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On the k-dimensionally covered n-dimensional lattice-parallelepiped

Ákos, G. Horváth
Department of Geometry
Technical University Budapest
1521

One of the most important questions in the n-dimensional discrete geometry, lattice geometry and convex optimization is the following:
How to find a lattice point which is the nearest one to a given point of the n-dimensional euclidean space. (It's the so-called „nearest lattice point problem" see [1], [2]).
From this problem is derived the following one:
Whether a nearest lattice point can be found among the vertices of ~~the~~ a given base-parallelepiped containing this point?
In this paper we introduce a new idea and bring it into connection with the second problem.

1. Definitions, notations and remarks

Let $\{\underline{e}_1,\dots,\underline{e}_n\}\subset E^n$ be linearly independent vectors. Let $L[\underline{e}_1,\dots,\underline{e}_k]$ denote the k-dimensional subspace in E^n which is spanned by the vectors $\{\underline{e}_1,\dots,\underline{e}_k\}$ $(k\leq n)$.
$\Gamma=\Gamma[\underline{e}_1,\dots,\underline{e}_n]$ is the n-lattice with the basis $\{\underline{e}_1,\dots,\underline{e}_n\}$ and $\Pi=\Pi[\underline{e}_1,\dots,\underline{e}_n]$ is the closed base parallelepiped of Γ spanned by the lattice vectors $\{\underline{e}_1,\dots,\underline{e}_n\}$. $D^n(0)$ denote the open Dirichlet-Voronoj cell of the lattice point 0, and $\bar{D}^n(0)$ is the closure of $D^n(0)$. Let $\mathcal{D}^n$ be the n-dimensional D-V cell-system, namely $\mathcal{D}^n=\cup\{\bar{D}^n(0)\ 0\in\Gamma\}$. Let $D_{\underline{e}_{i_1}\dots\underline{e}_{i_k}}^{k}(0)$ denote the k-dimensional D-V cell of the lattice point 0 in the lattice $\Gamma[\underline{e}_{i_1},\dots,\underline{e}_{i_k}]$.

DEFINITION 1:

The k-dimensional skeleton of the parallelepiped Π is the union of its k-dimensional ~~sur~~faces.

~~DEFINITION 2:~~

The parallelepiped Π of the n-lattice Γ is k-dimensionally covered ($0 \leq k \leq n$) if its k-dimensional skeleton is covered by the n-dimensional closed D-V cells $D^n(P)$, ~~where P is vertice of $\Pi[e_1,\dots,e_n]$~~. We give some remarks ~~to the definitions~~.

REMARK 1:

Every lattice-parallelepiped is 0-dimensionally covered.

REMARK 2:

We assume, that the parallelepiped $\Pi[e_1,\dots,e_n]$ of $\Gamma[e_1,\dots,e_n]$ is spanned by the edges of an L-simplex of $\Gamma[e_1,\dots,e_n]$. (The so called L-partitions can be found, for example in [3], [4] or [5].) Then it is easy to see that the parallelepiped Π is 1-dimensionally covered.

REMARK 3:

If a base-parallelepiped is k-dimensionally covered, where $0 \leq k \leq n$, then for arbitrary number $l (0 \leq l \leq k)$ it is l-dimensionally covered, too.

2. The n-dimensionally covered parallelepiped

From the definition of the D-V region (or D-V cell) it's obvions that an n-dimensional parallelepiped is n-dimensionally covered if and only if one of the nearest lattice points to a given point of the parallelepiped can be found among the vertices of this parallelepiped. For this reason it is important to examine that in what way the n-dimensional coveredness can be guaranteed. We have seen (see REMARK 3) that the k-dimensional coveredness follows from the n-dimensional coveredness, where $0 \leq k \leq n$. The question is that out of the less-dimensional one follows, too?

STATEMENT 1:

If Π is $(n-1)$-dimensionally covered then it is n-dimensionally covered, too.

PROOF:

Assume that $K = \bigcup\{\overline{D^n(R)} \mid R \text{ is a vertex of } \Pi\}$ doesn't cover the parallelepiped Π. Then there exists such a point in the interior of Π which is lying in the set $D^n(P) \setminus K$, where P isn't a vertex of Π. ($\Pi - \mathrm{int}\,\Pi \subset K$ holds, because $\Pi - \mathrm{int}\,\Pi$ is the $(n-1)$-dimensional skeleton of Π and Π is $(n-1)$-dimensionally covered.) But $D^n(P)$ is convex and its centre P isn't lying in Π, for this reason there exists such a point Q in the $(n-1)$-dimensional skeleton of Π, which is lying in $D^n(P)$. This contradicts with the $(n-1)$-dimensional coveredness. $\square$

STATEMENT 2:

Let $\Pi[\underline{e}_1,\dots,\underline{e}_n]$ and $\Gamma[\underline{e}_1,\dots,\underline{e}_n]$ be n-dimensional, too. From the k-dimensional coveredness doesn't follows the n-dimensional coveredness if $k < n-1$.

PROOF:

Assume that $n=3$ and $k=1$ and regard the following lattice (see: figure 1):

$\underline{e}_1(1,0,0)$

$\underline{e}_2(0,1,0)$

$\underline{e}_3(\frac{1}{3},\frac{1}{3},\frac{1}{3})$

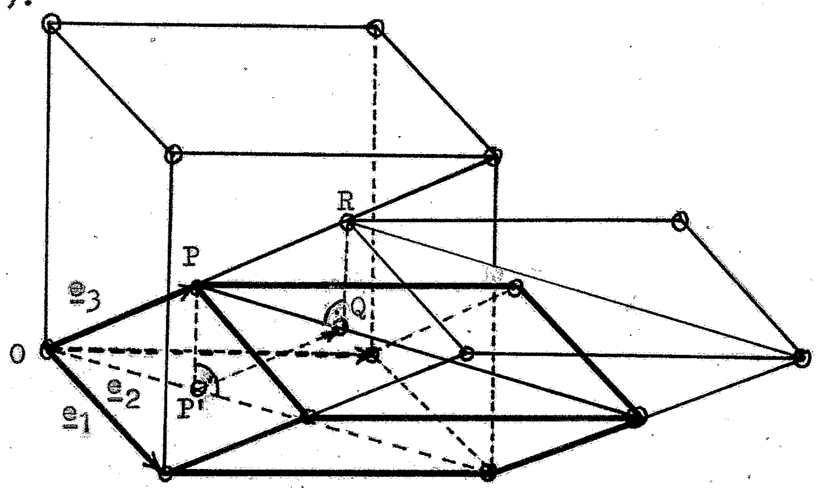

figure 1

It can readily verify that the orthogonal projection P' of the point P (where P is the end point of the vector $\underline{e}_3$) on the plane $L[\underline{e}_1, \underline{e}_2]$ is in the D-V region $D^n(P)$, from this reason the point Q is in the region $D^n(R)$ and $\Pi[\underline{e}_1, \underline{e}_2, \underline{e}_3]$ isn't 2-dimensionally covered (and isn't 3-dimensional, too). At the same time $\Pi[\underline{e}_1, \underline{e}_2, \underline{e}_3]$ is 1-dimensionally covered, because $\{e_3, e_1, e_2\}$ are successive minima of the lattice L. $\square$

Now we add a condition which can guarantee the n-dimensional coveredness. We define with induction a type of simplices, and prove that the parallelepiped which is spanned by such a simplex is n-dimensionally covered.

DEFINITION 3:

1) We say that the 2-dimensional simplex is super acute if it is an acute triangle.

2) Let k be an arbitrary natural number greater than two. The k-dimensional simplex is super acute if the following two conditions hold:

i) Every (k-1)-dimensional face of the simplex is a super acute one.

ii) Let $\underline{p}$ and $\underline{q}$ denote two vectors hawing common start point in the centre of the circumscribe ball of the simplex, and the end points of which are the centre of a (k-1)-dimensional face of the simplex, respectively. Then the angle of these vectors is obtuse. (If the centre of the simplex is on a (k-1)-dimensional face of the simplex let $\underline{p}$ denote that normal-vector of this face, which is showing outside of the simplex.)

REMARK 4:

Super acute simplex is for example the regular simplex.

LEMMA 1:

The super acute simplex includes the centre of its circumscribed ball.

PROOF:

We prove it by induction. It's obvious that in the case of $k=2$ the statement holds. Assume that there exists such face-hyperplane of the simplex, which separate the centre 0 of the simplex from the interior of the simplex. Let $\underline{p}$ denote the vector with ~~the~~ start point 0 and end point 0^{*} where 0^{*} is the centre of the $(k-1)$-dimensional separating face. Then the scalar product $\langle \underline{p} | \underline{q} \rangle$ is positive if $\underline{q}$ is such a vector which has common origin with $\underline{p}$ and is lying in that half-space which contains $\underline{p}$ and ~~which is~~ bounded by that hyperplane the normal-vector of which is $\underline{p}$ and contains the point 0. For this reason the condition ii) of the DEFINITION 3 does not hold, because the other $(k-1)$-dimensional faces of the simplex are $(k-1)$-dimensional super acute simplexes, ~~for this reason they~~ including ~~theirs~~ own centres. This is ~~then~~ a contradiction. □

STATEMENT 3:

The n-dimensional $\Pi[\underline{e}_1,\dots,\underline{e}_n]$ parallelepiped is n-dimensionally covered if the simplex $S[\underline{e}_1,\dots,\underline{e}_n]$ is super acute one.

PROOF:

Let P denote the end point of the vector $\underline{e}_n$. Then the region $D^n(P)$ is in that convex cone K, which is bounded by the perpendicular bisector hyperplanes of those edges of the simplex $S[\underline{e}_1,\dots,\underline{e}_n]$ which contain the point P. It's easy to see that the edges of K are the half-lines which have the common start point 0 (where 0 is the centre of the simplex S) and go to the centres of such

~~the~~ LEMMA 1 and the definition of the super acute simplex we have that $K \cap L[\underline{e}_1,\dots,\underline{e}_{n-1}] = \emptyset$, namely the region $D^n(P)$ ~~also~~, among the hyperplanes $L[\underline{e}_1,\dots,\underline{e}_{n-1}] + l\cdot\underline{e}_n$ $l=0,\pm1,\pm2\dots$ intersects only one. In a similar manner it can be seen that for every indices $i_1 \le \dots \le i_{n-1}$ $j \ne i_k$ $k=1,\dots,n-1$ the cell $D^n(P)$ among the hyperplanes $L[\underline{e}_{i_1},\dots,\underline{e}_{i_{n-1}}] + l\cdot\underline{e}_j$ $l=0,\pm1,\pm2\dots$ intersects only one. This means ~~that~~ ~~points~~ there is no point of Π ~~aren't~~ in the cell $D^n(Q)$ if Q isn't a vertice of Π. For this reason $\Pi[\underline{e}_1,\dots,\underline{e}_n]$ is n-dimensionally covered. This completes the proof of the statement. □

3. The connection between the (k-1)-dimensional and k-dimensional coverednesses

In this paragraph we'll prove a theorem which is characterising the k-dimensional coveredness.

LEMMA 2:

Assume, that the parallelepiped $\Pi[\underline{e}_1,\dots,\underline{e}_n]$ is k-dimensionally covered, where $0 \le k \le n$ and the commom start point of the vectors $\{\underline{e}_1,\dots,\underline{e}_n\}$ is the point O. Then the cell $D^n(O)$ can ~~be~~ intersect~~ed~~s only ~~by~~ those k-dimensional ~~sur~~faces of Π which contain the point O.

PROOF:

Assume that $D^n(O) \cap \Pi[\underline{e}_{i_1},\dots,\underline{e}_{i_k}] + \underline{e}_\ell \ne \emptyset$ where $i_1 \le \dots \le i_k$ $0 \le k < n$ and $\underline{e}_\ell \notin \{\underline{e}_{i_1},\dots,\underline{e}_{i_k}\}$. Let P denote a point of this intersection, and regard that point Q for which $\overrightarrow{OQ} = \overrightarrow{OP} - \underline{e}_\ell$. Then $Q \in \Pi[\underline{e}_{i_1},\dots,\underline{e}_{i_k}]$ and $Q \in D^n(O) - \underline{e}_\ell = D^n(R)$, where R is the end point of the vector $-\underline{e}_\ell$. But $D^n(R) \cap D^n(S) = \emptyset$ holds for every lattice point $S \ne R$, for this reason $\Pi[\underline{e}_1,\dots,\underline{e}_n]$ isn't k-dimensionally covered. This is a contradiction, and we proved ~~the~~ LEMMA 2. □

LEMMA 3:

Let O be the common start point of the vectors

$\{\underline{e}_1,\dots,\underline{e}_n\}$. Then

$$D^{k}_{\underline{e}_{i_1},\dots,\underline{e}_{i_k}}(0) \supseteq D^{n}(0) \cap L[\underline{e}_{i_1},\dots,\underline{e}_{i_k}]$$

where $0 \leq k \leq n$ is arbitrary, and $i_1 \leq \dots \leq i_k$.

PROOF:

It's easy to see using the definition of the region

$$D^{(k)}_{\underline{e}_{i_1},\dots,\underline{e}_{i_k}}(0) \quad \text{and} \quad D^{n}(0). \qquad \square$$

We can say that $\mathcal{D}^{k}_{\underline{e}_{i_1},\dots,\underline{e}_{i_k}} \supseteq \mathcal{D}^{n} \cap L[\underline{e}_{i_1},\dots,\underline{e}_{i_k}]$. It's easy to see that in the general case $\mathcal{D}^{k}_{\underline{e}_{i_1},\dots,\underline{e}_{i_k}} \supset \mathcal{D}^{n} \cap L[\underline{e}_{i_1},\dots,\underline{e}_{i_k}]$ (for example see the example of the STATEMENT 2.).

THEOREM: Assume that $0 \leq k < n$.

p~~oint 0, then~~ The following two statements are equivalent:

1) The parallelepiped Π is k-dimensionally covered in the lattice Γ.

2) Π is (k-1)-dimensionally covered and for every set of indices $1 \leq i_1 < \dots < i_k \leq n$ the following equality holds:

$$\mathcal{D}^{k}_{\underline{e}_{i_1},\dots,\underline{e}_{i_k}} = \mathcal{D}^{n} \cap L[\underline{e}_{i_1},\dots,\underline{e}_{i_k}].$$

PROOF:

1) => 2)

We have to show that

$$\underline{x} \in D^{k}_{\underline{e}_{i_1},\dots,\underline{e}_{i_k}}(0) \cap \Pi[\underline{e}_{i_1},\dots,\underline{e}_{i_k}] \Rightarrow \underline{x} \in D^{n}(0) \cap \Pi[\underline{e}_{i_1},\dots,\underline{e}_{i_k}].$$

Assume that ~~this~~ The point $\underline{x}$ isn't in the region $D^n(0) \cap \Pi[\underline{e}_{i_1}, \ldots, \underline{e}_{i_n}]$. But $\Pi[\underline{e}_1, \ldots, \underline{e}_n]$ is k-dimensionally covered and for this reason it is in ~~the~~ a cell $D^n(P)$, where P is a ~~vertice~~ vertex of the parallelepiped $\Pi[\underline{e}_1, \ldots, \underline{e}_n]$. Using Lemma 2 it follows that P is a ~~vertice~~ vertex of $\Pi[\underline{e}_{i_1}, \ldots, \underline{e}_{i_k}]$. Since $\underline{x} \in D^n(P)$ and $\underline{x} \in L[\underline{e}_{i_1}, \ldots, \underline{e}_{i_k}]$, so $\underline{x} \in D^n(P) \cap L[\underline{e}_{i_1}, \ldots, \underline{e}_{i_k}]$. Take into consideration Lemma 3 we have the following relation:

$\underline{x} \in \overline{D^{k}_{\underline{e}_{i_1}, \ldots, \underline{e}_{i_k}}(P)}$, namely $\underline{x} \in D^{k}_{\underline{e}_{i_1}, \ldots, \underline{e}_{i_k}}(0) \cap \overline{D_{\underline{e}_{i_1}, \ldots, \underline{e}_{i_k}}(P)}$.

This is a contradiction, and so we proved that the relation 1) $\Longrightarrow$ 2) holds.

Conversely, regarding a k-dimensional ~~surface~~ face containing the origin of the parallelepiped $\Pi[\underline{e}_1, \ldots, \underline{e}_n]$. Without loss of generality ~~We~~ we may assume that this ~~surface~~ face is ~~the~~ $\Pi[\underline{e}_1, \ldots, \underline{e}_k]$. Regard the k-lattice $\Gamma[\underline{e}_1, \ldots, \underline{e}_k]$ and the (k-1)-dimensional skeleton of $\Pi[\underline{e}_1, \ldots, \underline{e}_k]$. Since $\Pi[\underline{e}_1, \ldots, \underline{e}_n]$ is (k-1)-dimensionally covered, the (k-1)-dimensional skeleton of $\Pi[\underline{e}_1, \ldots, \underline{e}_k]$ is covered by the closure of that n-dimensional cell which is ~~put into~~ lying at the vertices of the parallelepiped $\Pi[\underline{e}_1, \ldots, \underline{e}_k]$. (Lemma 2.) The intersections of this cell with the plane $L[\underline{e}_1, \ldots, \underline{e}_k]$ are the D-V-regions of the lattice $\Gamma[\underline{e}_1, \ldots, \underline{e}_k]$, because the conditions $\mathfrak{D}^{(k)}_{\underline{e}_{i_1}, \ldots, \underline{e}_{i_k}} = \mathfrak{D}^n \cap L[\underline{e}_{i_1}, \ldots, \underline{e}_{i_k}]$ hold. This means that for the k-lattice $\Gamma[\underline{e}_1, \ldots, \underline{e}_k]$ the k-dimensional parallelepiped $\Pi[\underline{e}_1, \ldots, \underline{e}_k]$ is (k-1)-dimensionally covered. By Statement 1 it is also k-dimensionally covered~~, too.~~ So the parallelepiped $\Pi[\underline{e}_1, \ldots, \underline{e}_k]$ is covered by the cells $\mathfrak{D}^{(k)}_{\underline{e}_1, \ldots, \underline{e}_k}(P) = \mathfrak{D}^n \cap L[\underline{e}_1, \ldots, \underline{e}_k]$ of the vertices $\Pi[\underline{e}_1, \ldots, \underline{e}_k]$, and so we have that $\Pi[\underline{e}_1, \ldots, \underline{e}_n]$ is k-dimensionally covered. $\square$

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