Studies of the University of Žilina
Mathematical Series
Vol. 22/2008, 7–18

MAXIMAL CONVEX HULL OF CONNECTING SIMPLICES.

Á. G. HORVÁTH

ABSTRACT. This paper deals with the following question concerning the volume of the convex hull of two "connecting" simplices:

How do we have to place two simplices with common center in $\mathbb{E}^3$ as to maximize the volume of their convex hull? The answer to this simple question is depend on the dimension of the simplices and to prove the expected final results is not easy.

This paper contains the following two cases: either both of the simplices are regular triangles or are regular tetrahedra, respectively.

1. INTRODUCTION

The examined problem of this paper is strongly related with the following classical one:

How do we have to place v points on the unit sphere in E^3 so as to maximize the volume of their convex hull? For $v = 4, 6$ and 12 the solution is given by an inequality of L. Fejes Tóth (see in [4]) showing that the best configurations are the vertices of a regular tetrahedron, octahedron and icosahedron, respectively. The extremal configurations are known also for $v = 5, 7, 8$. In the case of $v = 5$ it is the triangular dipyramid as be seen. The other two cases was proved by Berman and Hanes in [1]. We remark that the optimal polyhedron in the case of $v = 8$ was discovered (earlier) by Grace (see in [3]). In the case of triangles we exclude the regular octahedron from the possibility. Our solution is a nonregular simplicial polyhedron combinatorially equivalent to an octahedron. In the c ase of tetrahedra our solution is a cube contrary to the optimal arrangements of eight points on the sphere. (See [2].)

The actuality of the present paper is argumented from an observation of L. Fejes Tóth (see p.127 in [5]) saying that "if P have maximal volume among all convex polyhedra with v vertices inscribed in a given sphere, P must be bounded solely by triangular faces". This means that the optimal arrangements of the above (seems to be more general than our) problem are different from that problem investigated in this paper.

Received April 16, 2008.
2000 *Mathematics Subject Classification.* Primary 52B60, 52B10.
Key words and phrases. simplex, convex hull, volume.

2. Two congruent regular triangle with common center

In this section we consider two congruent regular triangle with common center in the space, and search for that situation where the volume of the convex hull of their vertices are maximal. The respective vertices denoted by A, B, C and A', B', C' lie on the sphere of radius r around the common center O. It is easy to see that one of edges of the triangle $\Delta(A, B, C)$ intersects the other closed triangle $\Delta(A', B', C')$ and vice versa. Denote these two edges by BC and $B'C'$, respectively. If the convex hull of all vertices is a polyhedron (with non-empty interior) then the intersection of the two plane of two triangle is a line, say m. Denote by α and β the angles of the lines $(m, (AO))$ and $(m, (A'O))$, respectively, and by γ the angles of triangle planes. First of all observe that the examined convex body is the union of two disjoint non-convex double pyramids P and Q determined by the vertices A, B, C, B', C' and A', B', C', B, C, respectiv ely. (See on Fig. 1.) The volume of P is

$$v(P) = \frac{1}{3}\left[a(ABOC)(m_{B'} + m_{C'})\right],$$

where $a(ABOC)$ is the area of the (concave) quadrilateral $(ABOC)$ and $m_{B'}$ and $m_{C'}$ are the distances of the points B' and C' from the plane (ABC), respectively.

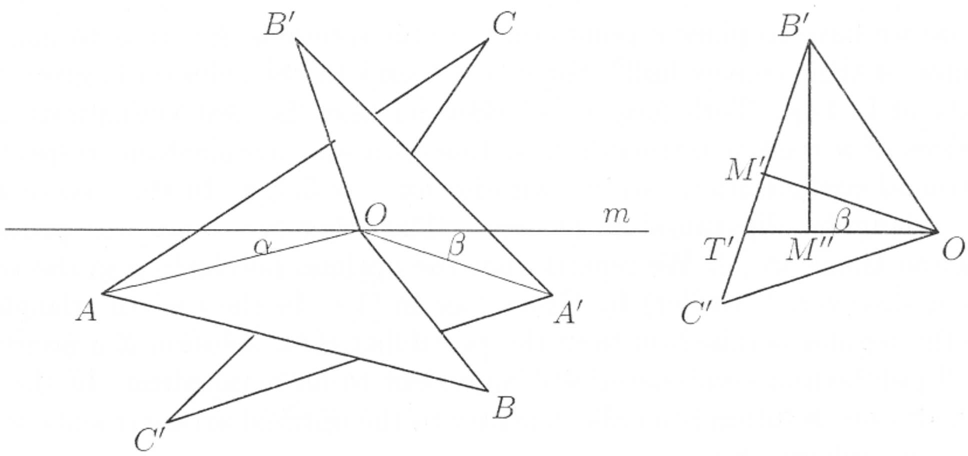

Figure 1. Two regular triangle with common center.

Let now T' be the intersection of m and $B'C'$ and denote by M' and M'' the midpoint of the edge $B'C'$ and the base point of the perpendicular line B' to m in the plane $(A'B'C')$, respectively (Fig. 1.). Then we have

$$m_{B'} = \sin\gamma|B'M'| = \sin\gamma(\sin(B'OM') + \beta)\cdot|OB'|,$$

and

$$m_{C'} = \sin\gamma(\sin(C'OM') - \beta)\cdot|OC'|.$$

Thus the general formula for the convex hull is:

$$v = v(P) + v(Q) = \frac{1}{3}\sin\gamma \cdot$$
$$\cdot \Big\{ \big[(\sin(B'OM') + \beta) \cdot |OB'| + +(\sin(C'OM') - \beta) \cdot |OC'|\big] a(ABOC) +$$
$$+ \big[(\sin(BOM) + \alpha) \cdot |OB| + (\sin(COM) - \alpha) \cdot |OC|\big] a(A'B'OC') \Big\}. \quad (1)$$

Remark that this formula is valid in all of the cases when O is a common point of the interior of the two triangles and both triangles have such edge, which intersects the other triangle in a point. If O is the common center of the triangles and the circumscribed circles have radii r and r', respectively, then we have

$$v = \frac{2}{3}\sin\gamma \Big\{ \Big[\sin\frac{1}{2}(B'OC')\cos\beta \cdot r' \Big] a(ABOC) +$$
$$+ \Big[(\sin\frac{1}{2}(BOC)\cos\alpha \cdot r \Big] a(A'B'OC') \Big\}. \quad (2)$$

If we assume that the triangles are congruent our formula becomes simpler:

$$v = \frac{2}{3}\sin\gamma \cdot r \cdot a(ABOC) \cdot \sin\frac{1}{2}(B'OC') \cdot (\cos\beta + \cos\alpha), \quad (3)$$

and finally, if the triangles are regular then we have:

$$v = \frac{2}{3}\sin\gamma \cdot r \cdot 2r^2\frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}(\cos\beta + \cos\alpha) = \frac{r^3}{2}\sin\gamma(\cos\alpha + \cos\beta). \quad (4)$$

The maximal value of the volume function is attained in the case when the parameters are $\alpha = \beta = 0$ and $\gamma = \frac{\pi}{2}$, respectively. The above calculation leads to the following theorem:

Theorem 2.1. *The volume of the convex hull of two regular triangle with a common center is maximal if and only if their planes are orthogonal to each other and one of their vertices is opposite position with respect to the common center O.*

Remark. *Our formula (3) applied to two congruent rectangular triangle with equal sides and common center, reproduces the maximal volume arrangement of six points on the sphere. In this case we have the volume formula:*

$$v = \frac{2r^3}{3}\sin\gamma(\cos\alpha + \cos\beta).$$

showing, that the optimal placing is the "opposite" one as in the previous theorem. (Of course the convex hull is the regular octahedron.)

3. Two congruent regular tetrahedra with a common center

3.1. The combinatorial description of the problem and the case of dual position.

The spherical triangles corresponding to the faces of the first tetrahedron divide the unit sphere into four congruent close domains. These contain the vertices of the second tetrahedron. We now have two possibilities:

1. All of the spherical triangles contain exactly one from the vertices of the other tetrahedron and changing the role of the tetrahedra we also get it. In this case we say that the two tetrahedra are in **dual position**. We remark that in a dual position the corresponding spherical edges cross to each other, respectively.
2. The tetrahedra are not in dual position.

In this section we will give a formula which determines the volume of the convex hull of two regular tetrahedra in dual position and having common center . From this formula we can establish the conjectured maximal volume and so we get a candidate the optimal placing of the tetrahedra. We note that quadrilateral faces of the convex hull are considered as degenerate tetrahedra in this paper.

3.1.1. Notations. We introduce the parameters which will be necessary to the calculation.

$\mathcal{I}=\{1,2,3,4\}$; $\mathcal{I}'=\{1',2',3',4'\}$: The vertex sets of the tetrahedra.
r: The radius of the common circumscribed ball.
O: The center of the common circumscribed ball, i.e. of both tetrahedra.
$S(1,2,3)$: The spherical triangle spanned by the points $\{1,2,3\}$.[1]
$C(1,2,3)$: The convex hull of $\{1,2,3\}$ in $\mathbb{E}^3$.
$\mathbf{n}_i$: The normal vector of the plane $(\mathcal{I}\setminus\{i\})$ defined by $\mathbf{n}_i:=-\vec{Oi}$.
α_i: The angle of the vectors $\vec{Oi'}$ and $\mathbf{n}_i$.
$\alpha_{i'}$: The angle of the vectors $\vec{Oi}$ and $\mathbf{n}_{i'}$.
$\alpha_{i,j}=\alpha_{i',j'}$: The angle of the vectors $\vec{i'j'}$ and $\vec{ij}$.
$\alpha_{i,j'}$: The angle of the vectors $\vec{Oi}$ and $\vec{Oj'}$.
$\alpha_{i',j}$: The angle of the vectors $\vec{Oi'}$ and $\vec{Oj}$.

3.1.2. Connections among the parameters. In this section we give some connections among the parameters above.

Statement 3.1. *The following equations hold for the parameters defined above:*

1. $\alpha_i=\alpha_{i'}$,
2. $\cos\alpha_i=\sum\limits_{j\neq i'}\cos\alpha_{j,i'}$,
3. $\sum\limits_{j\neq i'}\cos^2\alpha_{j,i'}+\sum\limits_{k<l;k,l\neq i'}\cos\alpha_{k,i'}\cos\alpha_{l,i'}=\dfrac{2}{3}$,
4. $\sum\limits_{j\neq i'}\cos^2\alpha_{j,i'}=\dfrac{4}{3}-\cos^2\alpha_i$,
5. $\cos\alpha_{k',l'}=-\dfrac{3}{8}(\cos\alpha_{k,l'}+\cos\alpha_k+\cos\alpha_l+\cos\alpha_{l,k'})$.

Proof. By definition we have

$$\cos\alpha_i\cdot r^2=\langle\vec{Oi'}|\mathbf{n}_i\rangle=\langle-\vec{Oi'}|-\mathbf{n}_i\rangle=\langle\vec{Oi}|\mathbf{n}_{i'}\rangle=\cos\alpha_{i'}\cdot r^2$$

[1]For the notation of the vertices also holds a duality, the vertex i' is in the spherical triangle $S(j,k,l)$ for every point i'.

showing 1. The second statement similarly follows from the equality $\mathbf{n}_i = \sum_{j\neq i} \vec{Oj}$. The third one is analogous to the well-known statement " the sum of the square of the direction cosines is equal to one". In fact, we can write for example the vector $\vec{O1'}$ as the linear combination of the vectors $\{\vec{O2}, \vec{O3}, \vec{O4}\}$ in the form:

$$\vec{O1'} = x\vec{O2} + y\vec{O3} + z\vec{O4}, \qquad (x, y, z \geq 0).$$

An easy computation shows that the coefficients are:

$$x = \frac{3}{4}(2\cos\alpha_{2,1'} + \cos\alpha_{3,1'} + \cos\alpha_{4,1'})$$

$$y = \frac{3}{4}(\cos\alpha_{2,1'} + 2\cos\alpha_{3,1'} + \cos\alpha_{4,1'})$$

$$z = \frac{3}{4}(\cos\alpha_{2,1'} + \cos\alpha_{3,1'} + 2\cos\alpha_{4,1'}).$$

The scalar product of this equality with the vector $\vec{O1'}$ gives the following equation:

$$\begin{aligned} r^2 = \frac{3}{4}r^2(2\cos^2\alpha_{2,1'} &+ \cos\alpha_{3,1'}\cos\alpha_{2,1'} + \cos\alpha_{4,1'}\cos\alpha_{2,1'} + \\ &+ \cos\alpha_{2,1'}\cos\alpha_{3,1'} + 2\cos^2\alpha_{3,1'} + \cos\alpha_{4,1'}\cos\alpha_{3,1'} + \\ &+ \cos\alpha_{2,1'}\cos\alpha_{4,1'} + \cos\alpha_{3,1'}\cos\alpha_{4,1'} + 2\cos^2\alpha_{4,1'}). \end{aligned}$$

Simplifying this equality and change the number 1 to the general value i, we get the third identity.

The fourth formula follows from the previous two ones. Obviously

$$\sum_{j\neq i'} \cos^2\alpha_{j,i'} = \Big(\sum_{j\neq i'} \cos\alpha_{j,i'}\Big)^2 - 2\sum_{k<l;k,l\neq i'} \cos\alpha_{k,i'}\cos\alpha_{l,i'},$$

and from the second and third equality we have:

$$\sum_{j\neq i'} \cos^2\alpha_{j,i'} = \cos^2\alpha_i - 2\Big(\frac{2}{3} - \sum_{j\neq i'} \cos^2\alpha_{j,i'}\Big),$$

yielding the required equality.

Using the regularity of the tetrahedra we get the last formula from the equality:

$$\cos\alpha_{k',l'} = \frac{\langle \mathbf{n}_k - \mathbf{n}_l | \vec{k'l'}\rangle}{|\mathbf{n}_k - \mathbf{n}_l||\vec{k'l'}|} = -\frac{r^2(\cos\alpha_{k,l'} + \cos\alpha_k + \cos\alpha_l + \cos\alpha_{l,k'})}{\frac{8}{3}r^2}.$$

3.1.3. The main lemma. The following lemma gives the base of the computation of volume of the convex hull examined in this paragraph.

Lemma 3.2. *Assume that the tetrahedra have dual position. If the edge $C(1,2)$ does not intersect the tetrahedron $C(1,2,3,4)$, then the edge $C(3,4)$ intersects the tetrahedron $C(1,2,3,4)$.*

Proof. In this case the Euclidean triangles $C(O,1,2)$ and $C(O,3',4')$ intersecting in a segment belongs to each of this triangle, respectively.

Denote by $F_{1,2}$ and $F_{3',4'}$ the midpoints of the edges $C(1,2)$ and $C(3',4')$, respectively. By our assumption the edge $C(3',4')$ meats the triangle $C(O,1,2)$ in the point S, meaning that the angle $SOF_{1,2}$ greater then the angle $SOF_{3',4'}$, but less or equal to $\frac{\Pi}{3}$. (See in Fig.2.)

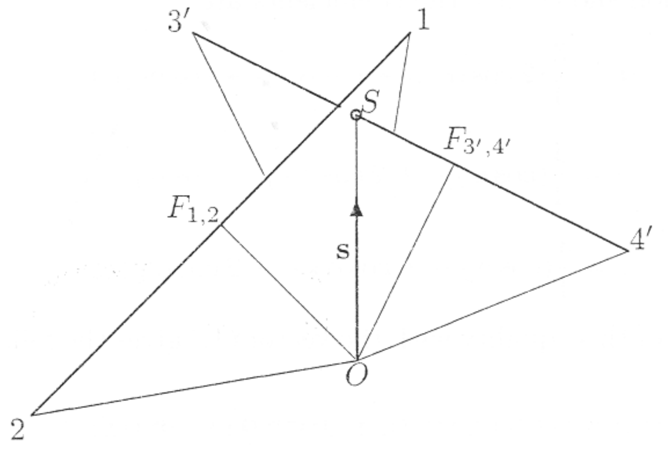

Figure 2. Connection between the opposite pair of edges.

Since the lines $(OF_{1,2})$ and $(OF_{3',4'})$ are the normal transversalis of the examined tetrahedra, respectively, they also contain the midpoints $F_{3,4}$ and $F_{1',2'}$ of the edges $C(3,4)$ and $C(1',2')$, respectively. On the other hand the planes of the triangles $C(O,3,4)$ and $C(O,1',2')$ are orthogonal to the planes of the triangle $C(O,1,2)$ and $C(O,3',4')$, resp. Let finally $S' \neq O$ be a point on the intersection line of the planes of the triangles $C(O,3,4)$ and $C(O,1',2')$. W e prove that the angle $S'OF_{1'2'}$ is greater than the angle $S'OF_{3,4}$. To this denote by $\mathbf{s}$, $\mathbf{s}'$, $\mathbf{f}_{1,2}$ and $\mathbf{f}_{3',4'}$ the unit vectors of the directions $\vec{OS}$, $\vec{OS}'$, $O\vec{F}_{1,2}$ and $O\vec{F}_{3',4'}$, respectively. The orthogonality of the above planes gives the following equations:

$$0 = \langle \mathbf{s} \times \mathbf{f}_{1,2} | \mathbf{s}' \times \mathbf{f}_{1,2} \rangle = \langle \mathbf{s} \times \mathbf{f}_{3',4'} | \mathbf{s}' \times \mathbf{f}_{3',4'} \rangle,$$

providing

$$\langle \mathbf{f}_{1,2} | \mathbf{s} \rangle \langle \mathbf{f}_{1,2} | \mathbf{s} \rangle = \langle \mathbf{f}_{1,2} | \mathbf{f}_{1,2} \rangle \langle \mathbf{s} | \mathbf{s}' \rangle = \langle \mathbf{f}_{3,4'} | \mathbf{f}_{3',4'} \rangle \langle \mathbf{s} | \mathbf{s}' \rangle = \langle \mathbf{f}_{3,4} | \mathbf{s} \rangle \langle \mathbf{f}_{3,4} | \mathbf{s} \rangle.$$

Using now the facts, that $O\vec{F}_{3,4} = -\mathbf{f}_{1,2}$ and $O\vec{F}_{1',2'} = -\mathbf{f}_{3',4'}$ we have

$$\cos(F_{1,2}OS)\cos(F_{3,4}OS') = \cos(F_{3',4'}OS)\cos(F_{1',2'}OS'),$$

i.e. the angle $(F_{3,4}OS')$ less than the angle $(F_{1',2'}OS')$. With respect to the duality condition this means that the segment $C(3,4)$ intersects the triangle $C(1',O,2')$. This also means that $C(3,4)$ intersects the tetrahedron $C(1',2',3',4')$, as we stated.

In a consequence of this lemma from a pair of opposite edges of the first tetrahedron exactly one intersects the other tetrahedron. Thus the examined convex hull is disjoint union of four double pyramids (with the common vertex O) and three tetrahedra spanned by a pair of complementary (with respect to the index sets) edges of the two tetrahedra. Among these six edges there are no two opposite ones belonging to one of the original tetrahedra.

3.1.4. Computation of the volume of the convex hull. In this section we use the functions $d(\cdot,\cdot)$, $l(\cdot)$, $a(\cdot)$ and $v(\cdot)$, which are the distance, length, area and volume functions of their arguments, respectively. Denote by $M_{1',2'}$ the intersection of the segment $C(1',2')$ by the plane of the triangle $C(O,3,4)$. We will compute the volume of the tetrahedron $C(1',2',3,4)$ as the volume of a double pyramid based on the triangle $C(M_{1',2'},3,4)$. The area of the triangle is:

$$a(C(M_{1',2'},3,4)) = \frac{1}{2}d(M_{1',2'},C(3,4))l(C(3,4)),$$

where, from the equality

$$d(M_{1',2'},C(3,4)) + \frac{r}{\sqrt{3}} = |\langle \vec{OM_{1',2'}}|\vec{O3}+\vec{O4}\rangle|\frac{\sqrt{3}}{2r} = \left|\langle \vec{OM_{1',2'}}| - \vec{O1} - \vec{O2}\rangle\right|\frac{\sqrt{3}}{2r}$$

using the representation

$$\vec{OM_{1',2'}} = \frac{\langle \vec{O2'}|\vec{12}\rangle \vec{O1'} - \langle \vec{O1'}|\vec{12}\rangle \vec{O2'}}{\langle \vec{1'2'}|\vec{12}\rangle}$$

of the vector $\vec{OM_{1',2'}}$, we have:

$$d(M_{1',2'},(34)) = \frac{3\sqrt{3}}{8}r\left(\frac{\cos\alpha_{1',2}\cos\alpha_{1,2'} - \cos\alpha_1\cos\alpha_2}{\cos\alpha_{1',2'}} - \frac{8}{9}\right).$$

Now the area is:

$$a(C(M_{1',2'},3,4)) = \frac{1}{2}\frac{2\sqrt{6}}{3}r\frac{3\sqrt{3}}{8}r\left(\frac{\cos\alpha_{1',2}\cos\alpha_{1,2'} - \cos\alpha_1\cos\alpha_2}{\cos\alpha_{1',2'}} - \frac{8}{9}\right) =$$
$$= \frac{3\sqrt{2}}{8}r^2\left(\frac{\cos\alpha_{1',2}\cos\alpha_{1,2'} - \cos\alpha_1\cos\alpha_2}{\cos\alpha_{1',2'}} - \frac{8}{9}\right).$$

Since the sum of the two heights are:

$$-\left(l(C(1'M_{1',2'}))\cos\alpha_{1',2'} + l(C(2'M_{1',2'}))\cos\alpha_{1',2'}\right) =$$
$$- = l(C(1'2'))\cos\alpha_{1',2'} = -\sqrt{\frac{8}{3}}r\cos\alpha_{1',2'}$$

we get for the volume:

$$v(C(1',2',3,4)) =$$
$$= \frac{1}{3}\cdot\frac{3\sqrt{2}}{8}r^2\left(-\frac{\cos\alpha_{1',2}\cos\alpha_{1,2'} + \cos\alpha_1\cos\alpha_2}{\cos\alpha_{1',2'}} - \frac{8}{9}\right)\cdot\sqrt{\frac{8}{3}}r\cdot\cos\alpha_{1',2'} =$$
$$= \frac{\sqrt{3}}{3}r^3\left[-\frac{1}{2}(\cos\alpha_{1',2}\cos\alpha_{1,2'} - \cos\alpha_1\cos\alpha_2) + \frac{4}{9}\cos\alpha_{1',2'}\right].$$

Analogously we get the respective volumes $v(C(M_{1',3'},2,4))$ and $v(C(M_{1',4'},2,3))$. On the basis of the results of the section (3.2) we can built the convex hull from four double pyramids with common vertex O up the faces and from three tetrahedra as above. We have to compute the following formula:

$$v=\frac{1}{3}\big[a(C(1,3,4))(\cos\alpha_1+\cos\alpha_2+\cos\alpha_3+\cos\alpha_4)r+$$
$$+v(C(M_{1',2'},3,4))+v(C(M_{1',3'},2,4))+v(C(M_{1',4'},2,3))\big]=$$
$$=\frac{\sqrt{3}}{3}r^3\Big\{\frac{2}{3}(\cos\alpha_1+\cos\alpha_2+\cos\alpha_3+\cos\alpha_4)-$$
$$-\frac{1}{2}\big[(\cos\alpha_{1',2}\cos\alpha_{1,2'}-\cos\alpha_1\cos\alpha_2)+(\cos\alpha_{1',3}\cos\alpha_{1,3'}-\cos\alpha_1\cos\alpha_3)+$$
$$+(\cos\alpha_{1',4}\cos\alpha_{1,4'}-\cos\alpha_1\cos\alpha_2)\big]-\frac{4}{9}(\cos\alpha_{1',2'}+\cos\alpha_{1',3'}+\cos\alpha_{1',4'})\Big\}=$$
$$=\frac{\sqrt{3}}{3}r^3\Big\{\frac{2}{3}(\cos\alpha_1+\cos\alpha_2+\cos\alpha_3+\cos\alpha_4)-$$
$$-\frac{1}{2}(\cos\alpha_{1',2}\cos\alpha_{1,2'}+\cos\alpha_{1',3}\cos\alpha_{1,3'}+\cos\alpha_{1',4}\cos\alpha_{1,4'})+$$
$$+\frac{1}{2}\cos\alpha_1(\cos\alpha_2+\cos\alpha_3+\cos\alpha_4)+\frac{4}{9}(\cos\alpha_{1',2'}+\cos\alpha_{1',3'}+\cos\alpha_{1',4'})\Big\}. \quad (5)$$

Using now the equalities $\alpha_i=\alpha_{i'}$, $\cos\alpha_i=\sum_{j\neq i'}\cos\alpha_{j,i'}$ and $\cos\alpha_{k',l'}=-\frac{3}{8}(\cos\alpha_{k,l'}+\cos\alpha_k+\cos\alpha_l+\cos\alpha_{l,k'})$ proved in the section (3.2), we can substitute the last term of (5) by another one, contains only the other angles in its argument. In fact

$$\frac{4}{9}(\cos\alpha_{1',2'}+\cos\alpha_{1',3'}+\cos\alpha_{1',4'})=$$
$$=\frac{4}{9}\left(-\frac{3}{8}\left[\sum_{k=2}^{4}(\cos\alpha_{k,1'}+\cos\alpha_k+\cos\alpha_1+\cos\alpha_{1,k'})\right]\right)=$$
$$=-\frac{1}{6}\left[\sum_{j=2}^{4}(\cos\alpha_{j,1'})+\sum_{i=2}^{4}(\cos\alpha_{1,i'})+3\cos\alpha_1+\cos\alpha_2+\cos\alpha_3+\cos\alpha_4\right]=$$
$$=-\frac{1}{6}\left[5\cos\alpha_1+(\cos\alpha_2+\cos\alpha_3+\cos\alpha_4)\right].$$

Now we can write the formula (5) in the form:

$$v=\frac{\sqrt{3}}{6}r^3\Big[(\cos\alpha_2+\cos\alpha_3+\cos\alpha_4)-\frac{1}{3}\cos\alpha_1-$$
$$-(\cos\alpha_{1',2}\cos\alpha_{1,2'}+\cos\alpha_{1',3}\cos\alpha_{1,3'}+\cos\alpha_{1',4}\cos\alpha_{1,4'})+$$
$$+\cos\alpha_1(\cos\alpha_2+\cos\alpha_3+\cos\alpha_4)\Big]. \quad (6)$$

This formula corresponds to the index 1 because it is symmetric with respect to the other indices. Since the volume is invariant under the permutation of the indices in $\mathcal{I}$ we have (analogously) further three formula for the volume corresponding to the other indices. The arithmetic mean of their is also the volume v, so we have a symmetric (with respect to its indices) form:

$$v = \frac{\sqrt{3}}{24} r^3 \left[\frac{8}{3}\sum_{i=1}^{4} \cos\alpha_i + \left(\sum_{i=1}^{4} \cos\alpha_i\right)^2 - \right.$$
$$\left. - \sum_{i=1}^{4} \cos^2\alpha_i - \sum_{k=1}^{4}\sum_{k\neq l} \cos\alpha_{k',l}\cos_{k,l'}\right]. \quad (7)$$

3.1.5. The extremal configuration. The equality (7) gives a possibility to determine those placing of the tetrahedra which convex hull has maximal volume. First we substitute the equality (4) into the third summand of our formula and write the formula (7) into the following new form:

$$v = \frac{\sqrt{3}}{24} r^3 \left[\left(\sum_{i=1}^{4} \cos\alpha_i\right)^2 + \frac{7}{3}\sum_{i=1}^{4} \cos\alpha_i - \frac{16}{3} + \right.$$
$$\left. + \frac{1}{3}\sum_{i=1}^{4} \cos\alpha_i + \sum_{k=1}^{4}\sum_{l\neq k} (\cos\alpha_{k,l'})^2 - \cos\alpha_{k',l}\cos\alpha_{k,l'})\right]. \quad (8)$$

Secondly substitute the equality (3) into the fourth summand of this formula and write it the following form:

$$v = \frac{\sqrt{3}}{24} r^3 \left[\left(\sum_{i=1}^{4} \cos\alpha_i\right)^2 + \frac{7}{3}\sum_{i=1}^{4} \cos\alpha_i - \frac{16}{3} + \right.$$
$$\left. + \sum_{k=1}^{4}\sum_{l\neq k} \left(\frac{1}{3}\cos\alpha_{k,l'} + (\cos\alpha_{k,l'})^2 - \cos\alpha_{k',l}\cos\alpha_{k,l'}\right)\right] =$$
$$= \frac{\sqrt{3}}{24} r^3 \left[\left(\sum_{i=1}^{4} \cos\alpha_i\right)^2 + \frac{7}{3}\sum_{i=1}^{4} \cos\alpha_i - \frac{16}{3} + \right.$$
$$\left. + \sum_{k=1}^{4}\sum_{l\neq k} \cos\alpha_{k,l'}\left(\frac{1}{3} + \cos\alpha_{k,l'} - \cos\alpha_{k',l}\right)\right]. \quad (9)$$

Now we can apply the inequality between the arithmetic and geometric means to the last part of this formula. Thus we get the inequality:

$$v \leq \frac{\sqrt{3}}{24} r^3 \left[\left(\sum_{i=1}^{4} \cos\alpha_i\right)^2 + \frac{7}{3}\sum_{i=1}^{4} \cos\alpha_i - \frac{16}{3} + \right.$$
$$\left. + \sum_{k=1}^{4}\sum_{l\neq k} \cos\alpha_{k,l'}\left(\frac{\frac{1}{3} + 2\cos\alpha_{k,l'} - \cos\alpha_{k',l}}{2}\right)^2\right]. \quad (10)$$

The equality holds if and only if all pairs of indices k, l fulfill:

$$\cos\alpha_{k,l'} = \frac{1}{3} + \cos\alpha_{k,l'} - \cos\alpha_{k',l}$$

that is $\cos\alpha_{k',l} = \frac{1}{3}$ for all k, l. In this case for all k we get that $\cos\alpha_k = 1$ implying that the eight vertices are the vertices of a cube. In this case we also get that the optimal value is

$$v = \frac{\sqrt{3}}{24}r^3\left(16 - \frac{16}{3} + \frac{28}{3} + 4\cdot 3\frac{4}{9}\right) = \frac{\sqrt{8}}{3\sqrt{3}}r^3.$$

Hence we get the following theorem:

Theorem 3.3. *The value $v = \frac{8}{3\sqrt{3}}r^3$ is an upper bound for the volume of the convex hull of two regular tetrahedra in dual position. It is attained if and only if the eight vertices of the two tetrahedra are the vertices of a cube inscribed in the common circumscribed sphere.*

3.1.6. The "placing matrix" of the placing. The last formula of the volume of the convex hull can be written in more simple form if we define a **placing matrix** G corresponding to the vertex sets of the tetrahedra. Let A and A' be the matrices corresponding to the first and second tetrahedra, respectively. The columns of these matrices are the coordinates of the vectors $\vec{Oi}$ and $\vec{Oi'}$, respectively. (We are using fixed orthonormed basis of the space.)

Definition 3.1. *We define* **the placing matrix G** *as the product*

$$G = \frac{1}{r^2}A^T \cdot A',$$

where A^T means the transpose of A.

By this notation our formula (8) can be written in the following short form:

$$v = \frac{\sqrt{3}}{24}r^3\left[-\frac{8}{3}\cdot Tr(G) + (Tr(G))^2 - Tr(G^2)\right] =$$
$$= \frac{\sqrt{3}}{9}r^3\left[-Tr(G) + \frac{3}{8}\left[(Tr(G))^2 - Tr(G^2)\right]\right], \quad (11)$$

where Tr(G) denotes the trace of the matrix G. We remark that there is an analogous formula for the area of convex hull of two regular triangle of the plane with a common center. Namely we have:

$$a = \frac{\sqrt{3}}{2}r^2\left[-Tr(G)\right]. \quad (12)$$

3.2. Some words about the other cases.

If the tetrahedra are not in dual position we can distinguish two subcases.

1. From the spherical domains corresponding to the first tetrahedron one contains two vertices of the second tetrahedron. (Or changing the notation of the tetrahedra get this combinatorial situation.)

2. Two domains contain two vertices, respectively.

In this paragraph first we prove that the second case implies the identity of the tetrahedra. Then we prove that in the first combinatorial but geometrically most symmetric case the convex hull is less than the volume of the cube. In general we can not prove that the optimal case without combinatorial restrictions is the case of the cube, but we think that it is the truth.

Statement 3.4. *Assume that the closed regular spherical tetrahedra $S(1,2,3)$ and $S(4,2,3)$ contain the vertices $2',4'$ and $1',3'$, respectively. Then the two tetrahedra are the same.*

Proof. Consider the following spherical polygon: $S(4',2')\cup S(2',3')\cup S(3',1')$. We remark that if a closed spherical triangle of the first tetrahedron contains three point from the vertices of the other tetrahedron then they agree. This means that the closed spherical segment $S(2,3)$ does not contain vertices of the second tetrahedron. Let denote by F the midpoint of the segment $S(2',3')$. If now $F\in S(1,2,3)\cup S(4,2,3)$ then the pair of points F and $1'$ or F and $4'$ belong to a spherical tetrahedron meaning that for example F and $1'$ are a vertices and the midpoint of the opposite side of the triangle $S(4,2,3)$, respectively. So the point F one of the vertices $2,3$ or the midpoint of this side. In this last case the two tetrahedra agree, so we can assume that e.g. $F=2$. This is also impossible, because in this case the points 1 and 4 are the midpoints of the sides $S(1,3)$ and $S(3,4)$, respectively, showing then its distance is not the same that the common edge lengthes of the regular spherical triangles. Similar argument shows that the midpoints of the sides $S(2',1')$, $S(4',1')$ and $S(4',3')$ are not in the union $S(1,2,3)\cup S(4,2,3)$. Consider the common plane of this center points. This plane contains the origin and intersects the triangles $S(1,2,3)$, $S(4,2,3)$, $S(1,2,4)$ and $S(4,2,3)$, respectively. The union of these intersection spherical segments give a great circle in which the above halving points distributed uniformly. A straightforwar d (but not short) argument shows that the above assumption on midpoints gives a contradiction.

Now examine the first combinatorial situation of the placing, precisely one spherical triangle of a tetrahedron contains precisely two vertices of the second one. So we can find an edge of the second tetrahedron separated from the first tetrahedron by a face-plane of it. This means that the convex hull has a piece which is the convex hull of a face of the first tetrahedron and an edge of the second one. The vertices of this double pyramid let be the set of points $1,2,3,4',1'$ meaning that the plane of the face $C(123)$ separates the vertex 4 and the edge $C(4'1')$, respectively.

The most symmetric case in this placing when the tetrahedra have common plane of symmetry. That is, we assume that the edge $C(4'1')$ lying in the bisector of the vertices $2,3$.

In this case the edges $C(23)$ and $C(2'3')$ are parallel to each other and there are two planes of symmetry of the convex hull: the common plane of the edges $C(4'1')$ and $C(14)$ and the common bisector plane of the pair of points $4',1$ and $1',4$, respectively. The volume of the convex hull can be counted using the fact, that the faces of this body are the triangles $C(1'4'2)$, $C(143')$, the trapezoids $C(1'423')$, $C(14'23')$ and their mirror images with respect to the first plane of symmetry, respectively. (This situation can be seen on Fig.3.)

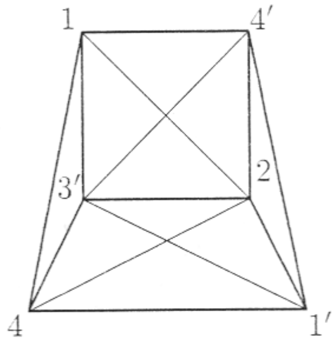

Figure 3. The symmetric case.

Let the common distance of the points $2, 3, 2', 3'$ and the plane of the other vertices denoted by $m(=\frac{\sqrt{6}}{3})$, and denoted by α and β the half of the angles $1'O4$ and $1O4'$, respectively. Then the area of the trapezoid $C(1'414')$ is $a = \sin(\alpha+\beta)+\frac{1}{2}(\sin(2\alpha)+\sin(2\beta))$. On the other hand, if we denote the area of the orthogonal projection of the faces $C(1'43'2)$, $C(14'23')$, $C(143')$ and $C(1'4'2)$ into the plane of the points $1', 4, 1, 4'$ by a_1, a_2, a_3 and a_4, respectively. Then we have:

$$v = 2m\left(\frac{1}{2}(a_1+a_2)+\frac{1}{3}(a_3+a_4)\right) =$$
$$= \frac{m}{3}\left(3(a_1+a_2+a_3+a_4)-(a_3+a_4)\right) = \frac{m}{3}\left(3a-(a_3+a_4)\right).$$

It is easy to check that the value of a is maximal if $\alpha=\beta$. This also implies that the value of (a_3+a_4) is minimal (it is equal to zero). So the optimal arrangement is also the case of the cube. On the other hand, our combinatorial assumption excludes this possibility, thus we get that in the examined cases the volume of the convex hull is strictly less than the volume of the cube, as we stated.

References

[1] Berman J. D. and Hanes K., *Volumes of polyhedra inscribed in the unit sphere in E^3*, Math. Ann., **188** (1970), 78–84.

[2] Croft H. T., Falconer K. J., Guy R. K., *Unsolved Problems in Geometry*, Springer Vol. II 1991.

[3] Grace, *Search for largest polyhedra*, Math. Comp., **17** (1963), 197–199.

[4] Tóth L. F., *An inequality concerning polyhedra*, Bull. Amer. Math. Soc., **54** (1948), 139–146.

[5] Tóth L. F., *Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer, Berlin, 2nd ed. 1972.

Á. G. Horváth, Department of Geometry, Budapest University of Technology and Economics (BME), H-1521 Budapest, Hungary