Periodica Mathematica Hungarica Vol. 24 (3), (1992), pp. 189–192

ON A QUESTION OF K. BEZDEK AND T. ODOR ON Γ-PARALLELOTOPS

Á. G. HORVÁTH (Budapest)

Abstract

K. Bezdek and T. Odor proved the following statement in [1]: If a covering of E^3 is a lattice packing of the convex compact body K with packing lattice Λ (K is a Λ-parallelotopes) then there exists such a 2-dimensional sublattice Λ' of Λ which is covered by the set $\cup(K+z|z\in\Lambda')$. ($K\cup L(\Lambda')$ is a Λ'-parallelotopes). We prove that the statement is not true in the case of the dimensions $n=6,7,8$.

Definitions

Let $e_1,\dots,e_n$ be n independent points in E^n. Then the set Λ of points $z=\sum_{i=1}^{n} x_i e_i, (x_1,\dots,x_n\in Z$ integers) is called a lattice. The lattice Λ^* is a sublattice of Λ iff $\Lambda^*\subset\Lambda$. The vector m is a minimum of Λ if for every lattice-vectors $v\in\Lambda, |m|\le|v|$. The number of the minima of Λ is denoted by $s(\Lambda)$ and let s_n be the following number:

$$s_n=\max(s(\Lambda)|\Lambda\subset E^n). \tag{1}$$

Let $D(v)$ be the Dirichlet–Voronoi cell [2] of the lattice point v and D denote that the cell $D(O)$. If H is a k-dimensional vector system in E^n we shall denote by $L[H]$ that k-dimensional subspace of E^n which is spanned by the set H.

2. The conjecture and the result

Let $\Lambda\subset E^n$ be an n-lattice and $K\subset E^n$ an n-dimensional convex O-symmetric compact set, for which:

$$\begin{aligned}&(v+\operatorname{int}K)\cap(w+\operatorname{int}K)=0\quad v,w\in\Lambda, v\ne w \text{ and}\\ &\cup(cl(K)+v|v\in\Lambda)=E^n\end{aligned} \tag{2}$$

($cl(K)$ is the closure of K). The question is the following:

Supported by Hung. Nat. Found for Sci. Research (OTKA) grant no. 1615 (1991).
Mathematics subject classification numbers, 1980/85. Primary 52A45; Secondary 52A20.
Key words and phrases. Γ-parallelotops, lattice packing.

Akadémiai Kiadó, Budapest
Kluwer Academic Publishers, Dordrecht

Does it exists such an $(n-1)$-dimensional sublattice Λ^* of Λ, for which

$$\cup(cl(K) + v | v \in \Lambda^*) \supset L[\Lambda^*]? \tag{3}$$

The answer is positive when $n \leq 3$ (see [1]). We shall prove that in the case of $n = 6, 7, 8$, the conjecture is not true, so if the O-symmetric compact convex set K is the $D - V$ cell D of $\mathit{0}$, then the following statement is true:

THEOREM. *If $n = 6, 7, 8$ then there exists such an n-lattice Λ in which there is no sublattice Λ^* satisfying the assumption (3) with the convex body D.*

3. Lemmas

LEMMA 1. *Let $m \in \Lambda$ be a minimum vector when $\Lambda \subset E^n$ is a n-lattice. Then there exists such an $(n-1)$-dimensional face of D which is orthogonal to m and contain the midpoint of m.*

LEMMA 2. *If in the n-lattice Λ there exists such a $(n-1)$-sublattice Λ^* for which (3) holds, and $e_n \in \Lambda$ such that $\Lambda = \cup\{ke_n + \Lambda^* | k \in Z\}$ then*

$$\dim\{(e_n + L(\Lambda^*)) \cap cl(D)\} = \dim\{(-e_n + L(\Lambda^*)) \cap cl(D)\} \leq n-2 \tag{4}$$

Since these assertions are trivial we don't prove it. From these lemma we have the following;

LEMMA 3. *If Λ^*, Λ, e_n satisfy the assumption of Lemma 2 and $m \in \Lambda$ is a primitive lattice vector, being a normal vector of a $(n-1)$-face of the $D - V$ cell D then $m \in \Lambda^*$ or $m \in \Lambda^* \pm e_n$.*

PROOF. If $m \in ke_n + \Lambda^*$, where $|k| \geq 3$ or $|k| = 2$ and m is not orthogonal to the subspace $L(\Lambda^*)$ then there exists such a real number $\alpha > 1$ for which the following set is not empty:

$$\{\alpha e_n + L(\Lambda^*)\} \cap D \tag{5}$$

so in these cases we get:

$$\dim(\{e_n + L(\Lambda^*)\} \cap D) = n-1 \tag{6}$$

and the assumption (4) is not valid. For this reason we may assumed that $|k| = 2$ and the original lattice vector m is orthogonal to the subspace Λ^* or $|k| = 1$. Since in the first case the examined face of D is lying in the $(n-1)$ hiperplane $e_n + L(\Lambda^*)$ the statement is proved. ■

4. The proof of the theorem

Assume that the n-lattice Λ has n linearly independent minima and it can be found $(n-1)$-sublattice Λ^* of Λ which satisfy the condition (3). The number $s(\Lambda)$ of minima of Λ is equal to:

$$s(\Lambda) = \sigma + s(\Lambda^*) \tag{7}$$

where σ is the number of those minima which are not in the hiperplane $L(\Lambda^*)$. From the Lemma 1 and 3 we get:

$$\sigma \leq 2M(n-2, \phi) \tag{8}$$

where $M(n-2, \phi)$ is the maximal number of points in S^{n-2} so that distance between any two of them is at least ϕ for any ϕ. (S^{n-2} is the surface of the $(n-1)$-dimensional unite ball). Let ρ be the distance between $L(\Lambda^*)$ and $L(\Lambda^*) + e_n$ and r the radius of the following $(n-1)$-dimensional ball B:

$$B = \{x \,|\, |x| \leq |m|, m \text{ is a minimum of } \Lambda\} \cap L(\Lambda^*) + e_n. \tag{9}$$

From the Lemma 3 it follows that the orthogonal projection of the origin O onto the $(n-1)$-hiperplane $L(\Lambda^*) + e_n$ is not an inner point of the $D-V$ cell D, so we have $r^2 \leq \rho^2 = |m|^2 - r^2$ and $r \leq 2^{-1/2}|m|$. From this condition we get $\pi/2 \leq \phi$. We now apply the result of Hajós and Davenport (see [3] and [4]) which says that:

$$M(n-2, \pi/2) = 2(n-1) \quad \text{and} \quad M(n-2, \phi) \leq n \quad \text{for} \quad \pi/2 > \phi. \tag{10}$$

This means that:

$$s(\Lambda) - s(\Lambda^*) \leq 4(n-1) \tag{11}$$

holds in this lattice. Let Λ be an n-lattice for which the number $s(\Lambda) = s_n$ is the maximal. But Λ^* is an $(n-1)$-lattice with $s(\Lambda^*) \leq s_{n-1}$, therefore the answer for the n-dimensional question is negative if $s_n - s_{n-1} > 4(n-1)$. For this reason it can be seen by help of the result of Watson [5] that in the case of $n = 6, 7, 8$ the conjecture is not true because the abovementioned differences are 32, 54, 114 respectively. So the statement of the theorem is true.

REMARK. From the proof of this Theorem it may be conjectured that the answer for the question is negative if $n \geq 6$, because it probable that s_n is not a polynomial function of the dimension n. At the same time I think that the conjecture is true in the case of $n = 4$ and 5.

REFERENCES

[1] K. BEZDEK and T. ODOR, Λ-parallelotopes, *Period. Math. Hung.*

[2] P. M. GRUBER and C. G. LEKKERKERKER, *Geometry of Numbers,* North–Holland, Amsterdam 1987.

[3] H. DAVENPORT and G. HAJÓS, Problem 35, *Mat. Lapok,* **2** (1951), 68

[4] G. FEJES TÓTH, New results in the Theory of Packing and Covering, *Convexity and its Applications*, edited by P. M. Gruber and J. M. Wills, Basel, Birkhauser, 1983, 318–359.
[5] G. L. WATSON, The number of minimum points of a positive quadratic form, *Dissertationes Math.* **84** (1971), 46.

(Received March 5, 1991)

TECHNICAL UNIVERSITY OF BUDAPEST
FACULTY OF MECHANICAL ENGINEER
DEPARTMENT OF GEOMETRY
1521 BUDAPEST