Studia Scientiarum Mathematicarum Hungarica 29 (1994), 67–70

SOME REMARKS CONNECTED WITH G. CSÓKA'S PAPER "ON AN EXTREMAL PROPERTY OF MINKOWSKI-REDUCED FORMS"

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Denote $f(x)=\sum_{i,j=1}^{n} a_{ij}x_ix_j$ an n-ary positive definite quadratic form. The symmetric matrix $A=[a_{ij}]$ is the matrix of this form. In the famous work of Minkowski [1] the following interesting statement can be found without proof: "*The values*

$$s_1=\sum_{i=1}^{n} a_{ii}, \quad s_2=\sum_{i\neq k} a_{ii}a_{kk}, \quad \ldots, \quad s_n=\prod_{i=1}^{n} a_{ii}, \tag{1}$$

are minimal for the Minkowski-reduced forms of the equivalent positive definite forms, if $n\leqq 5$."

In the paper [2] G. Csóka proved this theorem for the n-dimensional cases when $n\leqq 6$ and he verified that in the cases $n>6$ the Minkowski-reduced forms do not have the above mentioned property. Our question is the following: is there such an element in every equivalence class of the n-ary positive definite forms for which the values $s_1,s_2,\ldots,s_n$ are at the same time minimal? The answer is negative if $n>6$, I will give a counter-example in Paragraph 1. This means that minimalizing the values $s_1,s_2,\ldots,s_n$ on an equivalence class, after each other, we get in general different forms of this class so we have different types of reductions. In Paragraph 2 we take some further remarks which are connected with this problem.

§ 1. In the paper [3] C. C. Ryškov gave a set of forms from which one can choose some interesting counter-examples (e.g. such a form which is Hermite-reduced but not Venkov-reduced, Venkov-reduced but not Hermitean one; the suitable definitions can be found in [4] p. 149 and p. 160 and [5]). Consider now the following positive definite form:

$$f(\mathbf{x})=\alpha(x_1^2+\ldots+x_5^2)+(1-\alpha)(x_1+\ldots+x_5)^2+x_6^2+\beta x_7^2, \tag{2}$$

where $\frac{13}{15}<\alpha<\frac{11}{12}$, and $\beta\geqq 1$. Denote by $\{\mathbf{a}_1,\ldots,\mathbf{a}_7\}$ that vector system which corresponds to f, and take the following vectors:

$$\mathbf{e}_1=\mathbf{a}_1,\ldots,\mathbf{e}_6=\mathbf{a}_6,$$
$$\mathbf{e}_7=\frac{1}{3}(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_4)+\frac{1}{4}\mathbf{a}_5+\left(\frac{1}{2}-\gamma\right)\mathbf{a}_6+\frac{\sqrt{11}}{12}\mathbf{a}_7. \tag{3}$$

1991 *Mathematics Subject Classification.* Primary 11H55; Secondary 52C07.
Key words and phrases. Reduced form, euclidean point lattice

These vectors are linearly independent. Let L'_γ be the lattice that is spanned by $\{\mathbf{e}_1, \dots, \mathbf{e}_6\}$. It is obvious that the lattice L_γ that is spanned by the vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_7\}$ is the following: $U\{L'_\gamma + k\mathbf{e}_7 | k \in \mathbb{Z}\}$. From (2) it can be seen that the shortest vectors of L'_γ are the unit vectors $\pm\mathbf{e}_1, \dots, \pm\mathbf{e}_6$, and for an other vector $\mathbf{v} \in L'_\gamma$, $f(\mathbf{v}) > 2\alpha > \frac{26}{15}$. We shall examine the cases of $k \geqq 4$ and $k = 1, 2, 3$, respectively. If $\beta \geqq 1$ and $|k| \geqq 4$ then for a vector $\mathbf{v} \in$ $\in L'_\gamma + k\mathbf{e}_7$ we have $f(\mathbf{v}) \geqq \frac{11}{9}$. Assume that $\mathbf{v} \in L'_\gamma + \mathbf{e}_7$ and $\mathbf{v} \neq \mathbf{e}_7$. From (2) we see that:

$$(4) \qquad f(\mathbf{v}) > f(\mathbf{e}_7) = \frac{109}{144} + 2(1-\alpha) + \frac{11}{144}\beta - \gamma + \gamma^2 > 1$$

if $0 < \gamma$ is sufficiently small. Similarly we get for $\mathbf{v} \in L'_\gamma \pm 2\mathbf{e}_7$ that

$$(5) \qquad \mathbf{v} = \sum_{i=1}^{4} \left(-\frac{1}{3} + m_i \right) \mathbf{a}_i + \left(\frac{1}{2} + m_5 \right) \mathbf{a}_5 + (m_6 - 2\gamma)\mathbf{a}_6 + \frac{\sqrt{11}}{6}\mathbf{a}_7,$$

where the numbers m_i are integer.

In the case of $\mathbf{v} \neq 2\mathbf{e}_7 - \sum_{i=1}^{4} \mathbf{e}_i - \mathbf{e}_6 \stackrel{\text{def}}{=} \mathbf{e}_6^\star$

$$(6) \qquad f(\mathbf{v}) \geqq f(\mathbf{e}_6^\star) = \frac{100}{144} + \frac{44}{144}\beta + 4\gamma^2 > 1.$$

In the last case we suppose that

$$\mathbf{v} \neq \mathbf{e}_7^\star := 3\mathbf{e}_7 - \sum_{i=1}^{6} \mathbf{e}_i = -\frac{1}{4}\mathbf{a}_5 + \left(\frac{1}{2} - 3\gamma \right) \mathbf{a}_6 + \frac{\sqrt{11}}{4}\mathbf{a}_7$$

and so we have if $\beta > 10\gamma + 1$ that

$$(7) \qquad f(\mathbf{v}) > f(\mathbf{e}_7^\star) = \frac{45}{144} + \frac{99}{144}\beta - 3\gamma + 9\gamma^2 > 1.$$

It is clear that the vector systems $\{\mathbf{e}_1, \dots, \mathbf{e}_7\}$ and $\{\mathbf{e}_1, \dots, \mathbf{e}_6^\star, \mathbf{e}_7^\star\}$ are the bases of the same lattice L_γ so the corresponding forms are equivalent. If we take now the following parameters:

$$(8) \qquad \alpha = \frac{263}{288}, \quad \beta = \frac{133}{132}, \quad \gamma = 10^{-10},$$

then

$$(9) \qquad \begin{aligned} &\mathbf{e}_1^2 = \dots = \mathbf{e}_6^2 = 1 \quad \mathbf{e}_7^2 = 1 + \frac{1}{144} + \frac{11}{132 \cdot 144} - 10^{-10} + 10^{-20}, \\ &(\mathbf{e}_6^\star)^2 = 1 + \frac{44}{132 \cdot 144} + 4 \cdot 10^{-20}, \\ &(\mathbf{e}_7^\star)^2 = 1 + \frac{99}{132 \cdot 144} - 3 \cdot 10^{-10} + 9 \cdot 10^{-20}, \end{aligned}$$

and so

$$\mathbf{e}_6^{\star 2} + \mathbf{e}_7^{\star 2} < \mathbf{e}_6^2 + \mathbf{e}_7^2, \tag{10}$$

thus the basis of L_γ for which the value s_1 is minimal is the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_6^\star, \mathbf{e}_7^\star\}$, but

$$\begin{aligned} \mathbf{e}_6^{\star 2} \cdot \mathbf{e}_7^{\star 2} &> 1 + \frac{143}{132 \cdot 144} + \frac{44 \cdot 99}{(132 \cdot 144)^2} - 3 \cdot 10^{-10} \left[1 + \frac{44}{132 \cdot 144}\right] > \\ &> 1 + \frac{143}{132 \cdot 144} - 10^{-10} + 10^{-20} = \mathbf{e}_6^2 \cdot \mathbf{e}_7^2, \end{aligned} \tag{11}$$

for this reason the values $s_2, \ldots, s_7$ are not minimal for the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_6^\star, \mathbf{e}_7^\star\}$. So we have proved the following

THEOREM. *If $n \geqq 7$ then there exists such an equivalence class of the n-ary positive forms for which the functions $s_1, \ldots, s_n$ take their minima on different elements.*

§ **2.** First we note that in the above mentioned example the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_6, \mathbf{e}_7\}$ of L_γ is a Hermite-reduced one and the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_6^\star, \mathbf{e}_7^\star\}$ of L_γ is a Venkov-reduced form with respect to the form $\varphi = x_1^2 + \ldots + x_n^2$. Secondly, it can be seen that in L_γ there is no basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ for which $|\mathbf{f}_1| \leqq |\mathbf{f}_2| \leqq \ldots \leqq |\mathbf{f}_n|$ and if $\{\mathbf{g}_1, \ldots, \mathbf{g}_n\}$ is a basis of L_γ for which $|\mathbf{g}_1| \leqq \ldots \leqq |\mathbf{g}_n|$ then $|\mathbf{f}_i| \leqq |\mathbf{g}_i|$, $i = 1, \ldots, n$. (Such a basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is a Hermitean one, but in this lattice the length of the last vector of a Hermite basis is greater than $|\mathbf{e}_7^\star|$.) At third we remark that for the linearly independent vector system of a lattice L the following is true:

STATEMENT. *Let L be an n-lattice and $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset L$ a linearly independent vector system for which $|\mathbf{a}_1| \leqq \ldots \leqq |\mathbf{a}_n|$. Then the following statements are equivalent:*

(i) *$\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a successive minimum system of L;*
(ii) *if the vectors of the system $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are independent and $|\mathbf{b}_1| \leqq \ldots \leqq |\mathbf{b}_n|$ then $|\mathbf{a}_i| \leqq |\mathbf{b}_i|$;*
(iii) *the system $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is the common minimum of the functions $s_1, \ldots, s_n$ on the set of the independent systems containing n elements of L;*
(iv) *the system $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is the minimum of the function s_1.*

This statement follows, for example, from the Rado-Edmonds theorem for matroids (see [6]).

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(Received October 15, 1990)

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