BOLYAI SOCIETY
MATHEMATICAL STUDIES, 6

Intuitive Geometry
Budapest (Hungary), 1995
Budapest, 1997, pp. 327–334.

DISSECTIONS OF A CENTRALLY SYMMETRIC HEXAGON

Á. G. HORVÁTH

In this paper we prove that there are six possibilities for the decomposition of a centrally symmetric convex hexagon into centrally symmetric convex parts if the dissection is irreducible and face-to-face. We give an infinite sequence of irreducible dissections where the number of the components are $\frac{n(n+5)}{2}$, respectively.

1. INTRODUCTION

The problem is the following: How can we dissect a centrally symmetric convex hexagon into centrally symmetric convex parts? This question is motivated by the following problem of geometry of numbers: Discribe all combinatorial types of the Dirichlet-Voronoi cell (see [13], [5]) of the lattices of dimension n. It is known that such a cell consists of a tetrahedral or hexagonal zone (see e.g. [11], [8]) of $(n-1)$-dimensional faces and two congruant caps. Such a cap contains some centrally symmetric facets meeting in certain faces of dimension $(n-2)$. The centers of the facets of the zone determine a two dimensional plane. The orthogonal projection of the cells whose centers are on this plane form a plane-tiling which contains a lattice tiling (formed by the projection of the corresponding zones) of parallelograms or centrally symmetric convex hexagons, respectively. One of these polygons is decomposed by the projection of one of the corresponding caps to centrally symmetric convex parts thus knowing the dissections above, we have an important information about the combinatorial characterization of the Dirichlet-Voronoi cells. This is a central problem in geometry of numbers (see [2], [4], [6] or [7]).

Supported by Hung.Nat.Found for Sci.Reseach (OTKA) grant no.T.7351 (1994)

Similar dissectional problems on the plane were investigated by the authors Monsky [10], Stein [12] and Kasimatis [9].

2. Central symmetry and irreducibility

Lemma 1 basically determines the structure of a centrally symmetric convex decomposition of a convex polygon.

Lemma 1. *If the convex polygon P can be decomposed into convex centrally symmetric parts then it is centrally symmetric.*

The proof of this lemma is the first step of Alexandrov's theorem on convex polyhedra with centrally symmetric faces. (The theorem says that such a polyhedron has a centre of symmetry.) See e.g. [1] or [3]. An important observation of this proof that all of the edges of the dissection are parallel to an edge of the polygon P. Especially in the case when P is a hexagon we have only three possible directions for the "inner" edges of the decomposition.

It is possible that the union of some components of the dissection of P formes such a centrally symmetric convex polygon which is a proper part of P. In this case the dissection $\mathcal{P}$ can be reduced into another dissection of P. (The mentioned union as a new component reduce the number of the components.) This observation motivates the following definition:

Definition 1. *The dissection $\mathcal{P} = \{P_1, \dots, P_k\}$ of P is irreducible if and only if from the convexity of the set $\cup\{P_i \mid i \in \mathcal{I}\}$ it follows that $|\mathcal{I}| = 1$ or k.*

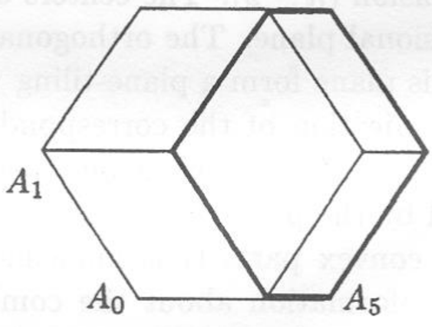

Fig. 1. Reducible dissection

We remark that from the convexity of Q by Lemma 1 follows that Q is centrally symmetric, so an irreducible dissection can not be simplified. (It can be seen a reducible dissection in Fig.1.) The following theorem shows that the assumption of irreducibility is not enough to guarantee the finitely many possibilities for the decomposition.

Theorem 1. *There is an infinite sequence of irreducible dissections of a centrally symmetric hexagon into finite number of centrally symmetric convex parts.*

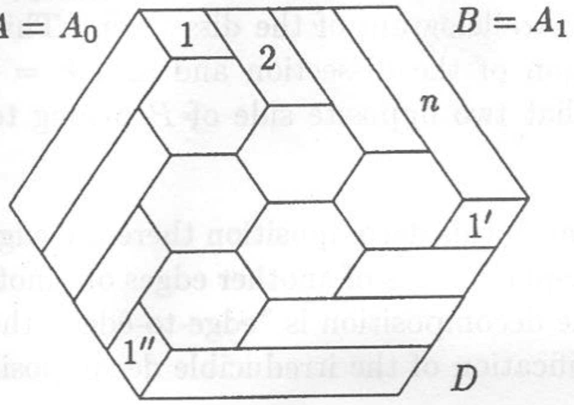

Fig. 2. Irreducible decomposition with $\frac{n(n+5)}{2}$ components

Proof. Consider a regular hexagon P and divide its edges into n segments of the same length. Those lines which parallel to the edges of the hexagon and contain an endpoint of the above segments give a decomposition of P into congruant regular triangles. Let AD be a main diagonal of the hexagon and AB be a side of P. We denote the points dividing AB by $A = A_0, \dots, B = A_n$, respectively. We define the first n components of the dissection by those parallelograms which have an edge parallel to AD, its other edge $A_0A_1, \dots, A_{n-1}A_n$ and consisting of $2, 4, \dots, 2n$ triangles, respectively. (See Fig.2). The $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$-rotations of these parallelograms about the centre of P give further $2n$ parallelograms which are also components of the dissection. It is easy to check that the uncovered part of the hexagon can be filled by $\frac{(n-1)n}{2}$ regular hexagons consisting of 6 triangles, respectively. So the number of the components:

$$3n + \frac{n(n-1)}{2} = \frac{n(n+5)}{2}.$$

On the other hand this decomposition is irreducible. Indeed, the union of the inner edges consist of three parallel pairs of edges of lengths s, $2 \le s \le n$, and some edges of length 1. (The unit is the length of the edges of a small triangle.) If Q is a convex proper subset of P consisting of the elements of the dissection $\{P_1,\ldots,P_k\}$ then it is a centrally symmetric polygon with at most three edge-directions. Thus there are only three possibilities:

1. Q is a parallelogram having edges shorter than n and thus $k=1$.
2. Q is a hexagon having edges shorter than n.
3. Q has an edge of length n.

In the second case the common lengths of the edges is 1 because two parallel edges with the same length $s\geq 2$ as the opposite sides of Q can determine only a parallelogram of the dissection. This means that in this case Q is a hexagon of the dissection and also $k=1$. In the last case it is easy to see that two opposite side of P belong to Q, so $Q=P$ and $k=\frac{n(n+5)}{2}$. ∎

We remark that in this decomposition there are edges of certain components which are proper subsets of another edges of (another) components. If we assume that the decomposition is "edge-to-edge" then we have a chance for the finite classification of the irreducible decompositions.

3. Edge-to-edge irreducible decompositions

In this section we describe all types of the edge-to-edge irreducible decompositions of a centrally symmetric convex hexagon.

Definition 2. *The dissection $\mathcal{P}$ of a convex polygon is edge-to-edge if each inner edge is an edge of precisely two distinct components of $\mathcal{P}$.*

We remark that the property "edge-to-edge" does not imply the property irreducibility as it can be seen in Fig.1.

Theorem 2. *There are only six combinatorial types of the edge-to-edge irreducible dissections of a centrally symmetric convex hexagon into finite number of centrally symmetric convex parts. One representant of each class can be seen in Fig.3 and Fig.4.*

Proof. Without loss of generality, we may assume that P is regular hexagon. First we notice that if an edge of the hexagon is divided by n segments by the components of the dissection $\mathcal{P}=\{P_1,\ldots,P_k\}$ then n is not greater than three. Indeed, let A_0A_1 be an edge of P and let e be an inner edge of the dissection ending in a relative inner point E of A_0A_1. Then the component P_1 is spanned by e and the segment s of A_0A_1, and having

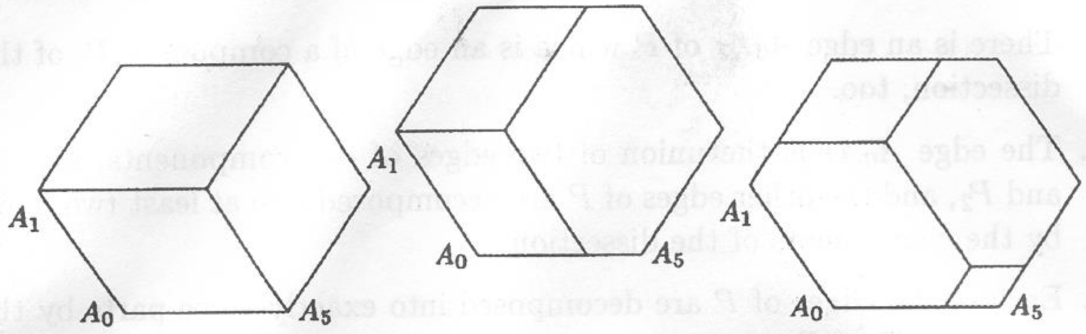

Fig. 3. Edge-to-edge irreducible dissections into $k = 3$ and $k = 4$ parts.

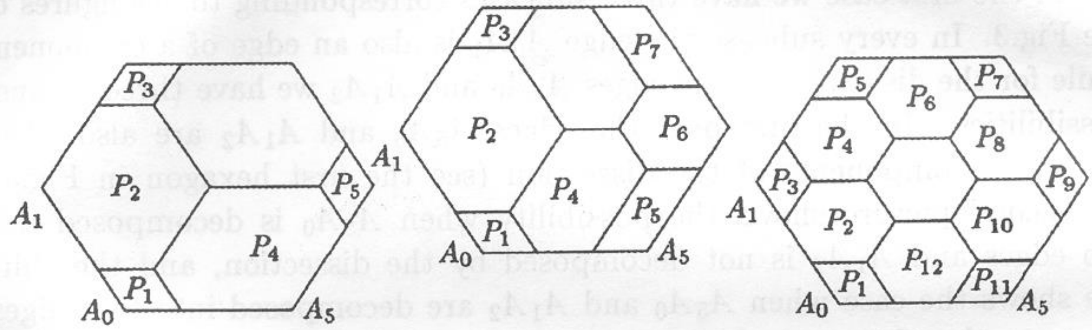

Fig. 4. Edge-to-edge irreducible dissections into $k = 6, 8$ and 13 parts.

an acute angle at E (see Fig.5) is a parallelogram. From the irreducibility we get that the other endpoint of s is an endpoint of the edge of P. (In the first picture of Fig.5 it is A_0.) Hence on the edge A_0A_1 there are at most two distinct vertices of the dissection. (Apart from A_0 and A_1.)

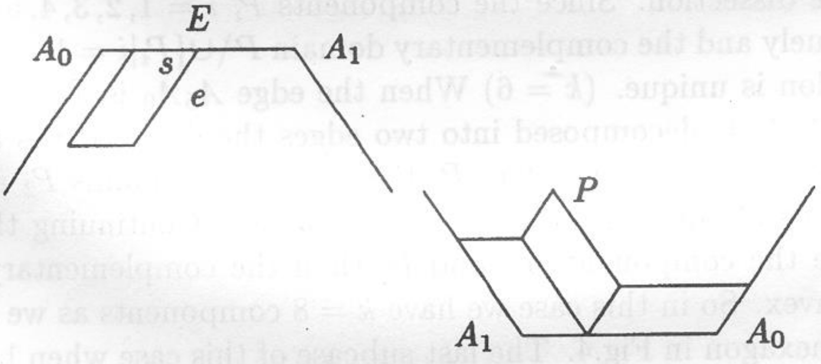

Fig. 5.

Another important observation is the fact that the configuration in the second picture of Fig.5 can not occur. Indeed because of irreducibility at the point P we can't continue the decomposition, all of the possible edges give such a convex part of the decomposition which contains at least two distinct components.

Hence we distinguish the following cases:

1. There is an edge A_0A_1 of P which is an edge of a component P_1 of the dissection, too.
2. The edge A_0A_1 is the union of two edges of two components, say, P_1 and P_2, and the other edges of P are decomposed into at least two parts by the components of the dissection.
3. Each of the edges of P are decomposed into exactly three parts by the dissection $\mathcal{P}$ of P.

In the first case we have three subcases corresponding to the figures of the Fig.3. In every subcase the edge A_0A_1 is also an edge of a component while for the dissection of the edges A_5A_0 and A_1A_2 we have three distinct possibilities. In the first one, the edges A_5A_0 and A_1A_2 are also edges of certain components of the dissection (see the first hexagon in Fig.3), the second picture shows the possibility when A_5A_0 is decomposed into two edges and A_1A_2 is not decomposed by the dissection, and the third one shows the case when A_5A_0 and A_1A_2 are decomposed into two edges. We note that because of the irreducibility, A_5A_0 and A_1A_2 can not be decomposed into three parts in this cases. The number of the components is $k = 3$ and $k = 4$, respectively.

The second case contains those possibilities when A_5A_0 and A_1A_2 decompose into two or three edges by the dissection. The first hexagon in Fig.4 gives that case when the edges A_5A_0 and A_1A_2 are also unions of two edges of the dissection. Since the components P_i $i = 1,2,3,4,5$ are determined uniquely and the complementary domain $P\setminus\cup\{P_i|i = 1...5\}$ is convex this dissection is unique. ($k = 6$) When the edge A_5A_0 is a union of three edges and A_1A_2 is decomposed into two edges the dissection is also determined, starting the parallelogram P_1 the contigous domains P_2 and P_4 are hexagons thus P_3 and P_5 are parallelograms, too. Continuing the process first we give the components P_6 and P_7 than the complementary domain, which is convex. So in this case we have $k = 8$ components as we can see in the second hexagon in Fig.4. The last subcase of this case when both of the edges A_5A_0 and A_1A_2 can decompose into three segments.(See Fig.6) First we draw the parallelogram P_1 and the hexagon P_2. P_2 can not be a parallelogram because $P_1 \cup P_2$ is not convex domain. This mean that the edge e of P_2 is parallel to the edge A_0A_5 so from the condition of irreducibility the point E is A_2. This is a contradiction thus in this case we don't have realizable dissection.

The last case gives the last dissection of Fig.4. Really, the first thirteen components $P_1, \ldots, P_{12}$ are determined unique by the dissections of the

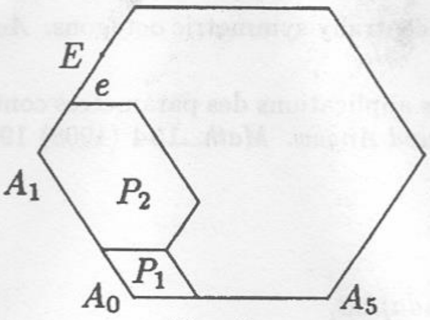

Fig. 6.

edges A_5A_0, A_0A_1 and A_1, A_2. The domain $P \setminus \cup\{P_i \quad i = 1, \ldots, 12\}$ is a convex hexagon so we have again only one possibility for the dissection.

Thus the list of Fig.3 and Fig.4 is complete. We note that the chains of parallel opposite edges contain segments with the same length thus the dissections in Fig.3 and Fig.4 contains centrally symmetric parts, which shows that it has six edge-to-edge irreducible dissections of a centrally symmetric convex hexagon into centrally symmetric convex parts. ■

REFERENCES

[1] A. D. Alexandrov, *Konvexe Polyeder*, Akademie-Verlag, Berlin, 1958.

[2] J. M. Conway, N. J. A. Sloane, *Sphere Packings, Lattices and Groups.* Springer-Verlag, 1988.

[3] N. N. Csencov, I. M. Jaglom, D. O. Skljarszkij, Válogatott feladatok és tételek az elemi matematika köréből, *Geometria II.* (Sztereometria), Tankönyvkiadó, 1968.

[4] B. N. Delone, Sur la partition reguliere de l'espace a 4 dimensions. *Izv. Akad. Nauk SSSR Otdel. Fiz.-Mat. Nauk* **7** (1929), 79–110, 147–164.

[5] G. L. Dirichlet, Über die Reduktion der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen. *J.Reine und Angew. Math.* **40** (1850), 209–227.

[6] L. Fejes Tóth, *Regular figures.* Pure and Applied Mathematics, Vol. 48, Pergamon Press, 1964.

[7] Á. G. Horváth, On Dirichlet-Voronoi cell Part I. Classical problems *Per. Poly. ser Mech. Eng.* **39/1** (1995), 25–42.

[8] P. M. Gruber, C. G. Lekkerkerker, *Geometry of numbers.* North-Holland, Amsterdam-New York-Oxford-Tokyo, 1987.

[9] Kasimatis, Dissections of regular polygons into triangles of equal areas. *Discr. and Comput. Geom* **4** (1989), 375–381.

[10] Monsky, On dividing a square into triangles. *Amer. Math. Monthly* **77** (1970) 161–164.

[11] P. McMullen, Convex bodies which tile space by translation. *Math.* **27** (1980), 113–121; **28** (1981), 191.

[12] Stein, Equidissectras of centrally symmetric octagons. *Aequatianes Math.* **37** (1989), 313–318.

[13] G. F. Voronoi, Nouvelles applications des parametres continus a la theorie des formes quadratiques. *J.Reine und Angew. Math.* **134** (1908) 198–287.

Ákos G. Horváth

Technical University of Budapest,
Department of Geometry,
1111 Budapest,
Egry J. u. 1. H. II. 22,
Hungary

ghorvath@vma.bme.hu