Extremal polygons with minimal perimeter *

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Abstract

In this paper we will investigate an isoperimetric type problem in lattices. If K is a bounded $\mathbf{0}$-symmetric (centrally symmetric with respect to the origin) convex body in E^n of volume $v(K) = 2^n \det L$ which does not contain non-zero lattice points in its interior, we say that K is *extremal* with respect to the given lattice L. There are two variations of the isoperimetric problem for this class of polyhedra. The first one is: *Which bodies have minimal surface area in the class of extremal bodies for a fixed n-dimensional lattice?* And the second one is: *Which bodies have minimal surface area in the class of extremal bodies with volume 1 of dimension n?* We characterize the solutions of these two problems in the plane. There is a consequence of these results, the solutions of the above problems in the plane give the solution of the lattice-like covering problem: Determine those centrally symmetric convex bodies whose translated copies (with respect to a fixed lattice L) cover the space and have minimal surface area.

1 Preliminary results, notation

A *lattice* L of the Euclidean n-space E^n is defined as the set of all integer linear combinations of the elements of a basis of E^n. The vector $\mathbf{m}$ is a *minimal* one if it is one of the shortest non-zero vectors of L. (The length of a vector is considered with respect to the usual Euclidean norm of E^n.)

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The Dirichlet-Voronoi cell (briefly *D-V cell*) $D(\mathbf{x})$ of a lattice point $\mathbf{x}$ is the collection of those points of the space which are not farther to $\mathbf{x}$ than to any other point of the lattice. It is clear from the definition above that L is invariant under the translations by the lattice vectors, and the reflections in a lattice point or in the midpoint of any lattice segment, respectively. From this immediately follows that any two D-V cells are translated copies of each other and, moreover, any cell and its $(n-1)$-dimensional faces (called *facets*) are centrally symmetric convex sets, respectively. The definition of $D(\mathbf{0})$ implies (by virtue of the fact that a lattice is a discrete point system) that it is a polyhedron defined as a finite intersection of certain half-spaces each of which contains the origin and is bounded by the midhyperplane of a lattice segment connecting the origin with a lattice point. The collection of the cells $D(\mathbf{x})$ for $\mathbf{x} \in L$ forms a so-called lattice tiling of the space E^n. Tiling means that their union covers the space and their interiors are mutually disjoint. This tiling is *face-to-face*, so in particular, any facet of the tile $D(\mathbf{x})$ is also a facet of another tile. It is obvious that the D-V cell D is bounded and we assume in this paper that D also is closed, so this region is compact. The volume of D is equal to the volume of a *basic parallelepiped* of L which is spanned by the vectors of a basis of the lattice. ($v(D) = \det L$.)

Minkowski proved the following basic theorem:

Theorem 1 ([?]) *A bounded centrally symmetric convex body K in E^n with centre at the origin $\mathbf{0}$ and volume $v(K) > 2^n v(D)$ contains at least one lattice point different from $\mathbf{0}$. (D is the D-V cell of L, $v(\cdot)$ is the n-dimensional volume function.)*

From this theorem immediately follows that an $\mathbf{0}$-symmetric convex body which does not contain non-zero lattice points in its interior (so-called *empty body*) has a volume at most $2^n v(D)$. H.MINKOWSKI introduced the concept of an *extremal body with respect to the lattice L* which is an empty $\mathbf{0}$-symmetric closed convex body with volume $2^n v(D)$. An example is $2D$, where D is the D-V cell of L.

In his book he investigated this class of bodies and proved some interesting theorems on it. First of all he characterized the elements of this class:

Theorem 2 ([?]) *Let K be a (bounded) $\mathbf{0}$-symmetric convex body. Then K is extremal if and only if the following two properties hold:*

a, *The space E^n is covered by the bodies $\frac{1}{2}K + \mathbf{u}$ where $\mathbf{u} \in L$.*

b, *Each point $\mathbf{x} \in E^n$ belongs to at most one body $\frac{1}{2} intK + \mathbf{u}$,* *where int$K$ means the interior of the body K.*

Second he proved the following statements:

Theorem 3 ([?]) *If the body K is an extremal one (with respect to a lattice L) then K satisfies the conditions:*

1. *K is a polytope.*
2. *K is centrally symmetric.*
3. *Each facet of K is centrally symmetric.*

Finally in [?] he showed that for an extremal body the following properties hold:

1. *At most $2(2^n - 1)$ lattice points belong to the relative interiors of the facets of K,*
2. *K has at most $2(2^n - 1)$ facets,*
3. *On the boundary of K there lie at least $2(2^n - 1)$ lattice points.*
4. *The relative interiors of the facets of K contain lattice points. -*

In his works [?] and [?] VORONOI also studies this class of polyhedra. He introduced the concept of *parallelohedron* as a convex polyhedron P whose translates by a lattice L cover E^n and they have disjoint interiors. So a polyhedron P is a parallelohedron if and only if $2P$ is extremal with respect to a lattice L. It is clear that e.g. the D-V cell D of the lattice L is a parallelohedron.

Later B.A.VENKOV [?] and P.McMULLEN [?] independently examined the class of those polyhedra whose translates cover E^n and have disjoint interiors. They introduced the concept of belt of such a polytope. This can be defined in the following way: If G is an $(n-2)$-face of K, then G lies in two facets of K, say F and F'. Since F is centrally symmetric, it has an $(n-2)$-face G' opposite to G, which is the intersection of F' with another facet F'', say. Carrying on in this way, we find a **belt** of facets $F, F', F'', \ldots, F^{(k)} = F$, say, such that each $F^{(i-1)} \cap F^{(i)}$ is a translate of G or of $-G$. Now there holds the following theorem of characterization:

Theorem 4 ([?],[?]) *The conditions*

1. *K is a polytope,*

2. K is centrally symmetric,

3. Each facet of K is centrally symmetric,

4. Each belt of K contains 4 or 6 facets,

are necessary and sufficient for a convex body to tile E^n with translations.

To show that the conditions are sufficient the authors described a suitable candidate for tiling of E^n by translates of K. Then they proved that the given candidate is a tiling, and from the definition it was clear that it is also a face-to-face and lattice-like one. This can be given by the following simple way: if F denotes a facet of K then there is a translation vector $\mathbf{t}_F$ carrying $-F$ into F, and the tiling is the family $\mathcal{K}$ of certain translates of K defined by $\mathcal{K} = \{K + \mathbf{t} | \mathbf{t} \in T\}$, where $T = \{\sum_F n_F \mathbf{t}_F | n_F \in Z\}$.

One of the most important concepts in the theory of the lattice covering and packing problems is the concept of *L-decomposition*. We say that an n-dimensional ball is a *solid ball* if it has the properties: It is an empty ball (does not contain lattice points in its interior), and on its surface there are at least $n+1$ independent lattice points. The convex hull of the lattice points lying on the boundary of a solid ball is an *L-polyhedron*, the collection of L-polyhedra gives the so-called *L-partition* (or L-decomposition) of the space with respect to the original lattice. This is in fact a tiling as we can see in Delone's papers [**?**, **?**]. In the 2-dimensional case the existence of an extremal body with minimal perimeter follows from compactness considerations (Blaschke selection theorem), since 0-symmetric compact convex sets having perimeters less than some constant lie in some fixed circle about 0. (We remark that also in the n-dimensional case there exists an extremal body of minimal surface area. This can be proved by compactness considerations, and standard estimates of the semiaxis of the John's ellipsoids, that yield that the diameters of compact convex sets with fixed volume and bounded surface area themselves are bounded.)

2 The problems

Our questions now are the following:

1. *Which bodies have minimal surface area in the class of extremal bodies for a fixed n-dimensional lattice?*

2. *Which bodies have minimal surface area in the class of extremal bodies with volume 1, of dimension n (i.e., when also the n-dimensional lattice is varying)?*

The purpose of this paper is to answer these questions in the case of the plane. From the theorem of McMullen and Venkov we know that in the 2-dimensional case the extremal polygons are either parallelograms or centrally symmetric hexagons. So the answer to the second question follows from the solution of the isoperimetric problem for convex polygons: the optimal one is the regular hexagon. The corresponding lattice is the regular triangular lattice. The answer to the first question is more complicated, it will be contained in the third pharagraph.

We conjecture that the solution of the first problem is the solution of the following one:

1'. *Give that* $\mathbf{0}$*-symmetric convex body whose translated copies by the given lattice* L *form a covering of the space and has minimal surface area in the class of all such bodies.*

We remark that in the plane the solution of problem 1 is the solution of problem 1' because in [**?**] the authors for sake of completeness described a proof of the fact, that if $\frac{1}{2}K$ is an $\mathbf{0}$-symmetric convex body whose translated copies by the given lattice L form a covering then there is a possibly degenerate centrally symmetric convex hexagon $K' \subset \frac{1}{2}K$ whose translated copies by the lattice L form a tiling. It is clear that the perimeter of this hexagon is not greater than the perimeter of $\frac{1}{2}K$.

If this conjecture is true then the solution of the second problem is the solution of the following one:

2'. *Give that* $\mathbf{0}$*-symmetric convex body with volume* 1 *whose translated copies by some lattice* L *form a covering of the space and has minimal surface area in the class of all such bodies.*

In fact a solution of the latter problem has a translation lattice L and by the conjecture above the optimal body is an extremal one with respect to this lattice.

3 The case of the plane

First we prove a lemma which says that the optimal polygon with respect to a fixed lattice gives an edge-to-edge tiling of the plane.

Lemma 1 *Denote by* L *a fixed lattice of the Euclidean plane and let* K *be*

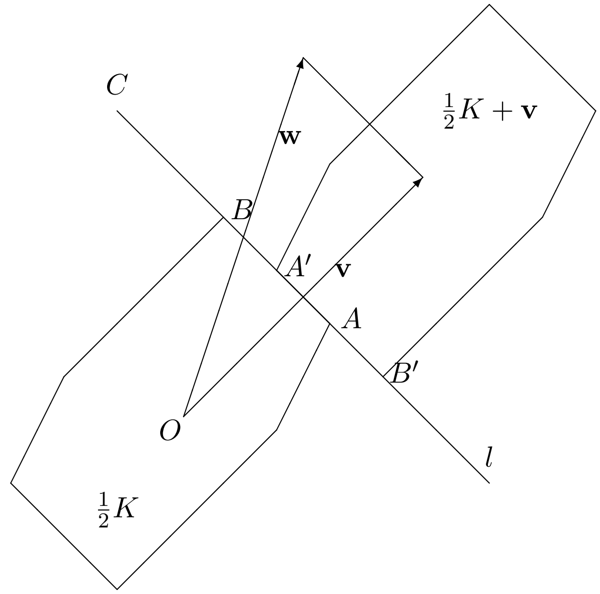

Figure 1:

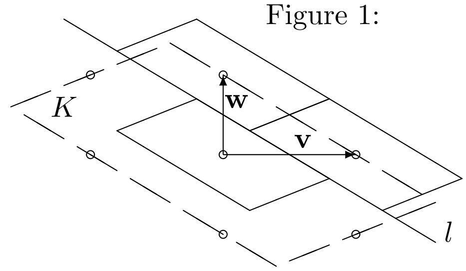

Figure 2:

an extremal polygon with respect to L with minimal perimeter. Then the midpoints of the edges of K are lattice points.

Proof: Using the statement 4 of the introduction we know that the relative interiors of the edges of K contain lattice points.

Suppose that there is an edge AB of K with lattice point $\mathbf{v}$ which is a relative inner point but it is not the midpoint of AB. There is a lattice translate $\frac{1}{2}K+\mathbf{v}$ of $\frac{1}{2}K$, that also is the mirror image of $\frac{1}{2}K$ in $\frac{1}{2}\mathbf{v}$. Let the mirror image of AB be $A'B'$. (See Fig.1.) Since $\frac{1}{2}\mathbf{v}$ is not the midpoint of AB, therefore e.g. A' is a relative inner point of AB. Then there is a lattice translate $\frac{1}{2}K+\mathbf{w}$ of $\frac{1}{2}K$, such that its vertex $B'+\mathbf{w}-\mathbf{v}$, that corresponds by the lattice translation to the veretex B' of $\frac{1}{2}K+\mathbf{v}$, coincides with A'. This is clearly impossible for $\frac{1}{2}K$ a hexagon. So $\frac{1}{2}K$ is a parallelogram, and the lattice tiling of translates of $\frac{1}{2}K$ is composed of rows, which correspond to one-dimensional lattices $\{n \cdot AB\}$, cf. Fig.2.

Now we prove that in this case K is not an optimal polygon. Consider the construction of Fig.3. Here the parallelogram partly drawn with broken lines is $K=(2A)(2B)(-2A)(-2B)$ and has on its boundary six lattice points, C, $D=\mathbf{v}$, $E=\mathbf{w}$, F, G, H. Let I be the mirror image of O in the idpoint of CD. Then the perimeter of K is $4((2A)C+(2A)D+(2A)I)$. However, in the case that each angle of the triangle CDI is less than $\frac{2\pi}{3}$, $(2A)$ is not the

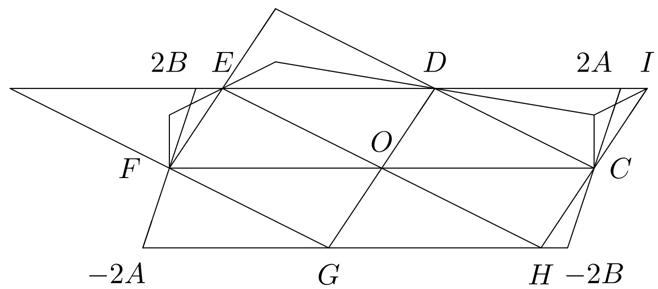

Figure 3: Definition of a tile better than a parallelogram one for a not edge-to-edge tiling

isogonal point of this triangle, and in the contrary case $(2A)$ is not a vertex of this triangle. Therefore there exists a point P_1 in this triangle, such that

$$(2A)C + (2A)D + (2A)I > P_1C + P_1D + P_1I.$$

Let us now reflect P_1 in D, obtaining P_2, P_2 in E, obtaining P_3, etc. Thus we obtain six points $P_1, \dots, P_6$, whose convex hull is a hexagon K', of perimeter $(P_1C + P_1D + P_1I)$, that is less than that of K. Hence K is not an optimal polygon, as claimed. $\square$

The corollary of this lemma is that the lattice tiling containing the optimal extremal polygon is an edge-to-edge one. From this we can prove that

Lemma 2 *For every lattice the optimal extremal polygon is a hexagon.*

Proof: By the above proved facts it suffices to exclude the case of an edge-to-edge lattice tiling of parallelograms. Suppose we have an edge-to-edge tiling of parallelograms. Then Fig.3. degenerates, and we may suppose $2B = E$, $2A = I$, and $(2B)(2A)(-2B) \leq \frac{\pi}{2}$. Than $2A = I$ is a vertex of the triangle CDI, with $CID\angle < \frac{2\pi}{3}$. Therefore there exists a point P_1 like in the proof of Lemma 1, which yields a contradiction. $\square$

Now we introduce some new notation. If H is an $\mathbf{0}$-symmetric, convex lattice hexagon whose centre is the origin, vertices are lattice points and which does not contain another lattice point in its interior or in its edges, then it can be decomposed into six empty lattice triangles, denoted by $1, 2, \dots, 6$. (See in Fig.4.) We denote the triangles which are the reflected images of the triangles $1, 2, \dots, 6$ in the centres of their edges opposite to the origin by $1', 2', \dots, 6'$, respectively.

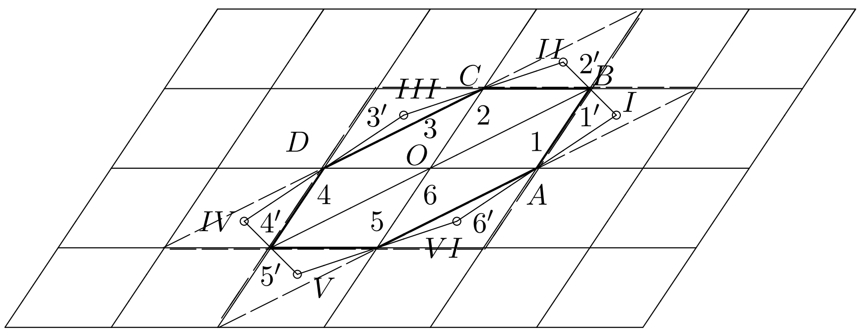

Figure 4: Extremal polygon is defined by reflections

Lemma 3 *Let H be an $\mathbf{0}$-symmetric, convex lattice-hexagon which does not contain another lattice point in its interior or in its edges. Now let K be an extremal polygon with the property that the midpoints of its edges are the vertices of H. Then the perimeter of K is minimal if and only if its vertices are the isogonal points of the triangles $1', 2', \ldots, 6'$. Provided each angle of these triangles is less than $\frac{2\pi}{3}$, and the vertices with angles at least $\frac{2\pi}{3}$, provided there are such vertices.*

Proof: From the property that the midpoints of the edges of K are the vertices of H we see that the vertices of K can be obtained by successive reflections in the vertices of H. Denote by $I, II, \ldots, VI$ the vertices of K corresponding to the triangles $1', 2', \ldots, 6'$. (The edges containing the vertex I also contain two vertices of the triangle $1'$.) If now the successive reflections take $1'$ into $2'$, $2'$ into $3'$,... then the image of I is II, that of II is III and so on. Now let the three vertices of the triangle $1'$ be P_1, P_2 and P_3, respectively. By the reflections we get that the perimeter of K is equal to

$$P(K) = 4(|IP_1| + |IP_2| + |IP_3|).$$

This proves the lemma. $\square$

We remark that in this lemma we did not use that the point I is a point of the closed triangle $1'$. It is easy to see that for a fixed hexagon H there is no such extremal polygon K which can arise from an outer point of $1'$ by successive reflections in the vertices of H. On the other hand if the point I is a point of the closed triangle $1'$, then the above method gives an extremal polygon. In fact, if the first four vertices of H are A, B, C, D respectively, then the half of the polygon K has area

$$a(\frac{1}{2}K) = 3a(\Delta_{0AB}) + a(\Delta_{AIB}) + a(\Delta_{BIIC}) + a(\Delta_{CIIID}) = 4a(\Delta_{0AB}).$$

So the area of K is $2^2 \det L$. (See Fig.4.)

Now we prove the main result of this paper. We note that in the case when the L-partition contains rectangles as L-polygons, we call L-triangle such a triangle whose three vertices are vertices of an L-rectangle. In this case there will be an ambiguity for K in the next theorem.

Theorem 5 *Let L be a fixed lattice of the plane, K be an extremal polygon with minimal perimeter. Then the vertices of K are the isogonal points of the triangles $1', 2', \dots, 6'$, which arise from a lattice-hexagon H consisting of six L-triangles of L having a common vertex (see Fig.4).*

Proof: From Lemma 2 we know that the optimal extremal polygon is a hexagon. By Lemma 1 we get that the midpoints of its edges are lattice points, and the convex hull of these points is an affine regular lattice-hexagon H containing only one lattice point in its interior and no lattice points in the relative interiors of its edges. (Assuming that there are at least two points in the interior of the hexagon, these points are in the interior of K, which is a contradiction. Assuming that e.g. the relative interior of AB contains a lattice point then also the relative interior of OC contains one a contradiction again.) The interior lattice point is the origin $\mathbf{0}$, and thus the common vertex of the empty lattice triangles $1, 2, \dots, 6$ with union H, is the origin. Now by Lemma 3 we get that the vertices of K are the isogonal points of the triangles $1', 2', \dots, 6'$ or their vertices with angles at least $\frac{2\pi}{3}$. (See Fig.4.) Since the perimeter of K is

$$P(K) = 4(|IP_1| + |IP_2| + |IP_3|)$$

we have to minimize the quantity $(|IP_1| + |IP_2| + |IP_3|)$.

First we assume that the point I is an inner point of the triangle $1'$. This means that triangle $1'$ has no angle greater than or equal to $\frac{2\pi}{3}$, and the angles $P_iIP_j\angle$ are equal to $\frac{2\pi}{3}$. Using the fact that if I is the isogonal point of the triangle Δ_{P_1,P_2,P_3}, then

$$|P_iP_j|^2 = |IP_i|^2 + |IP_j|^2 + |IP_i||IP_j|$$

for $i \neq j$, we get that

$(|IP_1| + |IP_2| + |IP_3|)^2 =$
$(|IP_1|^2 + |IP_2|^2 + |IP_3|^2) + 2(|IP_1||IP_2| + |IP_2||IP_3| + |IP_1||IP_3|) =$
$= \frac{1}{2}(|P_1P_2|^2 + |P_2P_3|^2 + |P_1P_3|^2) + \frac{3}{2}(|IP_1||IP_2| + |IP_2||IP_3| + |IP_1||IP_3|).$

We have

$$a(1') = \frac{1}{2}(|IP_1||IP_2| + |IP_2||IP_3| + |IP_1||IP_3|)\frac{\sqrt{3}}{2},$$

which means that the second examined sum

$$(|IP_1||IP_2| + |IP_2||IP_3| + |IP_1||IP_3|)$$

is constant. (We know that the areas of the empty lattice-triangles are equal.) So in this case the perimeter of K is minimal if and only if the sum $(|P_1P_2|^2 + |P_2P_3|^2 + |P_1P_3|^2)$ is minimal. But the triangle Δ_{P_1,P_2,P_3} is a lattice-triangle whose two edges give a basis of the lattice, so

$$(|P_1P_2|^2 + |P_2P_3|^2 + |P_1P_3|^2) \geq m_1 + m_2 + l,$$

where m_1 is the square of the length of a minimal vector $\mathbf{m}_1$ of the lattice, m_2 is the square of the length of such a shortest lattice vector $\mathbf{m}_2$ which is linearly independent from $\mathbf{m}_1$ and satisfies $< \mathbf{m}_1|\mathbf{m}_2 > \geq 0$, and l is the square of the length of the vector $\mathbf{m}_1 - \mathbf{m}_2$. In fact it can be proved easily that the lengths of the edges of the triangle above are not greater than the lengths of the corresponding edges of any other empty non-degenerate lattice triangle. (We compare the shortest edge with the shortest edge of the other triangle and so on. We distinguish two cases:

1,: $\Delta_{P_1P_2P_3}$ has a side parallel to $\mathbf{m}_1$. Then it suffices to investigate the case $P_1 = 0$, $P_2 = \mathbf{m}_1$, $P_3 = \mathbf{m}_2 + i\mathbf{m}_1$, i integer.

2,: $\Delta_{P_1P_2P_3}$ has no side parallel to $\mathbf{m}_1$. Then in the basis $\{\mathbf{m}_1, \mathbf{m}_2\}$ e.g. P_1, P_3 have second coordinates differing by atleast 2, and the minimal distance of any such points is greater than $\|\mathbf{m}_2 - \mathbf{m}_1\|$.)

In the **second** case, when I is a vertex of $1'$, the extremal polygon corresponding to the hexagon H is a parallelogram, because the respective image of the point I is left fixed by the first, second or third reflection. By virtue of Lemma 2 this extremal polygon is not an optimal one.

Thus we have proved the theorem. $\square$

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