) that two polygons with equal angles have the same area on the hyperbolic plane. This means that if $\mathcal{P} = \{A_1, \dots, A_n\}$ and $\mathcal{P}' = \{B_1, \dots, B_n\}$ are two n-sided polygons whose vertices are on the rays $r_i = OA_iB_i$ and whose angles are equal to the number α ($0 < \alpha < \frac{n-2}{n}\Pi$) then there is a pair of corresponding sides A_iA_{i+1}, B_iB_{i+1} intersecting each other. (See Fig.1.)

Consider now this pair of sides and assume that on the rays r_i and r_{i+1} the order of the points are (OA_iB_i) and $(OB_{i+1}A_{i+1})$, respectively. Let denote α_i and β_i the angles $A_{i-1}A_iO$ and $B_{i-1}B_iO$, respectively. Thus we have the inequalities

$$\beta_{i+1} > A_iB_{i+1}O > \alpha_{i+1}$$

and therefore

$$\alpha - \beta_{i+1} < \alpha - \alpha_{i+1} = OB_{i+1}B^* = OA_{i+1}A_{i+2}.$$

So, the point B_{i+2} is on the segment $OB^* \subset OA_{i+2}$. On the other hand, from the equality of the angles $OB_{i+1}B^*$ and $OA_{i+1}A_{i+2}$ follows the inequality $\alpha_{i+2} > B_{i+1}B^*O > \beta_{i+2}$ and we can use an inductive argument. This argument shows that in all of the rays we have the order of points (OB_jA_j), so on the ray r_i there are two points corresponding to the second polygon. This is a contradiction.

Thus we proved that for a fixed system of rays there is only one possibility to place of the vertices of a polygon with equal angles. If n is even the

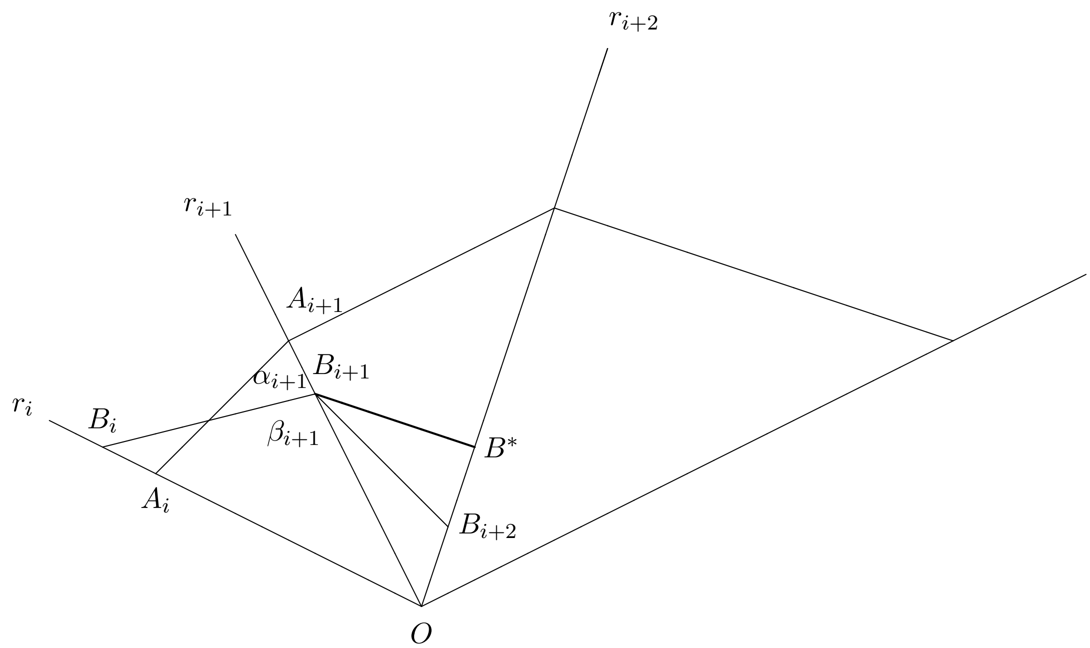

Figure 1: A pair of corresponding intersecting sides

reflected image of a required polygon in the center O is also n-sided polygon with equal angles and thus from the unicity we get that such a polygon is centrally symmetric one. Since the reflection in O changes the opposite sides of $\mathcal{P}$ the point O is intersection of the axe of symmetry and the common perpendicular of these sides. This proves the theorem.

Corollary: Assuming the conditions 1. or 2. to a n-sided polygon it is **regular** if and only if the distances from the point O to the sides are equal.

Remark: The proof of the fact that there is no two intersecting pairs use only the theorem on the outer (external) angle of a triangle which is true in the euclidean case, too. On the other hand a similarity does not change the angles of a polygon, so may assume that $\mathcal{P}$ and $\mathcal{P}'$ have the same area as in the hyperbolic case. Thus we get that in the **euclidean case the above unicity also hold apart from similarities**.

References

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