Lower bounds of the maximal coordinates of minimum vectors. *

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Dedicated to Prof. I.Reiman on the ocassion of his 70^{th} birthday

1 Introduction

This paper is connected with the question: How to find a basis of an n-lattice in E^n such that the absolute values of the coordinates belonging to the minima of the lattice with respect to this basis are "small enough". In the paper [2] the author has proved that in every lattice possessing n linearly independent minima one can find a basis so that the maximum of the absolute values of the coordinates to a minimum vector is not greater than the maximal index of the admissible centerings for the n-lattices and the exact value has been determined in the cases $n \geq 5$. (The basic definitions can be found e.g. in [4] or [2].)

However, in the known lower dimensional cases the value in question is always 1, the following question arises: Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ denote a lattice basis and $L(A)$ denote the number

$$L(A) := \max\left\{|x_i|; \mathbf{m} = \sum_{i=1}^{n} x_i \mathbf{a}_i \in \Lambda, |\mathbf{m}| = \min\Lambda, i = 1, \ldots, n\right\},$$

where the common length of the lattice minima is denoted by $\min\Lambda$. Is it true that the quantity:

$$L_n := \sup\{L(\Lambda) := \min\{L(A)|A \text{ is a basis of}\Lambda\}|\Lambda \subset E^n \text{ possessing } n \text{ independent minima}\}$$

tends to infinity when n grows up? (The supremum is taken for all lattices in E^n.)

The answer for this general question is unknown. In this paper we determine the value $L(\Lambda)$ for some lattices with a lot of minima e.g. A_n, D_n, E_6, E_7, E_8 (See [1]) and give a general lower bound for $L(\Lambda)$.

The lattice E_8 is very interesting because it is the "first" known instance for which $L(\Lambda = E_8)$ is greater then one.

2 Definitions, preliminary methods.

A lattice in E^n, defined by a basis $A = \{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ of E^n, is the set $\Lambda = [\mathbf{a}_1, \ldots, \mathbf{a}_n] = \{\sum_{i=1}^{n} x_i \mathbf{a}_i | x_i$ are integers $\}$. A minimum vector (or simply a minimum) of Λ is one of the shortest non-zero vectors in Λ. Let the numbers $L(A)$,$L(\Lambda)$ and L_n be defined as in the introduction. A lattice Λ' is a centering of the lattice Λ if $\Lambda' \supset \Lambda$. This centering is admissible iff $\min\Lambda' = \min\Lambda$. The index of the admissible centering is defined by the number $\mathrm{ind}(\Lambda'/\Lambda) = v(\Lambda)/v(\Lambda')$,where $v(\Lambda)$ is the volume of a basic parallelepiped in the lattice Λ. Let V_n be the maximum of the n-dimensional indices for all admissible centerings of all lattices Λ possessing n linearly independent minima. (See e.g. [5], [6].) Now we have the following two theorems:

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Theorem 1 **([2])** *For an arbitrary dimension n $L_n \leq V_n$ holds, i.e. in every n-lattice, possessing n linearly independent minima, there exists a basis such that the absolute values of the coordinates belonging to any minimum vector are not greater than V_n.*

Theorem 2 **([2])** *The estimate in theorem 1 is not "sharp". It is well-known that $V_1 = V_2 = V_3 = 1$ and $V_4 = V_5 = 2$, (see [6]), while $L_n = 1$ for $n \leq 5$.*

The proof of the second theorem was based on the observation that a row-subtraction operation on the so-called characteristic matrix of size $n \times \sigma$, consisting of the n coordinates of σ minimal vectors with respect to that basis, is equivalent to a basis-change of that. Thus suitable row-substractions can give a "better" basis than the original one. To consruct "good" bases we will follow that way.

The known lattices with a lot of minima are so-called *integral* ones which means that the scalar product of any two vectors from it is an integer. If the determinant of an integral lattice is one we say that it is *unimodular*. Let $\mathbf{x}$ be a vector of an integral lattice. Then there is a sublattice of dimension $(n-1)$ containing the origin and orthogonal to $\mathbf{x}$. The *dual (reciprocal or polar)* lattice Λ^{-1} of Λ can generally be defined as the collection of those points of position vectors have integral scalar products with all vectors of Λ. Then we have the following relations:

$$\Lambda \subseteq \Lambda^{-1} \subseteq \frac{1}{(\det \Lambda)^2}\Lambda,$$

where $\det \Lambda$ means the determinant of the lattice. If $\{\mathbf{a}_1 \ldots \mathbf{a}_n\}$ is a basis of Λ then the vector system $\{\mathbf{f}_1 \ldots \mathbf{f}_n\}$ defined by the equalities

$$\mathbf{a}_i \cdot \mathbf{f}_j = \delta_{ij}$$

is a basis of the dual lattice Λ^{-1}. (δ_{ij} is the so-called Kronecker function.) This is called the dual basis of $\{\mathbf{a}_1 \ldots \mathbf{a}_n\}$. (For the classical derivation and other details we refer e.g. to the books [1], [4] or the paper [3].)

In this paper we shall concentrate on the following integral lattices:

$$Z_n = \left\{ \sum_{i=1}^{n} x_i \mathbf{e}_i \mid \mathbf{x} = (x_1, \ldots, x_n) \in \mathrm{Z}^n \right\},$$

where $\{\mathbf{e}_i\}$ is an orthonormal base of the Euclidean space E^n. (In another denotation $Z_n = C_n$.)

$$A_n = \left\{ \sum_{i=1}^{n+1} x_i \mathbf{e}_i \mid \mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathrm{Z}^{n+1} \quad \sum_{i=1}^{n+1} x_i = 0 \right\}.$$

The minima of A_n are the vectors $\mathbf{e}_i - \mathbf{e}_j$ $(i \neq j)$. This lattice is a generalization of the regular triangle lattice of the plane, a base of it consists of the edge vectors (starting at a common vertex) of a regular simplex of E^3. The following lattices can be given as some admissible centerings of the above ones:

$$D_n = \left\{ \sum_{i=1}^{n} x_i \mathbf{e}_i \mid \mathbf{x} = (x_1, \ldots, x_n) \in \mathrm{Z}^n \quad \sum_{i=1}^{n} x_i \equiv 0 (\mathrm{mod}\ 2) \right\}$$

$$E_8 = \left\{ \sum_{i=1}^{8} x_i \mathbf{e}_i \mid \mathbf{x} = (x_1, \ldots, x_8) \in \mathrm{Z}^8 \text{ or } \mathbf{x} \in (\mathrm{Z} + \frac{1}{2})^8 \ \sum_{i=1}^{8} x_i \equiv 0 (\mathrm{mod}\ 2) \right\}$$

$$E_7 = \left\{ \sum_{i=1}^{8} x_i \mathbf{e}_i \mid \mathbf{x} = (x_1, \ldots, x_8) \in E_8 \quad \sum_{i=1}^{8} x_i = 0 \right\}$$

$$E_6 = \left\{ \sum_{i=1}^{8} x_i \mathbf{e}_i \mid \mathbf{x} = (x_1, \ldots, x_8) \in E_8 \ x_1 + x_8 = \sum_{i=2}^{7} x_i = 0 \right\}$$

and have the minimum systems as follows:

D_n: $\pm\mathbf{e}_i \pm \mathbf{e}_j$ $(i \neq j)$;

E_8: The vectors $(\pm\mathbf{e}_i \pm \mathbf{e}_j)$ $(i \neq j)$ and the vectors $\frac{1}{2}\sum_{i=1}^{8}(\pm\mathbf{e}_i)$;

E_7: The vectors $\mathbf{e}_i \pm \mathbf{e}_j$ and the vectors $\frac{1}{2}(\mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3} + \mathbf{e}_{i_4} \pm \mathbf{e}_{i_5} \pm \mathbf{e}_{i_6} \pm \mathbf{e}_{i_7} \pm \mathbf{e}_{i_8})$ of E_7, where $1 \leq i, j \leq 8$ and $\{i_1, \cdots i_8\}$ is a permutation of $\{1, \cdots, 8\}$;

E_6: The vectors $\mathbf{e}_i - \mathbf{e}_j$ $\quad (2 \leq i \neq j \leq 7)$ and the vectors $\pm\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_{i_2} + \mathbf{e}_{i_3} + \mathbf{e}_{i_4} - \mathbf{e}_{i_5} - \mathbf{e}_{i_6} - \mathbf{e}_{i_7} - \mathbf{e}_8)$, where $\{i_2, \cdots i_7\}$ a permutation of $\{2, \cdots, 7\}$, and the vectors $\pm(\mathbf{e}_1 - \mathbf{e}_8)$;

3 A lower bound of the number $L(\Lambda)$.

The following theorem Theorem 3 is a refined version of the intuitive observation that if a minimal vector of an integral lattice has a large coordinate with respect to a fixed base then the angle between that basis vector and the hyperplane spanned by the other basis vectors is also large, near $\frac{\pi}{2}$.

Let $\{\mathbf{a}_1 \ldots \mathbf{a}_n\}$ be a basis of the lattice and $\{\mathbf{f}_1 \ldots \mathbf{f}_n\}$ be its dual. By the remark at the end of the preceding paragraph we get that there exists a minimal number k_i with the property that

$$\mathbf{g}_i := k_i \mathbf{f}_i \in \Lambda \text{ for every } i.$$

Definition 1 *The vector system $\{\mathbf{g}_1, \ldots, \mathbf{g}_n\}$ is called integral imbedding of the dual basis into Λ.*

Since for the lattice

$$\Lambda' = [\mathbf{g}_1 \cdots \mathbf{g}_n] \subset \Lambda,$$

hence

$$k \det \Lambda =: \det \Lambda' := \det[\mathbf{g}_1 \cdots \mathbf{g}_n] = \left(\prod_{i=1}^{n} k_i\right) \det[\mathbf{f}_1 \ldots \mathbf{f}_n] = \left(\prod_{i=1}^{n} k_i\right) \frac{1}{\det \Lambda}.$$

From this equation we have the connections with k's:

$$\left(\prod_{i=1}^{n} k_i\right) = k\,(\det \Lambda)^2.$$

Theorem 3 *Let Λ be an integral lattice with the basis $\{\mathbf{a}_1 \ldots \mathbf{a}_n\}$ and dual basis $\{\mathbf{f}_1 \ldots \mathbf{f}_n\}$. Then we have:*

1. *$L(\Lambda) \geq \min\{\max\{|\mathbf{f}_i \cdot \mathbf{m}| | \mathbf{m}$ is a minimal vectorof $\Lambda\}$ and $\{\mathbf{a}_j\}$ a basis of $\Lambda\}$.*
2. *For arbitrary minimal vector m holds the inequality $|\mathbf{g}_i \cdot \mathbf{m}| \leq s_i(\det \Lambda)^2$, where s_i denotes the maximum of the absolute values of the i-th coordinates of the minima with respect to $\{\mathbf{a}_1 \ldots \mathbf{a}_n\}$.*

Proof: Let now i be fixed and $\mathbf{m}$ a minimal vector of Λ. The intersection of the ball around the origin with radius $|\mathbf{m}|$ and the hyperplane of the s_i-th layer of Λ is an $n-1$-dimensional ball with centre

$$\frac{s_i}{\mathbf{f}_i^2}\mathbf{f}_i.$$

(The distance between the neighbouring parallel layers is $c := \frac{1}{|\mathbf{f}_i|} = \frac{\det \Lambda^*}{\det \Lambda}$ with the sublattice $\Lambda^* = [\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_{i+1}, \ldots \mathbf{a}_n]$.) This means that

$$\left|\cos(\frac{s_i}{\mathbf{f}_i^2}\mathbf{f}_i, \mathbf{m})\right| \leq \frac{\left|\frac{s_i}{\mathbf{f}_i^2}\mathbf{f}_i\right|}{|\mathbf{m}|} = \frac{s_i}{|\mathbf{f}_i||\mathbf{m}|}$$

hence

$$|\mathbf{g}_i \cdot \mathbf{m}| \leq |\mathbf{g}_i||\mathbf{m}|\frac{s_i}{|\mathbf{f}_i||\mathbf{m}|} = s_i\frac{|\mathbf{g}_i|}{|\mathbf{f}_i|} = s_i k_i.$$

However, from the inclusion $(\det \Lambda)^2\Lambda^{-1} \subseteq \Lambda$ and the inversion formula for the determinant of a dual of a sublattice ([1] p.166 Theorem 4.) we get that

$$|\mathbf{g}_i| = \det\left(< \mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n >^{\perp} \cap \Lambda\right) \leq$$

$$\leq (\det \Lambda)^2 \det\left(< \mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n >^{\perp} \cap \Lambda\right) =$$

$$= (\det \Lambda)^2 \frac{\det([\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n])}{\det \Lambda} = \det \Lambda \det \Lambda^*.$$

So we get the second statement of the Theorem. The first one follows from the inequality:

$$|\mathbf{g}_i \cdot \mathbf{m}| \leq s_i k_i,$$

or equivalently

$$|\mathbf{f}_i \cdot \mathbf{m}| \leq s_i.$$

Hence for the number $s = \max s_i$ the inequality:

$$s \geq \max\{|\mathbf{f}_i \cdot \mathbf{m}| | \mathbf{m} \in \Lambda \text{ minimal vector}\}$$

holds and we obtain now the first statement of the theorem:

$$L(\Lambda) \geq \min\{\max\{|\mathbf{f}_i \cdot \mathbf{m}| | \mathbf{m} \in \Lambda \text{ minimal vector}\} | \{\mathbf{a}_i\} \subset \Lambda \text{ basis}\},$$

as it was stated. □

A consequence of this theorem: there exists a lattice Λ such that $L(\Lambda)$ is greater than one.

Theorem 4 *For the unimodular lattice E_8*

$$L(E_8) \geq 2.$$

Proof: In this case the lattice is unimodular which means that $E_8 = E_8^{-1}$ and above $\mathbf{f}_i = \mathbf{g}_i \in E_8$. Hence the two statements of Theorem 3 are equivalent. Let L^* be an arbitrary lattice hyperplane (of dimension 7) with an integral basis $\{\mathbf{a}_j\}$ for which $\{\mathbf{a}_j | j = 1 \ldots 7\} \subset L^*$. For the coordinates of the vector $\mathbf{g}_8$ by the definition of E_8 we have two possibilities. Either its coordinates are integers or they are congruent modulo $\frac{1}{2}$. The vectors $(\pm\mathbf{e}_i \pm \mathbf{e}_j)$ are all minimal vectors thus if g_{8i}, g_{8j} are the coordinates of $\mathbf{g}_8$ with maximal absolute values, then we have a minimum vector $\mathbf{m}$ for which

$$|\mathbf{g}_8 \cdot \mathbf{m}| = |g_{8i}| + |g_{8j}|.$$

If the numbers g_{8i}, g_{8j} are integers then the above scalar product is greater or equal to 2, while if they are congruent modulo $\frac{1}{2}$ then so are the other coordinates which means that we have the following two cases:

1. Either one of the coordinates g_{8i}, g_{8j} has absolute value greater than $\frac{1}{2}$ then the above scalar product is greater or equal to $\frac{3}{2} + \frac{1}{2} = 2$,

2. or all of the coordinates have absolute value $\frac{1}{2}$, which means that $\mathbf{g}_8$ is a minimal vector of E_8. However, the scalar product

$$|\mathbf{g}_8 \cdot \mathbf{m}| = |g_{8i}| + |g_{8j}| = 1$$

is not zero, thus there is a lattice hyperplane parallel to L^* between the hyperplanes L^* and $L^* + \mathbf{g}_n$. So the minimal vector $\mathbf{g}_n$ has a coordinate greater than one with respect to any basis $\{\mathbf{a}_i\}$. □

4 The lattices Z_n, A_n, D_n and E_n.

With the method of [2] we shall determine the exact values of $L(Z_n)$, $L(A_n)$, $L(D_n)$ and $L(E_n)$ defined in paragraph 2.

Theorem 5 $L(Z_n) = L(A_n) = L(D_n) = L(E_6) = L(E_7) = 1$ *and* $L(E_8) = 2$.

Proof: In all of these cases we can start with an integral basis $\{\mathbf{a}_i\}$ whose elements are only minimal vectors. This basis provides an integer matrix A with columns, each containing the coordinates of a basis vector with respect to a fixed orthonormal basis of the space. If it is necessery we write the so-called characteristic matrix of the minima, too. The case of Z_n is trivial.
1. In the case of A_n we have $n+1$ rows and n columns for

$$A = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The minima of this lattice can be written in the form: $\pm(\sum_{k=i}^{j} \mathbf{a}_k)$ where $i \leq j$. This means that the basis consisting of the columns of the above matrix A is just a suitable one.
2. An integral basis for D_n is given by the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ . & . & . & . & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

With respect to this basis the minima are of the forms $\pm(\sum_{k=i}^{j} \mathbf{a}_k)$ $(1 < i \leq j \leq n)$, $\pm(\mathbf{a}_1 + \sum_{k=3}^{j} \mathbf{a}_k)$ for $3 \leq j \leq n$ and $\pm\left(\mathbf{a}_1 + \mathbf{a}_2 + 2\sum_{k=3}^{i-1} \mathbf{a}_k + \sum_{k=i}^{j} \mathbf{a}_k\right)$, respectively. Introducing the new basis $\{\mathbf{a}_1^* := \mathbf{a}_1 + \sum_{k=3}^{n-1} \mathbf{a}_k, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, the minima change to $\pm(\sum_{k=i}^{j} \mathbf{a}_k)$ $(1 < i \leq j \leq n)$, $\pm(\mathbf{a}_1^* - \sum_{k=j+1}^{n-1} \mathbf{a}_k))$ if $j \leq n-2$, $\pm\mathbf{a}_1^*$ for $j = (n-1)$ and $\pm(\mathbf{a}_1^* + \mathbf{a}_n)$ if $j = n$, $\pm(\mathbf{a}_1^* + \mathbf{a}_2 + \sum_{k=3}^{i-1} \mathbf{a}_k + \sum_{k=j+1}^{n-1} \mathbf{a}_k)$ for

$j \leq n-1$ and $\pm(\mathbf{a}_1^* + \mathbf{a}_2 + \sum\limits_{k=3}^{n-1} \mathbf{a}_k + \mathbf{a}_n)$ when $i = j = n$, respectively. This proves the case of D_n.

3. In the cases of the lattices E_n, $n = 6, 7, 8$ we use the so-called characteristic matrix of the minima in its first n columns the elements of the basis A. The other columns of this matrix give the coordinates of one from the opposite pairs of minima with respect to A. So the number of the columns is the number of minima divided by two. First we investigate E_6 represented in the space E^8 by

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ -1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & -1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

The characteristic matrix with respect to this basis is:

$$\left(\begin{array}{cccccc|cccccccccc|ccccccccc|} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right.$$

$$\left.\begin{array}{|cccc|cc|cc|c|c|c} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{array}\right)$$

and we get that the new integral basis $\{\mathbf{a}_1^* = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_6, \mathbf{a}_2^* = \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\}$ carrying the characteristic matrix into the required form. To do this transformation we first subtract the second row from the third and fourth ones then we subtract the first row of the new matrix from the second and sixth ones, respectively. Since in the second step we apply only such rows which are equal to the corresponding rows of the original matrix, the columns consisting of 0's and 1's change to columns containing only 0's, 1's and -1's. It is easy to check that the other columns also contain 0's,1's and -1's.)

Second we examine E_7 in E^8. Regarding the large number of minima, we decompose the characteristic matrix into two parts. The trivial part contains those minima whose coordinates with respect to A are 0's and 1's. We keep in mind from any row we subtract at most one other, then the trivial part of the matrix consists of 0's,1's and -1's. We shall give only the rest part of the characteristic matrix. The matrix A is

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & -1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & -1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

The nontrivial part of the characteristic matrix is

$$\left(\begin{array}{ccccccc|ccc|ccc|ccc|ccc|ccc|ccc|cccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 1 & 2 & 1 \\
2 & 2 & 2 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 3 & 3 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)$$

A suitable new basis is $\{\mathbf{a}_1^* = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 + \mathbf{a}_6 + \mathbf{a}_7, \mathbf{a}_2^* = \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 + \mathbf{a}_7, \mathbf{a}_3^* = \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6, \mathbf{a}_7\}$ and the row-subtraction steps are: the third from the fourth and fifth, the second from the third and seventh and the first from the second and sixth, respectively.

Finally in the case of E_8 we have a characteristic matrix with 120 columns. The original basis matrix is

$$A = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}.$$

The non-trivial part of the characteristic matrix is

$$\left(\begin{array}{ccccccccccccccc|cccccc|}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1
\end{array}\right.$$

$$\left|\begin{array}{ccccccccccccccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 5 & 5 \\
1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 5 & 5 & 4 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 4 & 4 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & -2 & -2
\end{array}\right.$$

$$\begin{array}{cccc|ccccccc|ccccccc|}
3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
5 & 5 & 5 & 5 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 6 & 5 & 5 & 5 & 5 & 5 & 5 \\
4 & 4 & 4 & 4 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 5 & 5 & 4 & 4 & 4 & 4 & 4 \\
3 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 4 & 4 & 4 & 3 & 3 & 3 & 3 \\
3 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 2 & 2 & 2 \\
2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
-2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{array}$$

$$\left|\begin{array}{rrrrrrrrrrrrrrrrrr}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2\\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 3\\
4 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3\\
4 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3\\
3 & 3 & 2 & 2 & 2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 3 & 2 & 2\\
2 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 1\\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1\\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{array}\right.$$

$$\left.\begin{array}{rrrrrrrrrrrrrrrrr}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2\\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3\\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2\\
3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1\\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0\\
1 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0\\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{array}\right)$$

and after the transformations of rows adding the eighth to the first, subtracting the third from the second, the fourth from the third, the fifth from the fourth, the sixth from the fifth and the seventh from the sixth we have a new basis, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3^* = \mathbf{a}_3 + \mathbf{a}_2, \mathbf{a}_4^* = \mathbf{a}_4 + \mathbf{a}_3 + \mathbf{a}_2, \mathbf{a}_5^* = \mathbf{a}_5 + \mathbf{a}_4 + \mathbf{a}_3 + \mathbf{a}_2, \mathbf{a}_6^* = \mathbf{a}_6 + \mathbf{a}_5 + \mathbf{a}_4 + \mathbf{a}_3 + \mathbf{a}_2, \mathbf{a}_7^* = \mathbf{a}_7 + \mathbf{a}_6 + \mathbf{a}_5 + \mathbf{a}_4 + \mathbf{a}_3 + \mathbf{a}_2, \mathbf{a}_8^* = \mathbf{a}_8 - \mathbf{a}_1\}$. With respect to this basis the absolute value of the coordinates of minima do not exceed the value 2, thus we proved the theorem. □

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