HYPERBOLIC PLANE-GEOMETRY REVISITED

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Abstract. Using the method of C. Vörös, we establish several results on hyperbolic plane geometry, related to triangles and circles. We present a model independent construction for Malfatti's problem and several (more then fifty) trigonometric formulas for triangles.

1. Introduction

As J. W. Young, the editor of the book [11], wrote in his introduction: *There are fashions in mathematics as well as in clothes – and in both domains they have a tendency to repeat themselves.* During the last decade, "hyperbolic plane geometry" aroused much interest and was investigated vigorously by a considerable number of mathematicians, as we can see from the large number of Google Scholar items given for the same expression as key words (192,000 results in (0.05 sec)). Despite the large number of items, the number of hyperbolic trigonometric formulas that can be collected from them is fairly small, they can be written on a page of size B5. (We also present such a collection in the second half of this introduction.) This observation is very surprising if we compare it to the fact that already in 1889, a very extensive and elegant treatise of spherical trigonometry was written by John Casey [5]. For this, the reason, probably, is that the discussion of a problem in hyperbolic geometry is less pleasant than in the spherical one.

On the other hand, in the 19th century, an excellent mathematician – Cyrill Vörös[1] in Hungary made a big step to solve this problem. He introduced a method for the measurement of distances and angles in the case that the considered points or lines, respectively, are not real. Unfortunately, since he published his works mostly in Hungarian or in Esperanto, his method is not well-known to the mathematical community.

To fill this gap, we use the concept of distance extracted from his work and, translating the standard methods of Euclidean plane geometry into the hyperbolic plane, apply it for various configurations. We give a model independent construction for the famous problem of Malfatti (discussed in [8]) and prove more than fifty interesting formulas connected with the geometry of hyperbolic triangles. By the notion of distance introduced by Vörös, we obtain results on hyperbolic plane geometry which are not well-known.

1.1. **Well-known formulas on hyperbolic trigonometry.** In this paper, we use the following notations. The points A, B, C denote the vertices of a triangle. The lengths of the edges opposite to these vertices are a, b, c, respectively. The angles at A, B, C are denoted by α, β, γ, respectively. If the triangle has a right angle, it is always at C. The symbol δ denotes half of the area of the triangle; more precisely, we have $2\delta = \pi - (\alpha + \beta + \gamma)$.

- **Connections between the trigonometric and hyperbolic trigonometric functions:**
$$\sinh a = \frac{1}{i}\sin(ia), \quad \cosh a = \cos(ia), \quad \tanh a = \frac{1}{i}\tan(ia)$$
- **Law of sines:**
$$\sinh a : \sinh b : \sinh c = \sin\alpha : \sin\beta : \sin\gamma \tag{1}$$
- **Law of cosines:**
$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma \tag{2}$$
- **Law of cosines on the angles:**
$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cosh c \tag{3}$$

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[1]Cyrill Vörös (1868 –1948), piarist, teacher

- **The area of the triangle:**

$$T := 2\delta = \pi - (\alpha + \beta + \gamma) \text{ or } \tan\frac{T}{2} = \Big(\tanh\frac{a_1}{2} + \tanh\frac{a_1}{2}\Big)\tanh\frac{m_a}{2} \tag{4}$$

where m_a is the height of the triangle corresponding to A and a_1, a_2 are the signed lengths of the segments into which the foot point of the height divide the side BC.

- **Heron's formula:**

$$\tan\frac{T}{4} = \sqrt{\tanh\frac{s}{2}\tanh\frac{s-a}{2}\tanh\frac{s-b}{2}\tanh\frac{s-c}{2}} \tag{5}$$

- **Formulas on Lambert's quadrangle:** The vertices of the quadrangle are A, B, C, D and the lengths of the edges are $AB = a, BC = b, CD = c$ and $DA = d$, respectively. The only angle which is not right-angle is $BCD\measuredangle = \varphi$. Then, for the sides, we have:

$$\tanh b = \tanh d \cosh a, \quad \tanh c = \tanh a \cosh d,$$

and

$$\sinh b = \sinh d \cosh c, \quad \sinh c = \sinh a \cosh b,$$

moreover, for the angles, we have:

$$\cos\varphi = \tanh b \tanh c = \sinh a \sinh d \quad \sin\varphi = \frac{\cosh d}{\cosh b} = \frac{\cosh a}{\cosh c},$$

and

$$\tan\varphi = \frac{1}{\tanh a \sinh b} = \frac{1}{\tanh d \sinh c}.$$

2. The distance of the points and on the lengths of the segments

First we extract the concepts of the distance of real points following the method of the book of Cyrill Vörös ([17]). We extend the plane with two types of points, one type of the points at infinity and the other one the type of ideal points. In a projective model these are the boundary and external points of a model with respect to the embedding real projective plane. Two parallel lines determine a point at infinity and two ultraparallel lines an ideal point which is the pole of their common transversal. Now the concept of the line can be extended; a line is real if it has real points (in this case it also has two points at infinity and the other points on it are ideal points being the poles of the real lines orthogonal to the mentioned one). The extended real line is a closed compact set with finite length. We also distinguish the line at infinity which contains precisely one point at infinity and the so-called ideal line which contains only ideal points. By definition the common lengths of these lines are $\pi k i$, where k is a constant of the hyperbolic plane and i is the imaginary unit. In this paper we assume that $k = 1$. Two points on a line determine two segments AB and BA. The sum of the lengths of these segments is the lengths of the line $AB + BA = \pi i$. We define the length of a segment as an element of the linearly ordered set $\bar{\mathbb{C}} := \overline{\mathbb{R}} + \mathbb{R}\cdot i$. Here $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the linearly ordered set of real numbers extracted with two new numbers with the "real infinity" ∞ and its additive inverse $-\infty$. The infinities can be considered as new "numbers" having the properties that either "there is no real number greater or equal to ∞" or "there is no real number less or equal to $-\infty$". We also introduce the following operational rules: $\infty + \infty = \infty$, $-\infty + (-\infty) = -\infty$, $\infty + (-\infty) = 0$ and $\pm\infty + a = \pm\infty$ for real a. It is obvious that $\overline{\mathbb{R}}$ is not a group, the rule of associativity holds only such expressions which contain at most two new objects. In fact, $0 = \infty + (-\infty) = (\infty + \infty) + (-\infty) = \infty + (\infty + (-\infty)) = \infty$ is a contradiction. We also require that the equality $\pm\infty + bi = \pm\infty + 0i$ holds for every real number b and for brevity we introduce the respective notations $\infty := \infty + 0i$ and $-\infty := -\infty + 0i$. We extract the usual definition of hyperbolic function based on the complex exponential function by the following formulas

$$\cosh(\pm\infty) := \infty, \sinh(\pm\infty) := \pm\infty, \text{ and } \tanh(\pm\infty) := \pm 1.$$

We also assume that $\infty\cdot\infty = (-\infty)\cdot(-\infty) = \infty$, $\infty\cdot(-\infty) = -\infty$ and $\alpha\cdot(\pm\infty) = \pm\infty$.

Assuming that the trigonometric formulas of hyperbolic triangles are also valid with ideal vertices the definition of the mentioned lengths of the complementary segments of a line are given. For instance, consider a triangle with two real vertices (B and C) and an ideal one (A), respectively. The lengths of the segments between C and A are b and b', the lengths of the segments between B and A are c and c' and the lengths of that segment between C and B which contains only real points is a, respectively. Let the right angle be at the vertex C and denote by β the other real angle at B. (See in Fig. 1.)

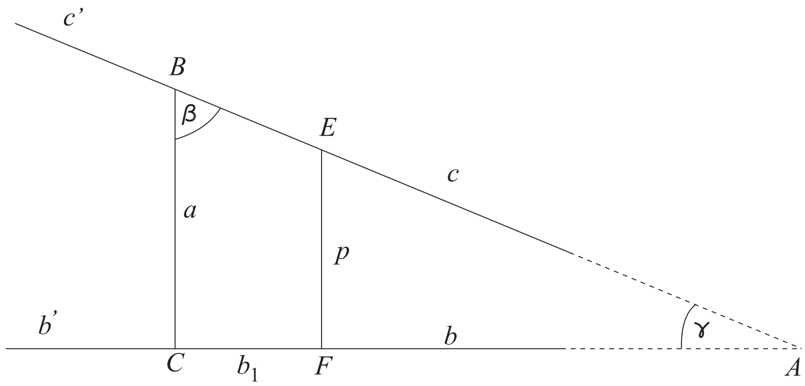

FIGURE 1. Length of the segments between a real and an ideal point

With respect to this triangle we have $\tanh b = \sinh a \cdot \tan\beta$ and since A is an ideal point, the parallel angle corresponding to the distance $\overline{BC} = a$ less or equal to β. Hence $\tan\beta > 1/\sinh a$ implying that $\tanh b > 1$. Hence b is a complex number. Let the polar of A is EF, then it is the common perpendicular of the lines AC and AB. The quadrangle $CFEB$ has three right angle. Denote by b_1 the length of that segment $\overline{CF}$ which contains real points only. Then we get

$$\tan\beta = \frac{1}{\tanh b_1 \sinh a},$$

meaning that

$$\sinh a \tan\beta = \frac{1}{\tanh b_1} = \tanh b.$$

Similarly we have that $\tanh b' = \sinh a \cdot \tan(\pi - \beta) = -\sinh a \cdot \tan\beta$ implying that $|\tanh b'| > 1$ hence b' is complex. Now we have that

$$\tanh b' = -\frac{1}{\tanh b_1}.$$

Using the formulas between the trigonometric and hyperbolic trigonometric functions we get that

$$\frac{1}{i}\tan ib = \frac{i}{\tan ib_1},$$

implying that

$$\tan ib = -\tan\left(\frac{\pi}{2} - ib_1\right)$$

so

$$b = -\frac{2n-1}{2}\pi i + b_1.$$

Analogously we get also that

$$b' = -\frac{2m+1}{2}\pi i - b_1.$$

Here n and m are arbitrary integers. On the other hand if $b_1 = 0$ then $AC = CA$ and so $b = b'$ meaning that $2n - 1 = 2m + 1$. For the half length of the complete line we can choose an odd multiplier of the number $\pi i/2$. The most simple choosing is when we assume that $n = 0$ and $m = -1$. Thus the lengths of the segments AC and CA can be defined as

$$b = b_1 + \frac{\pi}{2} \text{ and } b' = -b_1 + \frac{\pi}{2}$$

hold, respectively.

We now define all of the possible lengths of a segment on the basis of the type of the line contains them.

2.1. **The points A and B are on a real line.** We can distinguish six subcases. The definitions of the respective cases can be found in Table 1. In this table d means a real (positive) distance of the corresponding usual real elements which are a real point or the real polar line of an ideal point, respectively. Every box in the table contains two numbers which are the lengths of the two segments determined by the two points. For example, the distance of a real and an ideal point is a complex number. Its real part is the distance of the real point to the polar of the ideal point with a sign, this sign is positive in the case when the polar line intersects the segment between the real and ideal points, and is negative

		B		
	(AB) is real	real	infinite	ideal
A	real	$AB = d$ $BA = -d + \pi i$	$AB = \infty$ $BA = -\infty$	$AB = d + \frac{\pi}{2} i$ $BA = -d + \frac{\pi}{2} i$
	infinite		$AB = \infty$ $BA = -\infty$	$AB = \infty$ $BA = -\infty$
	ideal			$AB = d + \pi i$ $BA = -d$

TABLE 1. Distances on the real line.

		B	
	(AB) is at infinity	infinite	ideal
A	infinite	$AB = 0$ $BA = \pi i$	$AB = \frac{\pi}{2} i$ $BA = \frac{\pi}{2} i$
	ideal		$AB = 0$ $BA = \pi i$

TABLE 2. Distances on the line at infinity.

otherwise. The imaginary part of the length is $(\pi/2)i$, implying that the sum of the lengths of two complementary segments of this projective line has total length πi. Consider now an infinite point. This point can also be considered as the limit of real points or limit of ideal points of this line. By definition the distance from a point at infinity of a real line to any other real or infinite point of this line is $\pm\infty$ according to that it contains or not ideal points. If, for instance, A is an infinite point and B is a real one, then the segment AB contains only real points has length ∞. It is clear that with respect to the segments on a real line the length-function is continuous.

2.2. **The points A and B are on a line at infinity.** We can check that the length of a segment for which A or B is an infinite point is indeterminable. To see this, let the real point C be a vertex of a right-angled triangle which other vertices A and B are on a line at infinity with infinite point B. Then we get that $\cosh c = \cosh a \cdot \cosh b$ for the corresponding sides of this triangle. But from the result of the previous subsection

$$\cosh a = \cosh \infty = \infty \text{ and } \cosh b = \cosh\left(0 + \frac{\pi}{2} i\right) = \cos\left(-\frac{\pi}{2}\right) = 0,$$

showing that their product is undeterminable. On the other hand if we consider the polar of the ideal point A we get a real line through B. The length of a segment connecting the (ideal) point A and one of the points of its polar is $(\pi/2)i$. This means that we can define the length of a segment between A and B also as this common value. Now if we want to preserve the additivity property of the lengths of segments on a line at infinity, too then we must give the pair of values $0, \pi i$ for the lengths of segment with ideal ends. The Table 2 collects these definitions.

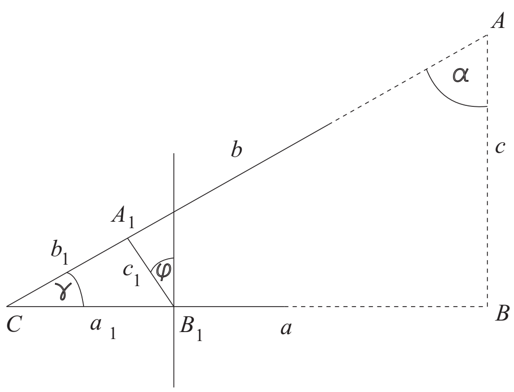

FIGURE 2. The length of an ideal segment

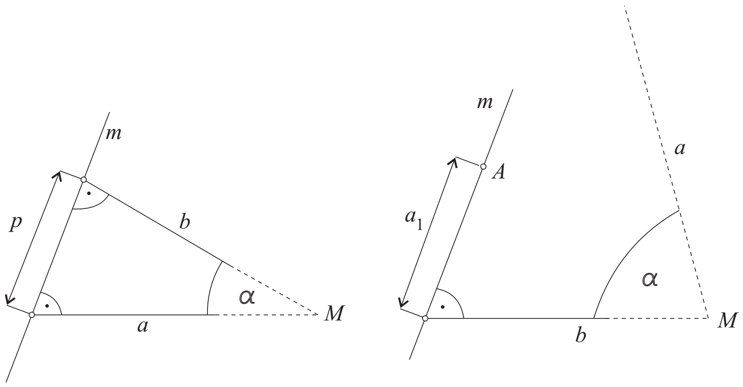

FIGURE 3. Angles at an ideal point

2.3. **The points A and B are on an ideal line.** This situation contains only one case: A, B and AB are ideal elements, respectively. Use the notation of Fig.1. Then $\cos\alpha = \cosh a \cdot \sin\beta$, and since β is greater than the parallel angle corresponding to the segment a we get that $\cosh a \cdot \sin\beta > 1$ so $\cos\alpha > 1$. Hence α is an imaginary number. From the Lambert's quadrangle $BCEF$ we get

$$\cosh a \sin\beta = \cosh p,$$

thus $\cosh p = \cos\alpha$ and so $\alpha = 2n\pi \pm pi$. Now an elementary analysis of the figure shows that the continuity property requires the choice $n = 0$. If we also assume that we choose the negative sign then the measure is $\alpha = -pi = p/i$, where p is the length of that segment of the common perpendicular which points are real.

Consider now an ideal line and its two ideal points A and B, respectively. The polars of these points intersect each other in a real point B_1. Consider a further real point C of the line BB_1 and denote by A_1 the intersection point of the polar of A and the real line AC (see Fig. 2).

Observe that A_1B_1 is perpendicular to AC thus we have $\tanh b_1 = \tanh a_1 \cdot \cos\gamma$. On the other hand $a = \pm a_1 + (\pi i)/2$ and $b = \pm b_1 + (\pi i)/2$ implying that $\tanh b = \tanh a \cdot \cos\gamma$. Hence the angle between the real line CB and the ideal line AB can be considered to $\pi/2$, too. Now from the triangle ABC we get that

$$\cosh c = \frac{\cosh b}{\cosh a} = \frac{\pm i \sinh b_1}{\pm i \sinh a_1} = \frac{\sinh b_1}{\sinh a_1} = \sin\left(\frac{\pi}{2} - \varphi\right) = \cos\varphi,$$

where φ is the angle of the two polars. From this we get $c = 2n\pi \pm \varphi/i = 2n\pi \mp \varphi i$. We choose $n = 0$ since at this time $\varphi = 0$ implies $c = 0$ and the positive sign because the length of the line is πi.

The length of an ideal segment on an ideal line is the angle of their polars multiplied by the imaginary unit i.

		a					
		real			at infinity		ideal
		M			M		M
		real	infinite	ideal	infinite	ideal	ideal
b	real	φ $\pi - \varphi$	0 π	$\frac{p}{i}$ $\pi - \frac{p}{i}$	$\frac{\pi}{2}$ $\frac{\pi}{2}$	∞ $-\infty$	$\frac{\pi}{2} + \frac{a_1}{i}$ $\frac{\pi}{2} - \frac{a_1}{i}$
	at infinity					M: ideal ∞ $-\infty$	M: ideal ∞ $-\infty$
	ideal						M: ideal $\frac{p}{i}$ $\pi - \frac{p}{i}$

TABLE 3. Angles of lines.

2.4. **Angles of lines.** Similarly we can deduce the angle between arbitrary kind of lines. We can find it in Table 3, where a and b are the given lines, $M = a \cap b$ is their intersection point, m is the polar of M and A and B is the poles of a and b, respectively. The numbers p and a_1 represent real distances, respectively, can be seen on Fig. 3. The general connection between the angles and distances is the following:

Every distance of a pair of points is the measure of the angle of their polars multiplied by i. The domain of the angle can be chosen on such a way, that we are going through the segment by a moving point and look at the domain which described by the moving polar of this point.

2.5. **The extracted hyperbolic theorem of Sines.** We note that with the above definition of the length of a segment the known formulas of hyperbolic trigonometry extracted to the formulas of general objects with infinite or ideal vertices. For example, we prove the hyperbolic theorem of Sines which has the following form for a right-angled triangle

$$\sinh a = \sinh c \cdot \sin \alpha.$$

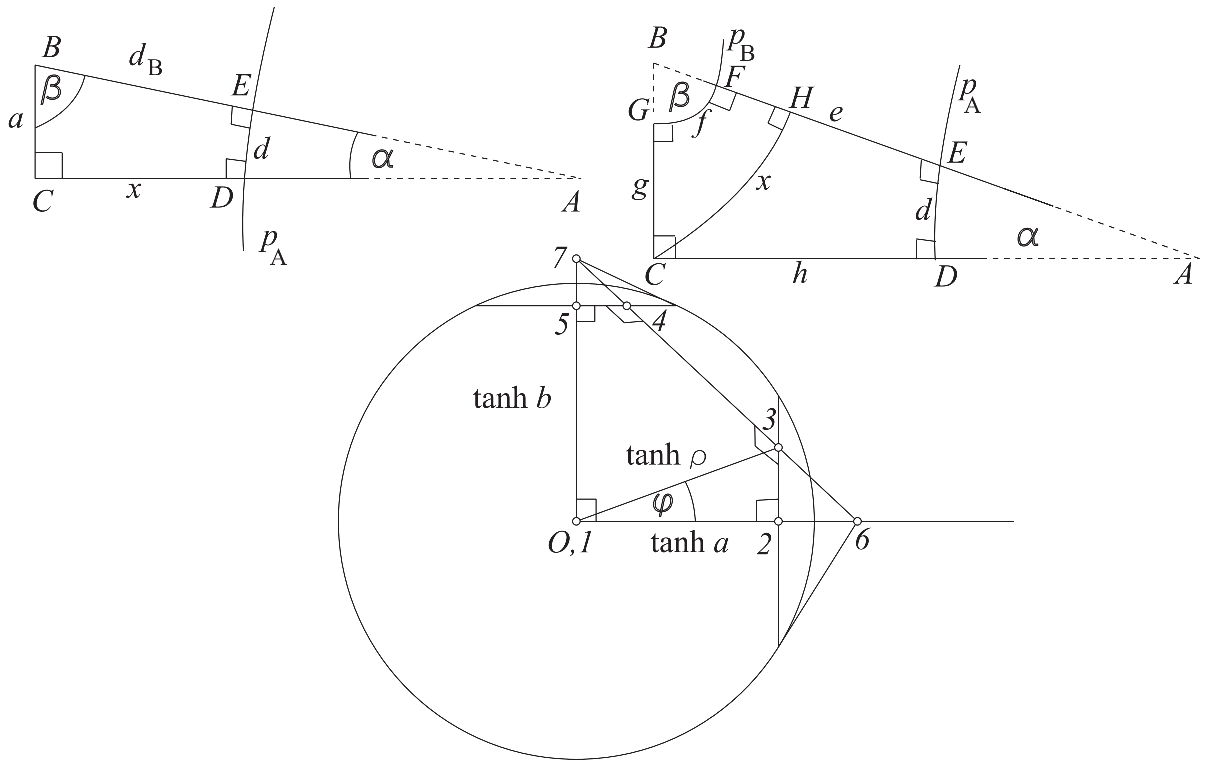

FIGURE 4. Hyperbolic theorem of sines with non-real vertices

First we prove that cases when the sides of the triangle lie on real lines, respectively. We assume that the right angle is at C and the side which opposite to a vertex noted to the same small letter as of the vertex. The angle at A or B is α or β, respectively. We remark that the angle at C is real because of our definition on the extracted angles (see Table 3).

- If A is an infinite point B and C are real ones then the product $\sinh c \cdot \sin \alpha = \infty \cdot 0$ is indeterminable and we can consider that the equality is true. The relation $\sinh b \cdot \sin \beta = \infty \cdot \sin \beta = \infty$ is also true by our agreement. If A, B are at infinity then $\alpha = \beta = 0$ and we can consider that holds the equality, too.
- In the case, when B, C are real points and A is an ideal point, let the polar of A is p_A. Then by definition $\sinh c = \sinh(d_B + (i\pi/2)) = \cosh(d_B)\sinh(i\pi/2) = i\cosh(d_B)$ where d_B is the distance of B and p_a and $\sin \alpha = \sin(d/i) = i(1/i)\sin(-id) = -i\sinh(d)$ where d is the length of the segment between the lines of the sides AC and BC. If p_A intersects AC and BC in the points D and E, respectively then $BCDE$ is a quadrangle with three right angles and with the sides a, x,

d and d_B (see the left figure in Fig. 4). This implies that $\sinh c \sin\alpha = \cosh(d_B)\sinh(d) = \sinh a$, as we stated.

- If C is a real point A is at infinity and B is an ideal point, then $\alpha = 0$ and the right-hand side $\sinh c \cdot \sin\alpha$ is undeterminable. If we consider $\sinh c \cdot \sin\beta = \infty \sin\beta$ it is infinite by our agreement and the statement is true, again.
- Very interesting the last case when C is a real point, A and B are ideal points, respectively, and the line AB is a real line (see the right-hand side picture in Fig. 4). Then $\sinh a = i\cosh g$, $\sinh c = \sinh(-e)$ and $\sin\alpha = -i\sinh d$ thus $\sinh c \sin\alpha = i\sinh e \sinh d$ and the theorem holds if and only if in the real pentagon $CDEFG$ with five right angles holds that $\sinh e \sinh d = \cosh g$. In Statement 1 we can find the proof of this nice connection among the sides of a pentagon with five right angles.

Statement 1. *Denote by a, b, c, d, e the edge lengths of the successive sides of a pentagon with five right angles on the hyperbolic plane. Then we have the following formulas:*

$$\cosh d = \sinh a \sinh b \quad \sinh c = \frac{\cosh a}{\sqrt{\sinh^2 a \sinh^2 b - 1}} \quad \sinh e = \frac{\cosh b}{\sqrt{\sinh^2 a \sinh^2 b - 1}}.$$

We prove the statement using Weierstrass homogeneous coordinates of the hyperbolic plane. Before the proof we recall the formula of (usual) distance of points with respect to such homogeneous coordinates. Consider the hyperboloid model of the hyperbolic plane H embedded into a 3-dimensional pseudo-Euclidean space with indefinite inner product with signature $(-,-,+)$. The points of the plane can be considered as the unit sphere of this space containing those elements which scalar square is equal to 1 and last coordinates are positives, respectively. It can be seen that the distance between two points $X = (x, y, z)^T$ and $X' = (x', y', z')^T$ holds the following formula:

$$\cosh|XX'| = -xx' - yy' + zz'.$$

Consider now the projection of H into the plane $z = 1$ from the origin. Then we get a projective (Cayley-Klein) model of H with the usual metric.

Proof. Assume that a pentagon 12345 with five right angles lies in this model as in Fig. 4 (bottom) the vertex 1 is the origin and the edges 12 and 51 lies on the first two axes of the coordinate system. Now we have to determine the length of the edge 34 using as parameter the respective lengths a and b of the edges 12 and 51. To this we can determine the coordinates of the points III, IV of H which mapped into the points 3,4, respectively. Consider the point X and its image 3. We have to determine first the Euclidean distance $\rho := |03|$ and the angle $\varphi := (2O3)_\measuredangle$ and then the coordinates of X are $\sinh\rho\cos\phi, \sinh\rho\sin\varphi, \cosh\rho$, respectively. If the hyperbolic length of 12 and 51 are a and b, respectively, then their Euclidean distances are $\tanh a$ and $\tanh b$, respectively. Obvious that the line 34 intersects the axes in such points 6 and 7, whose distances from the origin are $1/\tanh a$ and $1/\tanh b$, respectively. From this we get that

$$\cosh\rho = \frac{\cosh^2 a \tanh b}{\sqrt{\cosh^2 a \tanh^2 b - 1}} \quad \sinh\rho = \frac{\sqrt{\sinh^2 a \cosh^2 a \tanh^2 b + 1}}{\sqrt{\cosh^2 a \tanh^2 b - 1}}$$

and

$$\cos\varphi = \frac{\sqrt{\sinh^2 a \cosh^2 a \tanh^2 b}}{\sqrt{\sinh^2 a \cosh^2 a \tanh^2 b + 1}} \quad \sin\varphi = \frac{1}{\sqrt{\sinh^2 a \cosh^2 a \tanh^2 b + 1}}.$$

From these quantities we get

$$x = \frac{\sinh a \cosh a \tanh b}{\sqrt{\cosh^2 a \tanh^2 b - 1}}, \quad y = \frac{1}{\sqrt{\cosh^2 a \tanh^2 b - 1}}, \quad z = \frac{\cosh^2 a \tanh b}{\sqrt{\cosh^2 a \tanh^2 b - 1}},$$

and similarly for the pre-image X' of the point 4 we get

$$x' = \frac{1}{\sqrt{\cosh^2 b \tanh^2 a - 1}} \quad y' = \frac{\sinh b \cosh b \tanh a}{\sqrt{\cosh^2 b \tanh^2 a - 1}} \quad z' = \frac{\cosh^2 b \tanh a}{\sqrt{\cosh^2 b \tanh^2 a - 1}}.$$

Finally the inner product of these vectors gives the first required formula

$$\cosh d = \cosh|XX'| = \sinh a \sinh b.$$

The other two formulas of the statement are simple consequences of this first one. $\square$

In the second case we assume that either there is an ideal line or there is a line at infinity among the lines of the sides. Since C is a real point, the line which could be non-real one is the line of the hypotenuse AB. Now if it is at infinity and at least one vertex of it is an infinite point the statement evidently true. Assume that A, B and its line are ideal elements, respectively. Then the length c is equal to $(\pi/2)i$ the angle α is equal to $(\pi/2)+d/i$ where d is the distance between C and the polar of B and the length of a is equal to $d+(\pi/2)i$, respectively. The equality $\sinh(\pi/2)i\cdot\sin((\pi/2)+d/i) = (1/i)\sin(-(\pi/2))\cos(d/i) = -(1/i)\cosh d = i\cosh d = \sinh(d+(\pi/2)i)$ proves the statement in this case, too.

3. Power, inversion and centres of similitude

It is not clear who investigated first the concept of inversion with respect to hyperbolic geometry. A synthetic approach can be found in [13] using reflections in Bachmann's metric plane. To our purpose it is more convenient to use an analytic approach in which the concepts of centres of similitude and axis of similitude can be defined. We consider – as an analogy – the spherical approach of these concepts can be found in Chapter VI and Chapter VII in [5].

3.1. **Spherical concept.** It can be proved (§97. in [5]) that if an arc of a great circle (line) passing through a fixed point O cuts a fixed small circle in the variable points A,B, then

$$\tan\frac{1}{2}OA\cdot\tan\frac{1}{2}OB$$

is constant. This product is called the *spherical power* of O with respect to circle. It is positive or negative, according to whether O is exterior or interior to the circle. If from any point O outside a small circle two great circle arcs are drawn to it, of which one, OD, is a tangent, and the other a secant, meeting the small circle in the points A, B; then

$$\tan^2\frac{1}{2}OD = \tan\frac{1}{2}AO\cdot\tan\frac{1}{2}OB.$$

If we have two small circles on the sphere then the locus of points P for which the tangent segments to these circles are equal is a great circle called the *radical circle (axis of power)* of them. The radical circles of any two of three small circles are concurrent. The common point is the *power point* of the three small circles. This is the centre of the circle orthogonal to each of them.

For two small circles there are two *centres of similitude*. These are the points on the line connecting their centres, which divide the segment joining the centers of the two circles externally or internally in the spherical ratio of the sines of the radii. The common tangent lines to the circles pass through the centres of similitude, namely, the direct common tangent lines through the external centre and the inverse common tangent lines through the internal center. If the two small circles have intersecting interiors, the internal center of similitude exists, but inverse common tangent lines do not exist. If through a centre of similitude we draw a secant cutting the circles, then the pairs of points M, M';N, N' of Fig.5 are said to be *homothetic* and M, N'; M', N are *inverse*.

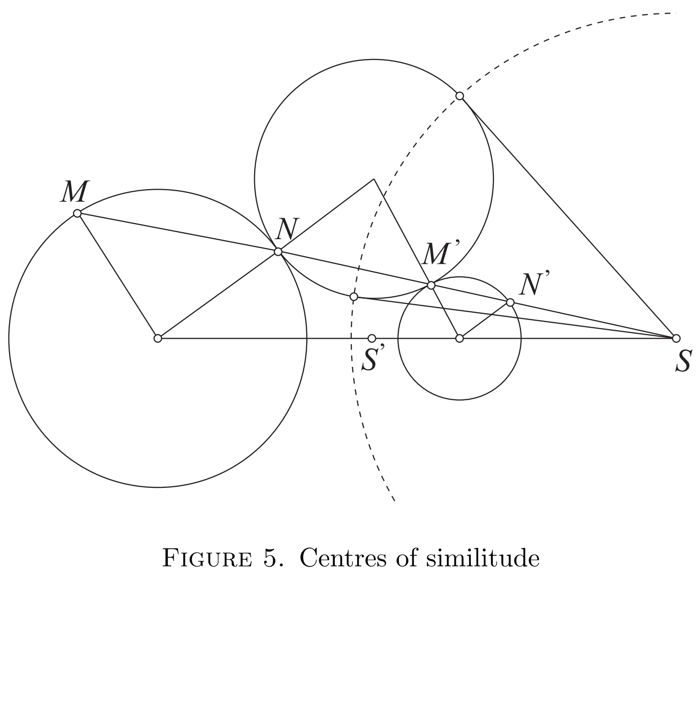

Figure 5. Centres of similitude

Then for the homothetic points M, M' the ratio

$$\tan\frac{SM}{2} : \tan\frac{SM'}{2}$$

is independent of M (see §97. in [5]). Moreover, also

$$\tan\frac{SM}{2}\tan\frac{SN'}{2} = \tan\frac{SM'}{2}\cdot\tan\frac{SN}{2}$$

is independent of M (see Cor. in §97. in [5]). Let us have three small circles c_1, c_2, c_3. For $k, l \in \{1,2,3\}$, we denote by O^i_{kl}, or O^o_{kl} the inner, or outer centers of similitude of c_k and c_l. Then the following four triples of points are collinear:

$$\{O^o_{12}, O^o_{23}, O^o_{31}\}, \{O^o_{12}, O^i_{23}, O^i_{31}\}, \{O^i_{12}, O^o_{23}, O^i_{31}\}, \{O^i_{12}, O^i_{23}, O^o_{31}\}.$$

All the four lines containing these triples of points are called *axes of similitude of the circles* c_1, c_2, c_3. Cf. §98 in [5]. Consequently if a variable circle touches two fixed circles, the line passing through the points of contact passes through a fixed point, namely, a centre of similitude of the two fixed circles; for the points of contact are centres of similitude. Moreover if a variable circle touches the two fixed circles, then the length of the tangent segment drawn to it from the respective center of similitude, for which the chord joining the two points of contact, passes, is constant. Thus if being given a fixed point S and any curve γ, on the sphere, if on the line segment joining S to any point M of γ a point N' is taken, such that $\tan(|SM|/2)\tan(|SM'|/2)$ is constant, the locus of N' is called the *inverse* of γ.

3.2. **Hyperbolic concept.** Returning to the hyperbolic case we have a new situation, namely two lines do not intersect in every case. For example, if we consider three points A, B, C on a line (with this order) then the ratio defined by

$$\frac{\sinh AC}{\sinh BC}$$

is equal to

$$\frac{\sinh AC}{\sinh BC} = \frac{\sinh(AB+BC)}{\sinh BC} = \cosh AB + \coth BC \sinh AB,$$

and by the assumption $\coth BC > 1$ it is greater then e^{AB}. Therefore a ratio can be attained by a real point C only if this ratio is greater than e^{AB}. (Obviously, this quantity depends on the distance of the points A, B). On the other hand every number greater or equal to 1 could be the ratio of hyperbolic sines of the radii of circles with centers A and B, respectively. Using the extracted concepts of lengths of segments this problem solved.

First we prove a lemma on which based our theory.

Lemma 1. *The product* $\tanh(PA)/2\cdot\tanh(PB)/2$ *is constant if* P *is a fixed (but arbitrary) point (real, at infinity or ideal),* P, A, B *are collinear and* A, B *are on a cycle of the hyperbolic plane (meaning that in the fixed projective model of the real projective plane it has a proper part).*

Proof. To prove this we have to consider three cases with respect to the type of the cycle with the necessary subcases with respect to the possible types of the points P, A, B.

(A): In the case of a circle we have more cases.

- P is a real point A, B are real points. In this case the center O of the circle is real and we can consider the real line through O and perpendicular to the line AB. The intersection of these lines is the real point C. Consider the triangles ACO and PCO, respectively. These have a common side OC and a respective right angle at C. For the pair of points choose such segments from the pair of possible segments, that the relation $AB = AC \cup CB$ be valid (see Fig. 6). From the Pythagorean Theorem we have $\cosh AC/\cosh CP = \cosh OA/\cosh PO$. Hence

$$\tanh\frac{AP}{2}\tanh\frac{BP}{2} = \tanh\frac{AC+CP}{2}\tanh\frac{BC-PC}{2} = \tanh\frac{AC+CP}{2}\tanh\frac{(AC-CP)}{2} =$$

$$= \frac{\sinh\frac{AC+CP}{2}}{\cosh\frac{AC+CP}{2}}\frac{\sinh\frac{AC-CP}{2}}{\cosh\frac{AC-CP}{2}} = \frac{\cosh AC - \cosh CP}{\cosh AC + \cosh CP} = \frac{\cosh OA - \cosh PO}{\cosh OA + \cosh PO} =$$

$$= \tanh\frac{OA+PO}{2}\tanh\frac{OA-PO}{2} = \text{constant} = c.$$

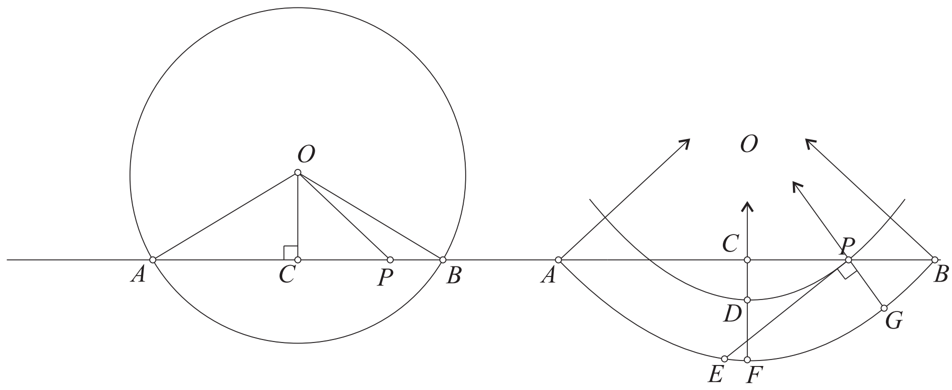

FIGURE 6. Power of a point into a cycle

We note that the absolute value of c is less or equal to 1 and the sign of c depends only on the fact that P is a point in the interior or a point of the exterior of the given circle. Additionally it is equal to zero if and only if either $P = A$ or $P = B$, holds.

- P is an infinite point A, B are real points. According to our agreements on the length of a segment and using of the symbols $\pm\infty$ the required product is either 1 or -1.
- Finally if P is an ideal point and A, B are real points, then using the enumeration above originating from the extracted Pythagorean Theorem we get that

$$c = \tanh\frac{OA+PO}{2}\tanh\frac{OA-PO}{2} = \tanh\frac{OA+d+(\pi/2)i}{2}\tanh\frac{OA-d-(\pi/2)i}{2} =$$

$$= \frac{\cosh OA - \cosh(d+(\pi/2)i)}{\cosh OA + \cosh(d+(\pi/2)i)} = \frac{\cosh OA + \sinh d}{\cosh OA - \sinh d},$$

showing that the absolute value of c is greater than 1, and the sign of c depends on the ratio of the radius of the circle and the distance d (between the polar of P and the center of the circle).

(B): In the case of paracycle the point O is at infinite. In Fig.6 we can see that if P is real then there is an unique paracycle through P with the same pencil of parallel lines. Now if $C \neq P$ we have the following calculation:

$$\tanh\frac{AP}{2}\tanh\frac{BP}{2} = \tanh\frac{AC+CP}{2}\tanh\frac{BC-PC}{2} = \tanh\frac{AC+CP}{2}\tanh\frac{(AC-CP)}{2} =$$

$$= \frac{\sinh\frac{AC+CP}{2}\sinh\frac{AC-CP}{2}}{\cosh\frac{AC+CP}{2}\cosh\frac{AC-CP}{2}} = \frac{\cosh AC - \cosh CP}{\cosh AC + \cosh CP} = \frac{\frac{\cosh AC}{\cosh CP}-1}{\frac{\cosh AC}{\cosh CP}+1}.$$

But using the equality on the diameter and height of a segment of a paracycle (*see also* eg. [9]) we get

$$\frac{\cosh AC}{\cosh CP} = \frac{e^{CF}}{e^{CD}} = e^{CF-CD} = e^{PG} = \cosh EP$$

showing that it is independent from the position of the secant AB. For $C = P$ this value is ± 1 and it is the result in that case, too, if P is at infinity. The absolute value of c is less then 1 for real P and greater than 1 for ideal P.

(C): In the case of hypercycle we have again more cases. First we assume that A, B and P are real points, respectively. O is an ideal point and C is the halving point of the segment AB ($AB = AC \cup CB = AP \cup PB$ as on Fig. 7). Let FG be the basic line of the hypercycle with distance b. Then all of the radiuses are orthogonal to FG. The minimal distance of a point of the segment AB from the line FG attained at the radius through E (and C). As in the case of paracycles we get that

$$\tanh\frac{AP}{2}\tanh\frac{BP}{2} = \frac{\frac{\cosh AC}{\cosh CP}-1}{\frac{\cosh AC}{\cosh CP}+1},$$

and from the quadrangle $AFGC$ with three right-angle we get that

$$\frac{\cosh AC}{\cosh CP} = \frac{\sinh AF}{\sinh GC} : \frac{\sinh PR}{\sinh GC} = \frac{\sinh b}{\sinh d},$$

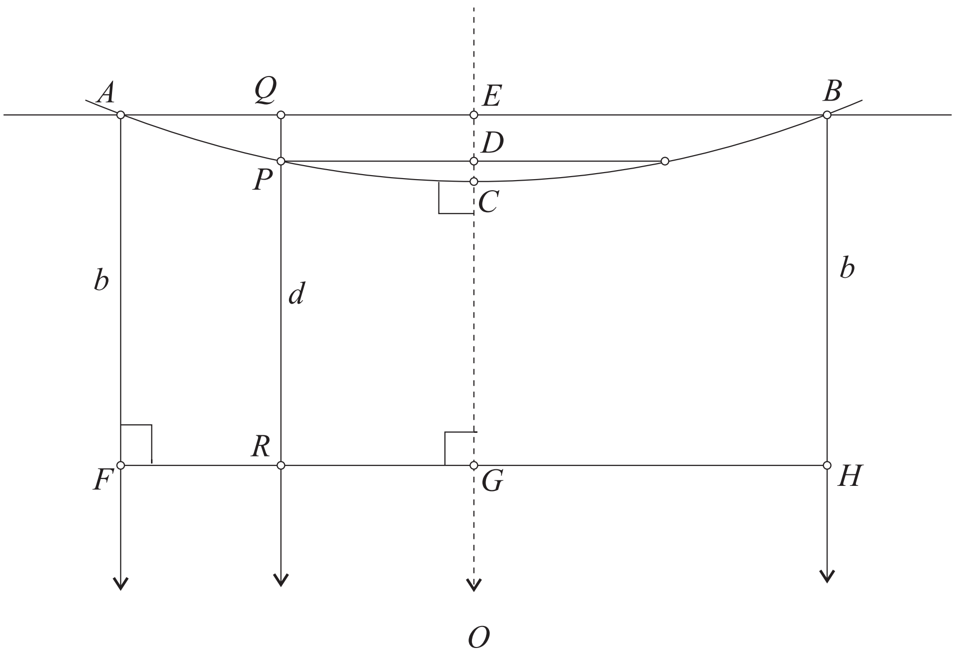

FIGURE 7. Power of a point into a hypercycle

where d is the distance of the point P from the basic line of the hypercycle. Thus the latter term is independent from the choice of the points A, B on the hypercycle implying that the examined value has the same property. Denote by c this constant. Of course $b \gtreqless d$ implies that $c \gtreqless 0$ and the absolute value of c is less than 1. If A, B are real points and P at infinity then $c = \pm 1$. The result in the case when A, B, P are distinct, non-ideal points and at least one among is at infinity can be gotten analogousy.

Finally, we have to consider all cases when at least one point is ideal (and by our assumption at least one from A and B is real). Of course, from the definitions of the length of a general segment we can use complex numbers as in (A) to prove our statement. For instance, assume that P and O are ideal points such that the line PO is also ideal and A, B are a real points (see Fig. 8). The examined expression is

$$c = \tanh\frac{AP}{2}\tanh\frac{BP}{2} = \frac{\cosh AC - \cosh CP}{\cosh AC + \cosh CP} = \frac{\sinh AF - \sinh PR}{\sinh AF + \sinh PR} = \frac{\sinh b - \sinh i\varphi}{\sinh b + \sinh i\varphi} =$$
$$= \frac{i\sinh b + \sin\varphi}{i\sinh b - \sin\varphi}.$$

where φ is the angle of the respective polars of P and R. This proves the statement, again. The remaining cases can be proved analogously and we omit their proofs.

□

On the basis of Lemma 1 we can define the power of a point with respect to a given cycle.

Definition 1. *The* power *of a point P with respect to a given cycle is the value*

$$c := \tanh\frac{1}{2}PA \cdot \tanh\frac{1}{2}PB,$$

where the points A, B are on the cycle, such that the line AB passes through the point P. With respect to Lemma 1 this point could be real, infinite or ideal one. The axis of power *of two cycles is the locus of points having the same powers with respect to the cycles.*

The usual statements are valid on the Euclidean or spherical power is valid also in the hyperbolic plane. The power of a point can be positive, negative or complex. (For example, in the case when A, B are real points we have the following possibilities: it is positive if P is a real point and it is in the exterior of the cycle; it is negative if P is real and it is in the interior of the cycle, it is infinite if P is a point at infinity, or complex if P is an ideal point.)

We can also introduce the concept of similarity center of cycles.

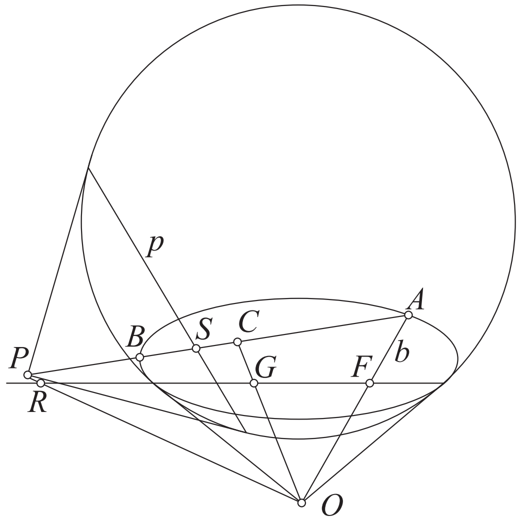

FIGURE 8. Power with ideal point P.

Definition 2. *The* centres of similitude *of two cycles with non-overlapping interiors are the common points of their pairs of tangents touching directly or inversely (i.e., they do not separate, or separate the circles), respectively. The first point is the* external center of similitude *the second one is* the internal center of similitude.

For intersecting cycles separating tangent lines do not exist, but the internal center of similitude is defined as on the sphere, but replacing sin by sinh. More precisely we have

Lemma 2. *Two points S, S' which divide the segments OO' and $O'O$, joining the centers of the two cycles in the hyperbolic ratio of the hyperbolic sines of the radii r, r' are the centers of similitude of the cycles. By formula, if*

$$\sinh OS : \sinh SO' = \sinh O'S' : \sinh S'O = \sinh r : \sinh r'$$

then the points S, S' are the centers of similitude of the given cycles.

Proof. Consider a line through the point S which intersects the cycles in M and M'. Consider also the triangles OMS and $O'M'S$, respectively. Since $OSM\measuredangle = O'SM'\measuredangle$ from our assumption (using the general hyperbolic theorem of sines) follows the other equality $OMS\measuredangle = O'M'S\measuredangle$. This implies that a tangent from S to one of the cycles is also a tangent to the other one. This means that S (and analogously S') is a center of similitude of the cycles. $\square$

We also have the following

Lemma 3. *If the secant through a centre of similitude S meets the cycles in the corresponding points M, M' then $\tanh \frac{1}{2}SM$ and $\tanh \frac{1}{2}SM'$ are in a given ratio.*

Proof. First we have to prove the hyperbolic analogy of the formula known as "Napier's analogy" (see in [5]) in spherical trigonometry. Consider the identity

$$\tanh \frac{a+b}{2} \coth \frac{c}{2} = \frac{\tanh \frac{a}{2} \coth \frac{c}{2} + \tanh \frac{b}{2} \coth \frac{c}{2}}{1 + \tanh \frac{a}{2} \tanh \frac{b}{2}}$$

and substitute to this equality the equalities

$$\tanh \frac{a}{2} \coth \frac{c}{2} = \frac{\sin(\alpha+\delta)}{\sin(\gamma+\delta)} \quad \tanh \frac{b}{2} \coth \frac{c}{2} = \frac{\sin(\beta+\delta)}{\sin(\gamma+\delta)},$$

where 2δ is the defect of the triangle defined by $2\delta = \pi - (\alpha+\beta+\gamma)$. (This equality can be shown in the following way. Add to the hyperbolic theorem of cosine for angle $\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cosh a$ the identity $\cos(\beta+\gamma) = \cos\beta\cos\gamma - \sin\beta\sin\gamma$ and use the formulas on the half of a distance then we get $\sinh \frac{a}{2} = \sqrt{(\sin\delta\sin(\alpha+\delta))/(\sin\beta\sin C)}$. Similarly, we get that

$$\cosh \frac{a}{2} = \sqrt{(\sin(\beta+\delta)\sin(\gamma+\delta))/(\sin\beta\sin\gamma)}$$

and the required equality follows.) Then we get

$$\tanh\frac{a+b}{2}\coth\frac{c}{2}=\frac{\sin(\alpha+\delta)+\sin(\beta+\delta)}{\sin(\gamma+\delta)+\sin\delta}=\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}},$$

or equivalently

$$\tanh\frac{a+b}{2}=\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}\tanh\frac{c}{2}.$$

Using this formula we have that

$$\tanh\frac{1}{2}SM:\tanh\frac{1}{2}SM'=\tanh\frac{1}{2}(SO+r):\tanh\frac{1}{2}(SO'+r')=\text{const.}$$

□

We now give the discussion of the cases for the possible centers of similitude. We have six possibilities.

(i): *The two cycles are circles.* To get the centers of similitude we have to solve an equation in x. Here d means the distance of the centers of the circles, $r\le R$ denotes the respective radii, and x is the distance of the center of similitude to the center of the circle with radius r.

$$\sinh(d\pm x):\sinh x=\sinh R:\sinh r$$

from which we get that

$$\coth x=\frac{\sinh R\mp\cosh d\sinh r}{\sinh r\sinh d}$$

or equivalently

$$e^x=\sqrt{\frac{\coth x+1}{\coth x-1}}=\sqrt{\frac{\sinh R\mp\cosh d\sinh r+\sinh r\sinh d}{\sinh R\mp\cosh d\sinh r-\sinh r\sinh d}}=\sqrt{\frac{(\sinh R)/(\sinh r)\mp e^{\mp d}}{(\sinh R)/(\sinh r)\mp e^{\pm d}}}$$

The two centers corresponding to the two cases of possible signs. If we assume that

$$e^x=\sqrt{\frac{(\sinh R)/(\sinh r)-e^{-d}}{(\sinh R)/(\sinh r)-e^{d}}}$$

then the center is an ideal point, point at infinity or a real point according to the cases

$$\sinh R/\sinh r<e^d,\quad \sinh R/\sinh r=e^d,\ \text{or}\ \sinh R/\sinh r>e^d,$$

respectively. The corresponding center is the external center of similitude. In the other case we have

$$e^x=\sqrt{\frac{(\sinh R)/(\sinh r)+e^{d}}{(\sinh R)/(\sinh r)+e^{-d}}},$$

and the corresponding center is always a real point. This is the internal center of similitude.

(ii): *One of the cycles is a circle and the other one is a paracycle.* The line joining their centers (which we call axis of symmetry) is a real line, but the respective ratio is zero or infinite. To determine the centres we have to decide the common tangents and their points of intersections, respectively. The external centre is a real, infinite ar ideal point and the internal centre is a real point.

(iii): *One of the cycles is a circle and the other one is a hypercycle.* The axis of symmetry is a real line such that the ratio of the hyperbolic sines of the radii is complex. The external center is a real, infinite or ideal point, the internal one is always real point. Each of them can be determined as in the case of two circles.

(iv): *Each of them is a paracycle.* The axis of symmetry is a real line and the internal centre is a real point. The external centre is an ideal point.

(v): *One of them is a paracycle and the other one is a hypercycle.* The axis of symmetry (in the Poincaré model, with the hypercycle replaced by the circular line containing it, and the axis containing the two apparent centers) is a real line. The internal centre is a real point. The external centre is a real, infinite or ideal point.

(vi): *Both of them are hypercycles.* The axis of symmetry (in the Poincaré model, with the hypercycle replaced by the circular line containing it, and the axis containing the two apparent centers) can be a real line, ideal line or a line at infinity. For the internal centre we have three possibilities as above as well as for the external centre.

Since using the extended concepts two points always determine a line and two lines always determine a point, all concepts defined on the sphere also can be used on the hyperbolic plane. Thus we use the concepts of "axis of similitude", "inverse and homothetic pair of points", "homothetic to and inverse of a curve γ with respect to a fixed point S (which "can be real point, a point at infinity, or an ideal point, respectively") as in the case of the sphere. More precisely we have:

Lemma 4. *The six centers of similitude of three cycles taken in pairs lie three by three on four lines, called* axes of similitude *of the cycles.*

Proof. If A, B, C their centers and a, b, c the corresponding radii of the cycles, A', B', C' the internal centers of similitude, and A'', B'', C'' the externals; then we have by definitions (see [17] p.70 or [16])

$$(ABC'') := \sinh AC'' : \sinh C''B = \sinh a : \sinh b,$$

and similarly

$$(BCA'') = \sinh b : \sinh c, \quad (CAB'') = \sinh c : \sinh a.$$

Hence

$$(ABC'')(BCA'')(CAB'') = 1.$$

Now the convers of the Menelaos-theorem is also valid (see [16] p.169) implying that the points A'', B'', C'' are collinear. Similarly, it may be shown that any two internal centers and an external center lie on a line. □

From Lemma 3 immediately follows that if the other corresponding intersection points of a line through S with the cycles is N, N' then $\tanh \frac{1}{2}SM \cdot \tanh \frac{1}{2}SN'$ is independent from the choosing of the line (see Fig.5). Thus being given a fixed point S (which is the center of the cycle for which we would like to invert) and any curve γ, on the hyperbolic plane, if on the halfline joining S (the endpoint of the halfline) to any point M of γ a point N' is taken, such that

$$\tanh \frac{SM}{2} \cdot \tanh \frac{SN'}{2}$$

is constant, the locus of N' is *called the inverse* of γ. We also use the name *cycle of inversion* for the locus of the points whose squared distance from S is

$$\tanh \frac{SM}{2} \cdot \tanh \frac{SN'}{2}.$$

Among the projective elements of the pole and its polar either one of them is always real or both of them are at infinity. Thus in a construction the common point of two lines is well-defined, and in every situation it can be joined with another point; for example, if both of them are ideal points they can be given by their polars (which are constructible real lines) and the required line is the polar of the intersection point of these two real lines. Thus the lengths in the definition of the inverse can be constructed. This implies that the inverse of a point can be constructed on the hyperbolic plane, too.

Remark. Finally we remark that all of the concepts and results of inversion with respect to a sphere of the Euclidean space can be defined also in the hyperbolic space, the "basic sphere" could be a hypersphere, parasphere or sphere, respectively. We can use also the concept of ideal elements and the concept of elements at infinity, if it is necessary. It can be proved (using Poincaré's ball-model) that every hyperbolic plane of the hyperbolic space can be inverted to a sphere by such a general inversion. This map sends the cycles of the plane to circles of the sphere.

3.3. **Applications of the theory for constructions.** In the books [17] and [16] there are many applications to the concept of general points, general lines and general distances. For example in [17] we can find the complete characterization of generalized conic sections and in [16] we can write the extracted theorems of Ceva and Menelaos, respectively. In this section we give some further applications some of them have analogous on the sphere but the knowledge of the author there is not known as a hyperbolic theorem and others are completely new observations.

3.3.1. *Construction of Gergonne.* Gergonne's construction (see e.g. [7] and see in Fig. 4) solve the following problem in the Euclidean plane:

Construct a circle touching three given circles of the Euclidean plane.

A nice construction is the following:

- Draw the point P of power of the given circles c_1, c_2, c_3 and an axis of similitude of certain three centres of similitude.
- Join the poles P_1, P_2, P_3 of this axis of similitude with respect to the circles c_1, c_2, c_3 with the point P by straight lines. Then the lines PP_i cut the circles c_i in two points Q_{i1} and Q_{i2}.
- A suitable choice $Q_{1j(1)}, Q_{2j(2)}, Q_{3j(3)}$ will give the touching points of some sought circle and c_1, c_2, c_3. More exactly, there are two such choices $Q_{1j(1)}, Q_{2j(2)}, Q_{3j(3)}$ and $Q_{1k(1)}, Q_{2k(2)}, Q_{3k(3)}$, satisfying $j(i) \neq k(i)$ for $1 \leq i \leq 3$, where $|PP_{ij(i)}| \leq |PP_{ik(i)}|$.

By the results of the preceding section we can say this construction on the hyperbolic plane too. We note that in the paper [8] this construction was proved by the conformal model. In this section we can give a proof without using any models.

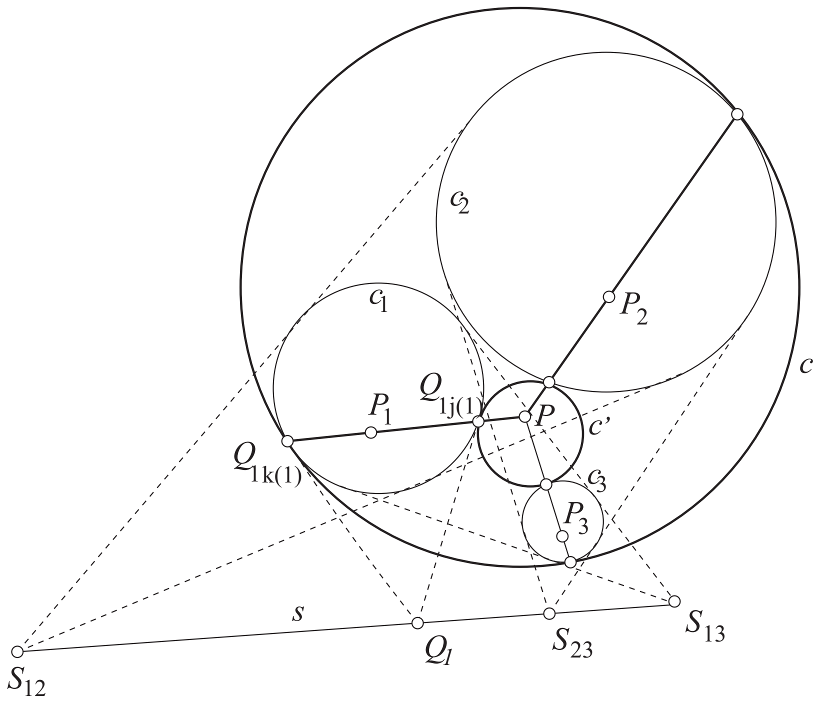

FIGURE 9. The construction of Gergonne

In Fig.9 the axis of similitude contains the three outer centers of similitude, in which case, choosing for $Q_{ij(i)}$ the intersection points closer to P, we obtain the common outward touching cycle, and for choosing the farther intersection points to P we obtain the common touching cycle that contains c_1, c_2, c_3. We denoted these circles in Fig.9 by c' and c'', respectively.

Choosing, e.g., for c_1, c_3 and c_2, c_3 the inner centers of similitude, and then for c_1, c_2 the outer center of similitude, we obtain another axis of similitude (by permuting the indices we obtain still two more similar cases). Then defining the points P_i and $P_{ij(i)}$ analogously like above, if $PQ_{1j(1)} \leq PQ_{1k(1)}$, $PQ_{2j(2)} \leq PQ_{2k(2)}$, and $PQ_{3j(3)} \geq PQ_{3k(3)}$, then the circle $Q_{1j(1)}Q_{2j(2)}Q_{3j(3)}$ touches c_1, c_2, c_3, contains c_3 and touches c_1, c_2 externally, while the circle $Q_{1k(1)}Q_{2k(2)}Q_{3k(3)}$ touches c_1, c_2, c_3, contains c_1, c_2, and touches c_3 externally.

Summing up: there are eight cycles touching each of c_1, c_2, c_3.

An Euclidean proof of the pertinence of this construction on circles can be rewritten also by hyperbolic terminology.

Gergonne's construction. Consider the cycles c' and c'' touching c_1, c_2 and c_3, in any of the four above described cases; in Fig. c' touches each of c_1, c_2, c_3 externally, and c'' touches each of c_1, c_2, c_3 internally. Then the line joining the touching points $Q_{ij(i)}$ and $Q_{ik(i)}$ passes through one of the centers of similitude P of c' and c''. Thus P is the point of power of c_1, c_2 and c_3. On the other hand, two of the three given cycles (say c_1 and c_2) give a touching pair with respect to c' and c'', hence its outer center of similitude S_{12} has the same power with respect to c' and c''. So the three outer centers of similitude S_{12}, S_{13} and S_{23} are on the axis of power of c' and c''. (It is also (by definition) an axis of similitude with respect to c_1, c_2 and c_3, say s. For c', c'' being another pair of touching circles, in the other three cases, the respective changes have to be made in the choice.) Since the pole Q_i (with respect to the cycle c_i) of the line joining $Q_{ij(i)}$ and $Q_{ik(i)}$ is the intersection point of the common tangents of c' and c_i at $Q_{ij(i)}$, and c'' and c_i at $Q_{ik(i)}$, respectively, it is also on s. By the theorem of pole-polar we get that the pole P_i of s with respect to c_i lies on the line $Q_{ij(i)}Q_{ik(i)}$. This proves the construction. □

3.3.2. *Steiner's construction on Malfatti's construction problem.* Malfatti (see [12]) raised and solved the following problem: *construct three circles into a triangle so that each of them touches the two others from outside moreover touches two sides of the triangle too.*

The first nice moment was Steiner's construction. He gave an elegant method (without proof) to construct the given circles. He also extended the problem and his construction to the case of three given circles instead of the sides of a triangle (see in [14], [15]). Cayley referred to this problem in [3] as *Steiner's extension of Malfatti's problem.* We note that Cayley investigated and solved its generalization in [3], he called it also Steiner's extension of Malfatti's problem. His problem is *to determine three conic sections so that each of them touches the two others, and also touches two of three more given conic sections.* Since the case of circles on the sphere is a generalization of the case of circles of the plane (as it can be seen easily by stereographic projection) Cayley indirectly proved Steiner's second construction. We also have to mention Hart's nice geometric proof for Steiner's construction which was published in [10]. (It can be found in various textbooks e.g. [4] and also on the web.)

In the paper [8] we presented a possible form of Steiner's construction which best meet the original problem. We note (see the discussion in the proof) that our theorem has a more general form giving all possible solutions of the problem, however for simplicity we restrict ourself to the most plausible case, when the cycles touch each other from outside. In [8] we used the fact that cycles represented by circles in the conformal model of Poincaré. The Euclidean constructions on circles of this model gives hyperbolic constructions on cycles in the hyperbolic plane. To do these constructions manually we have to use special rulers and calipers to draw the distinct types of cycles. For brevity, we think for a fixed conformal model of the embedding Euclidean plane and preserve the name of the known Euclidean concepts with respect to the corresponding concept of the hyperbolic plane, too. We now interpret this proof without using models, too.

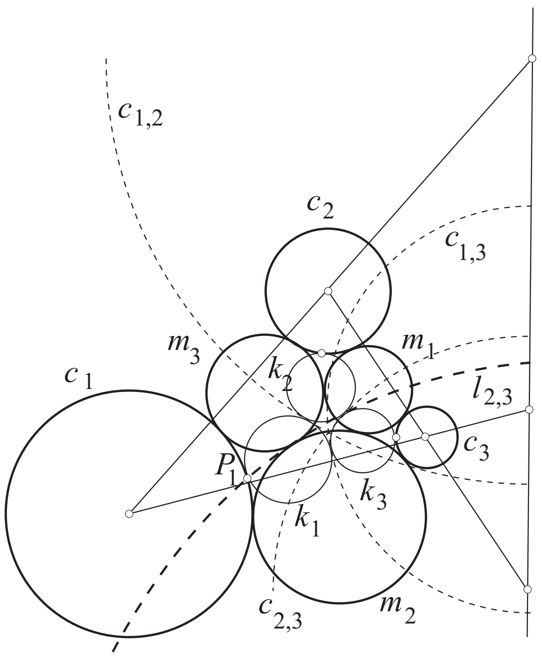

FIGURE 10. Steiner's construction.

Theorem 1 ([8]). *Steiner's construction can be done also in the hyperbolic plane. More precisely, for three given non-overlapping cycles there can be constructed three other cycles, each of them touches the two other ones from outside and also touches two of the three given cycles from outside.*

Proof. Denote by c_i the given cycles. Now the steps of Steiner's construction are the following.

(1) Construct the cycle of inversion $c_{i,j}$, for the given cycles c_i and c_j, where the center of inversion is the external centre of similitude of them. (I.e., the center of $c_{i,j}$ is the center of the above inversion, and c_i, c_j are images of each other with respect to inversion with respect to c_{ij}. Observe that c_{ij} separates c_i and c_j.)
(2) Construct cycle k_j touching two cycles $c_{i,j}, c_{j,k}$ and the given cycle c_j, in such a way that k_j, c_j touch from outside, and k_{ij}, c_{ij} (or c_{jk}) touch in such a way that k_j lies on that side of c_{ij} (or c_{ik}) on which side of them c_j lies.
(3) Construct the cycle $l_{i,j}$ touching k_i and k_j through the point $P_k = k_k \cap c_k$.
(4) Construct Malfatti's cycle m_j as the common touching cycle of the four cycles $l_{i,j}$, $l_{j,k}$, c_i, c_k.

The first step is the hyperbolic interpretation of the analogous well-known Euclidean construction on circles.

To the second step we follow Gergonne's construction which we did in the previous section. The third step is a special case of the second one. (A given cycle is a point now.) Obviously the general construction can be done in this case, too.

The fourth step is again the second one choosing three arbitrary cycles from the four ones since the quadrangles determined by the cycles have incircles.

Finally we have to prove that this construction gives the Malfatti's cycles. As we saw the Malfatti's cycles are exist (see in [8] Theorem 1). We also know that in an embedding hyperbolic space the examined plane can be inverted to a sphere. The trigonometry of the sphere is absolute implying that the possibility of a construction which can be checked by trigonometric calculations, is independent of the fact that the embedding space is a hyperbolic space or a Euclidean one. Of course, the Steiner construction is just such a construction, the touching position of circles on the sphere can be checked by spherical trigonometry. So we may assume that the examined sphere is a sphere of the Euclidean space and we can apply Cayley's analytical research (see in [3]) in which he proved that Steiner's construction works on a surface of second order. Hence the above construction produces the required touches. □

4. Applications for triangle centers

In this section we give formulas on triangle centers using the analogies between the spherical and hyperbolic geometry. The extracted concept of distances give the possibility to avoid the lengthy discussions of the existence, respectively. We substitute the concept of circle with to concept of cycle, and also use the concepts of similarity and inversion introduced in the previous section. The notation of this subsection follows the previous part of this paper: the vertices of the triangle are A, B, C, the corresponding angles are α, β, γ and the lengths of the sides opposite to the vertices are a, b, c, respectively. We also use the notion $2s = a + b + c$ for the perimeter of the triangle. Let denote R, r, r_A, r_B, r_C the radius of the circumscribed cycle, the radius of the inscribed cycle (shortly incycle), and the radiuses of the escribed cycles opposite to the vertices A, B, C, respectively. We do not assume that the points A, B, C are real and the distances are positive numbers. In most cases the formulas are valid for ideal elements and elements at infinity and when the distances are complex numbers, respectively. The only exception when the operation which needs to the examined formula is understandable. Before the examination of hyperbolic triangle centers we collect some further important formulas on hyperbolic triangles. We can consider them in our extracted manner.

4.1. **Staudtian of a hyperbolic triangle:** The *Staudtian of a hyperbolic triangle* something-like similar (but definitely distinct) to the concept of the Euclidean area. In spherical trigonometry the twice of this very important quantity called by Staudt the sine of the trihedral angle $O - ABC$ and later Neuberg suggested the names (first) "Staudtian" and the "Norm of the sides", respectively. We prefer in this paper the name "Staudtian" as a token of our respect for the great geometer Staudt. Let

$$n = n(ABC) := \sqrt{\sinh s \sinh(s-a) \sinh(s-b) \sinh(s-c)},$$

then we have

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = \frac{n^2}{\sinh s \sinh a \sinh b \sinh c}. \tag{6}$$

The proof of this equality is the following. From (2) we get

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma = \cosh(a-b) + \sinh a \sinh b (1 - \cos\gamma),$$

implying first that

$$\sin^2\frac{\gamma}{2}=\frac{1-\cos\gamma}{2}=\frac{-\cosh(a-b)+\cosh c}{2\sinh a\sinh b}=\frac{\sinh\frac{a-b+c}{2}\sinh\frac{-a+b+c}{2}}{\sinh a\sinh b}=$$
$$=\frac{\sinh(s-a)\sinh(s-b)}{\sinh a\sinh b},$$

and the statement follows immediately. Similarly we also have that

$$\cosh c=\cosh a\cosh b-\sinh a\sinh b\cos\gamma=\cosh(a-b)-\sinh a\sinh b(1+\cos\gamma),$$

implying that

$$\cos^2\frac{\gamma}{2}=\frac{1+\cos\gamma}{2}=\frac{1}{2}\frac{\cosh(a+b)-\cosh c}{\sinh a\sinh b}=\frac{\sinh s\sinh(s-c)}{\sinh a\sinh b}.$$

This observation leads to the following formulas on the Staudtian:

$$\sin\alpha=\frac{2n}{\sinh b\sinh c},\quad \sin\beta=\frac{2n}{\sinh a\sinh c},\quad \sin\gamma=\frac{2n}{\sinh a\sinh b}. \tag{7}$$

From the first equality of (7) we get that

$$n=\frac{1}{2}\sin\alpha\sinh b\sinh c=\frac{1}{2}\sinh h_C\sinh c, \tag{8}$$

where h_C is the height of the triangle corresponding to the vertex C. As a consequence of this concept we can give homogeneous coordinates for the points of the plane with respect to a basic triangle as follows:

Definition 3. *Let ABC be a non-degenerated reference triangle of the hyperbolic plane. If X is an arbitrary point we define its coordinates by the ratio of the Staudtian*

$$X:=(n_A(X):n_B(X):n_C(X))$$

where $n_A(X)$, $n_B(X)$ and $n_C(X)$ means the Staudtian of the triangle XBC, XCA and XAB, respectively. This triple of coordinates is the triangular coordinates *of the point X with respect to the triangle ABC.*

Consider finally the ratio of section (BX_AC) where X_A is the foot of the transversal AX on the line BC. If $n(BX_AA)$, $n(CX_AA)$ mean the Staudtian of the triangles BX_AA, CX_AA, respectively then using (8) we have

$$(BX_AC)=\frac{\sinh BX_A}{\sinh X_AC}=\frac{\frac{1}{2}\sinh h_C\sinh BX_A}{\frac{1}{2}\sinh h_C\sinh X_AC}=\frac{n(BX_AA)}{n(CX_AA)}=$$
$$=\frac{\frac{1}{2}\sinh c\sinh AX_A\sin(BAX_A)\measuredangle}{\frac{1}{2}\sinh b\sinh AX_A\sin(CAX_A)\measuredangle}=\frac{\sinh c\sinh AX\sin(BAX_A)\measuredangle}{\sinh b\sinh AX\sin(CAX_A)\measuredangle}=\frac{n_C(X)}{n_B(X)},$$

proving that

$$(BX_AC)=n_C(X):n_B(X),(CX_BA)=n_A(X):n_C(X),(AX_CB)=n_B(X):n_A(X). \tag{9}$$

4.2. **Angular Staudtian of a hyperbolic triangle.** In hyperbolic trigonometric formulas we also have a duality between side-lengths and angles. Thus naturally giving the idea to define the "dual concept" of the Staudtian. We call the getting quantity the *angular Staudtian* of the triangle defined by the equality:

$$N=N(ABC):=\sqrt{\sin\delta\sin(\delta+\alpha)\sin(\delta+\beta)\sin(\delta+\gamma)}.$$

On the angular Staudtian we have analogous formulas as on the Staudtian. Use now the law of cosines on the angles. Then we have

$$\cos\gamma=-\cos\alpha\cos\beta+\sin\alpha\sin\beta\cosh c$$

and adding to this equation the addition formula of the cosine function we get that

$$\sin\alpha\sin\beta(\cosh c-1)=\cos\gamma+\cos(\alpha+\beta)=2\cos\frac{\alpha+\beta+\gamma}{2}\cos\frac{\alpha+\beta-\gamma}{2}.$$

From this we get that

$$\sinh\frac{c}{2}=\sqrt{\frac{\sin\delta\sin(\delta+\gamma)}{\sin\alpha\sin\beta}}. \tag{10}$$

Analogously we get that

$$\sin\alpha\sin\beta(\cosh c+1)=\cos\gamma+\cos(\alpha-\beta)=2\cos\frac{\alpha-\beta+\gamma}{2}\cos\frac{-\alpha+\beta+\gamma}{2},$$

implying that

$$\cosh\frac{c}{2}=\sqrt{\frac{\sin(\delta+\beta)\sin(\delta+\alpha)}{\sin\alpha\sin\beta}}. \tag{11}$$

From these we get

$$\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}=\frac{N^2}{\sin\alpha\sin\beta\sin\gamma\sin\delta}. \tag{12}$$

Finally we also have that

$$\sinh a=\frac{2N}{\sin\beta\sin\gamma},\quad \sinh b=\frac{2N}{\sin\alpha\sin\gamma},\quad \sinh c=\frac{2N}{\sin\alpha\sin\beta}, \tag{13}$$

and from the first equality of (13) we get that

$$N=\frac{1}{2}\sinh a\sin\beta\sin\gamma=\frac{1}{2}\sinh h_C\sin\gamma, \tag{14}$$

where h_C is the height of the triangle corresponding to the vertex C. The connection between the two Staudtians gives by the formula

$$2n^2=N\sinh a\sinh b\sinh c. \tag{15}$$

In fact, from (7) and (13) we get that

$$\sin\alpha\sinh a=\frac{4nN}{\sin\beta\sin\gamma\sinh b\sinh c}$$

implying that

$$\sin\alpha\sin\beta\sin\gamma\sinh a\sinh b\sinh c=4nN.$$

On the other hand from (7) we get immediately that

$$\sin\alpha\sin\beta\sin\gamma=\frac{8n^3}{\sinh^2 a\sinh^2 b\sinh^2 c}$$

and thus

$$2n^2=\sinh a\sinh b\sinh cN,$$

as we stated. The connection between the two types of the Staudtian can be understood if we dived to the first equality of (7) by the analogous one in (19). Then we have

$$\frac{\sin\alpha}{\sinh a}=\frac{n}{N}\frac{\sin\beta}{\sinh b}\frac{\sin\gamma}{\sinh c}$$

which using the hyperbolic theorem of sines leads to the equality

$$\frac{N}{n}=\frac{\sin\alpha}{\sinh a}. \tag{16}$$

4.3. **On the centroid (or median point) of a triangle.** We denote the medians of the triangle by AM_A, BM_B and CM_C, respectively. The feet of the medians M_A,M_B and M_C. The existence of their common point M follows from the Menelaos-theorem. For instance if AB, BC and AC are real lines and the points A, B and C are ideal points then we have that $AM_C=M_CB=d=a/2$ implies that M_C is the middle point of the real segment lying on the line AB between the intersection points of the polars of A and B with AB, respectively (see Fig. 11).

The fact that the centroid is exist implies new real statements, e.g. *Consider a real hexagon with six right angles. Then the lines containing the middle points of a side and perpendicular to the opposite sides of the hexagon are concurrent.*

Theorem 2. *We have the following formulas connected with the centroid:*

- **Property of equal Staudtians.**

$$n_A(M)=n_B(M)=n_C(M) \tag{17}$$

- **The ratio of section** (AM_AM) **depends on the vertex.**

$$\frac{\sinh AM}{\sinh MM_A}=2\cosh\frac{a}{2},\quad \frac{\sinh BM}{\sinh MM_B}=2\cosh\frac{b}{2},\quad \frac{\sinh CM}{\sinh MM_C}=2\cosh\frac{c}{2} \tag{18}$$

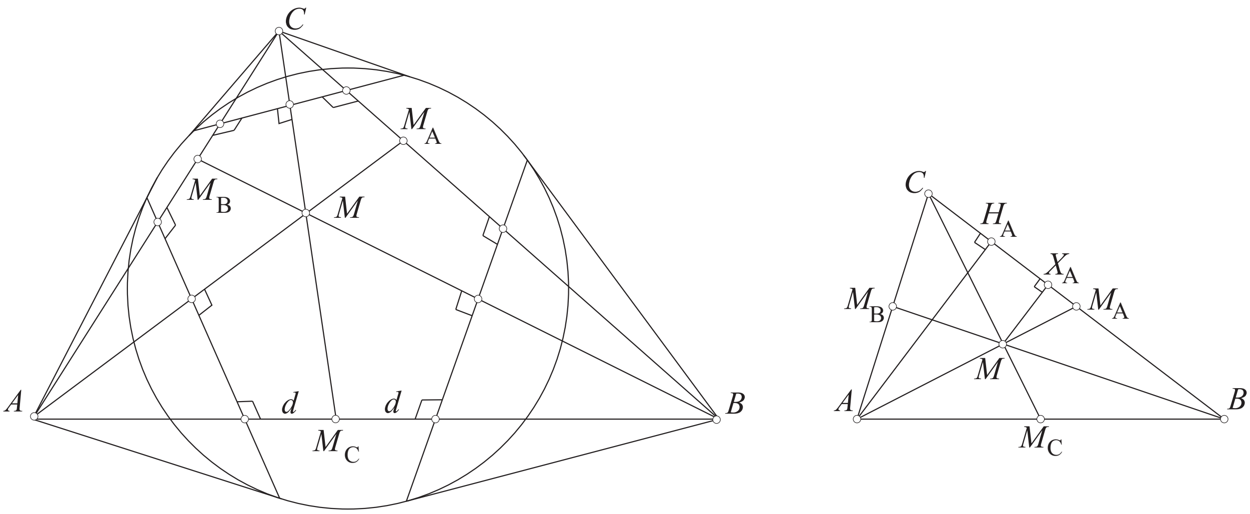

FIGURE 11. Centroid of a triangle with ideal vertices.

- **The ratio of section** (AMM_A) **is independent from the vertex.**

$$\frac{\sinh AM_A}{\sinh MM_A}=\frac{\sinh BM_B}{\sinh MM_B}=\frac{\sinh CM_C}{\sinh MM_C}=\frac{n}{n_A(M)}. \tag{19}$$

- **The "center of gravity" property of** M**.** *If y is any line of the plane then we have*

$$\sinh d'_M=\frac{\sinh d'_A+\sinh d'_B+\sinh d'_C}{\sqrt{1+2(1+\cosh a+\cosh b+\cosh c)}}. \tag{20}$$

where d'_A, d'_B, d'_C, d'_M mean the signed distances of the points A,B,C,M to the line y, respectively.

- **The "minimality" property of** M**.** *If Y is any point of the plane then we have*

$$\cosh YM=\frac{\cosh YA+\cosh YB+\cosh YC}{\frac{n}{n_A(M)}}=\frac{\cosh YA+\cosh YB+\cosh YC}{\sqrt{1+2(1+\cosh a+\cosh b+\cosh c)}}. \tag{21}$$

Remark. Using the first order approximation of the hyperbolic functions by their Taylor polynomial of order 1, we get from this formula the following one:

$$d'_M=\frac{d'_A+d'_B+d'_C}{3}$$

which associates the centroid with the physical concept of center of gravity and shows that the center of gravity of three equal weights at the vertices of a triangle is at M.
Remark. The minimality property of M for $Y=M$ says that

$$\cosh MA+\cosh MB+\cosh MC=\sqrt{1+2(1+\cosh a+\cosh b+\cosh c)}.$$

This implies that

$$\cosh YA+\cosh YB+\cosh YC=(\cosh MA+\cosh MB+\cosh MC)\cosh YM.$$

From the second-order approximation of $\cosh x$ we get that

$$3+\frac{1}{2}\left(YA^2+YB^2+YC^2\right)=\left(3+\frac{1}{2}\left(MA^2+MB^2+MC^2\right)\right)\left(1+\frac{1}{2}YM^2\right).$$

From this (take into consideration only such terms which order are less or equal to 2) we get an Euclidean identity characterizing the centroid:

$$YA^2+YB^2+YC^2=MA^2+MB^2+MC^2+3YM^2.$$

As a further consequence we can see immediately that the value $\cosh YA+\cosh YB+\cosh YC$ is minimal if and only if Y is the centroid.

Proof. The property (17) is a simple consequence of (9). Thus the centroid is the unit point with respect to the triangular coordinate system. Let the feet of the perpendiculars from M and the altitudes are X_A, X_B, X_C, and H_A,H_B, H_C, respectively. (19) follows from (17) since

$$\frac{\sinh AM_A}{\sinh MM_A} = \frac{\sinh AH_A}{\sinh MX_A} = \frac{n}{n_A(M)} = \frac{n}{n_B(M)} = \frac{\sinh BM_B}{\sinh MM_B}.$$

From (1) we get

$$\frac{\sinh MM_A}{\sinh MC} = \frac{\sin M_ACM\measuredangle}{\sin CM_AA\measuredangle} \text{ and } \frac{\sinh AM}{\sinh MC} = \frac{\sin ACM\measuredangle}{\sin CAM_A\measuredangle}$$

implying

$$\frac{\sinh AM}{\sinh MM_A} = \frac{\sin ACM\measuredangle \sin CM_AA\measuredangle}{\sin M_ACM\measuredangle \sin CAM_A\measuredangle} = \frac{\sin ACM\measuredangle}{\sin M_ACM\measuredangle} \frac{\sinh b}{\sinh \frac{a}{2}}.$$

On the other hand the equalities

$$\frac{\sin ACM\measuredangle}{\sin CM_CA\measuredangle} = \frac{\sinh \frac{c}{2}}{\sinh b} \text{ and } \frac{\sin BCM\measuredangle}{\sin BM_CA\measuredangle} = \frac{\sinh \frac{c}{2}}{\sinh a}$$

imply the equalities

$$\frac{\sin ACM\measuredangle}{\sin M_ACM\measuredangle} = \frac{\sin ACM\measuredangle}{\sin BCM\measuredangle} = \frac{\sinh a}{\sinh b}.$$

Hence we get

$$\frac{\sinh AM}{\sinh MM_A} = \frac{\sinh a}{\sinh b}\frac{\sinh b}{\sinh \frac{a}{2}} = 2\cosh\frac{a}{2}$$

proving (18). To prove (21), observe that in the triangle ABC holds the equality

$$\cosh a + \cosh b = 2\cosh\frac{c}{2}\cosh CM_C. \tag{22}$$

In fact, the law of cosines (2) with respect to the triangles ACM_C and BCM_C gives the equalities

$$\cosh a = \cosh\frac{c}{2}\cosh MM_C - \sinh\frac{c}{2}\sinh MM_C \cos CM_CB\measuredangle$$

and

$$\cosh b = \cosh\frac{c}{2}\cosh MM_C - \sinh\frac{c}{2}\sinh MM_C \cos CM_CA\measuredangle =$$
$$= \cosh\frac{c}{2}\cosh MM_C + \sinh\frac{c}{2}\sinh MM_C \cos CM_CB\measuredangle.$$

Adding these equalities we give the required one. Hence we have

$$\cosh YA + \cosh YB = 2\cosh\frac{c}{2}\cosh YM_C.$$

Consider now the triangles YCM and YM_CM, respectively. Using the law of cosines as in the previous formula we have that

$$\cosh YC = \cosh MY \cosh MC - \sinh MY \sinh MC \cos YMC\measuredangle$$

and

$$\cosh YM_C = \cosh MY \cosh M_CM + \sinh MY \sinh M_CM \cos YMC\measuredangle.$$

From these equations we get

$$\sinh M_CM \cosh YC + \sinh MC \cosh YM_C =$$
$$= \cosh YM(\cosh MC \sinh M_CM + \cosh M_CM \sinh MC) = \cosh YM \sinh M_CC.$$

Now

$$\cosh YA + \cosh YB = 2\cosh\frac{c}{2}\left(\frac{\cosh YM \sinh M_CC}{\sinh MC} - \frac{\sinh M_CM \cosh YC}{\sinh MC}\right) =$$
$$= \frac{\sinh MC}{\sinh M_CM}\left(\frac{\cosh YM \sinh M_CC}{\sinh MC} - \frac{\sinh M_CM \cosh YC}{\sinh MC}\right) = \cosh YM\frac{\sinh M_CC}{\sinh M_CM} - \cosh YC,$$

proves the first equality of (21). The second equality in (21) can be gotten from the equations

$$\frac{\sinh CM_C}{\sinh MM_C} = \frac{n}{n_A(M)}, \quad \frac{\sinh(CM_C - MM_C)}{\sinh MM_C} = 2\cosh\frac{c}{2}, \quad \cosh a + \cosh b = 2\cosh\frac{c}{2}\cosh CM_C,$$

eliminating CM_C and MM_C between these equations. We leave the calculation to the reader.

Finally, consider the minimality property (21) in the case when Y is an ideal point and A, B, C are real ones, respectively. Now M is also a real point and we have to consider the polar of Y which is a real line

y. Denote by the real (and positive) geometric distances of the points A, B, C, M to y is d_A, d_B, d_C, d_M, respectively. (21) says that

$$\cosh\left(d_M+\varepsilon_M i\frac{\pi}{2}\right)=\frac{\cosh\left(d_A+\varepsilon_A i\frac{\pi}{2}\right)+\cosh\left(d_B+\varepsilon_B i\frac{\pi}{2}\right)+\cosh\left(d_C+\varepsilon_C i\frac{\pi}{2}\right)}{\sqrt{1+2(1+\cosh a+\cosh b+\cosh c)}},$$

where ε_M is a sign depending on the positions of Y, M and $Y_M := y\cap YM$ on its line YM. It is $+$ if the segment $MY_M \subset MY$ and $-$ if this relation does not hold. (Similar definition are valid for $\varepsilon_A, \varepsilon_B$ and ε_C, respectively.) It is clear that these signs give the same value if the corresponding points lie on the same half-plane of the line y. Thus if we fixed the sign of one of the points (which distinct to zero) then the other signs have to be determined uniquely, too. Hence we have the equality

$$\varepsilon_M\sinh d_M=\frac{\varepsilon_A\sinh d_A+\varepsilon_B\sinh d_B+\varepsilon_C\sinh d_C}{\sqrt{1+2(1+\cosh a+\cosh b+\cosh c)}}$$

or equivalently

$$\sinh d'_M=\frac{\sinh d'_A+\sinh d'_B+\sinh d'_C}{\sqrt{1+2(1+\cosh a+\cosh b+\cosh c)}}$$

as we stated in (20). □

Corollary 1. *Assume that every two pairs of points which contain points at infinite have equal distances. We note that by our definition it is hold and the common value is ∞. We also assumed that $\infty/\infty = 1$. Then it follows the congruency of asymptotic triangles with three vertices at infinity. Really, assume that the vertices A, B or C tend to a points at infinity I_A, I_B or I_C, respectively, and at the same time M tends to the point M_∞. Assume also that the limit process sends X_A, X_B, X_C to X_{I_A}, X_{I_B} and X_{I_C}, respectively. Then (by the notation of the previous subsection) (8) yields that*

$$1=\lim\frac{n_A(M)}{n_B(M)}=\frac{1}{2}\lim\frac{\sinh MX_A}{\sinh MX_B}\lim\frac{\sinh a}{\sinh b}=\frac{\lim\sinh MX_A}{\lim\sinh MX_B}=\frac{\sinh M_\infty X_{I_A}}{\sinh M_\infty X_{I_B}}$$

implying that M_∞ is not only the centroid of the triangle $I_AI_BI_C$ but it is also the center of the incircle of this triangle. Hence that medians are also bisectors and altitudes implying that $M_A = X_A$, $M_B = X_B$ and $M_C = X_C$, respectively. Thus the triangle has a rotational symmetry of angle $2\pi/3$ at the center M. From this immediately follows the fact: Every two triangle with three vertices at infinity are congruent.

4.4. **On the center of the circumscribed cycle.** Denote by O and R the center and the radius of the circumscribed cycle of the triangle ABC, respectively. The midpoint of the sides AB, BC and AC are M_C, M_A and M_B, respectively. In the extracted plane O always exists and could be a real point, point at infinity or ideal point, respectively. Since we have two possibilities to choose the segments AB, BC and AC on their respective lines, we also have four possibilities to get a circumscribed cycle. One of them corresponds to the segments with real lengths and the others can be gotten if we choose one segment with real length and two segments with complex lengths, respectively. If A, B, C are real points the first cycle could be circle, paracycle or hypercycle, but the other three are always hypercycles, respectively. For example, let $a' = a = BC$ is a real length and $b' = -b+\pi i$, $c' = -c+\pi i$ are complex lengths, respectively. Then we denote by O_A the corresponding (ideal) center and by R_A the corresponding (complex) radius. We also note that the latter three hypercycle have geometric meaning. These are those hypercycles which fundamental lines contain a pair from the midpoints of the edge-segments and contain that vertex of the triangle which is the meeting point of the corresponding edges.

Theorem 3. *The following formulas are valid on the circumradiuses R, R_A, R_B and R_C, respectively.*

- **Formulas by the angular Staudtian of the triangle are:**

$$\tanh R=\frac{\sin\delta}{N},\quad \tanh R_A=\frac{\sin(\delta+\alpha)}{N},\quad \tanh R_B=\frac{\sin(\delta+\beta)}{N},\quad \tanh R_C=\frac{\sin(\delta+\gamma)}{N} \tag{23}$$

- **Formulas by the lengths of the edges are:**

$$\tanh R = \tanh\frac{a}{2}\tanh\frac{b}{2}\tanh\frac{c}{2}\cos\frac{\alpha+\beta+\gamma}{2} = \frac{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}{n} \tag{24}$$
$$\tanh R_A = \tanh\frac{a}{2}\coth\frac{b}{2}\coth\frac{c}{2}\cos\frac{-\alpha+\beta+\gamma}{2} = \frac{2\sinh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}}{n}$$
$$\tanh R_B = \coth\frac{a}{2}\tanh\frac{b}{2}\coth\frac{c}{2}\cos\frac{\alpha-\beta+\gamma}{2} = \frac{2\cosh\frac{a}{2}\sinh\frac{b}{2}\cosh\frac{c}{2}}{n}$$
$$\tanh R_C = \coth\frac{a}{2}\coth\frac{b}{2}\tanh\frac{c}{2}\cos\frac{\alpha+\beta-\gamma}{2} = \frac{2\cosh\frac{a}{2}\cosh\frac{b}{2}\sinh\frac{c}{2}}{n}$$

- **The ratio of the triangular coordinates of the circumcenter O is***:*

$$n_A(0) : n_B(O) : n_C(O) = \cos(\delta+\alpha)\sinh a : \cos(\delta+\beta)\sinh b : \cos(\delta+\gamma)\sinh c \tag{25}$$

Proof. Assume that the radius CO divides the angle γ at C into the angles γ_1 and γ_2, respectively (see Fig. 11). Then we have $OCA\measuredangle = OAC\measuredangle = \gamma_1$, $OCB\measuredangle = OBC\measuredangle = \gamma_2$, hence $OAB\measuredangle = \alpha - \gamma_1$ and $OBA\measuredangle = \beta - \gamma_2$. Since $OAB\measuredangle = OBA\measuredangle$ we get that $OAB\measuredangle = \frac{1}{2}(\alpha+\beta-\gamma) = \pi/2 - (\delta+\gamma)$.

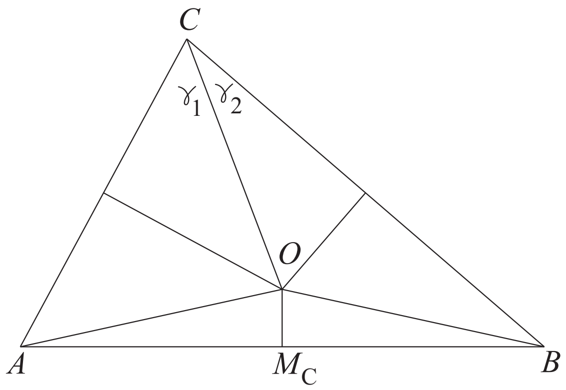

FIGURE 12. The circumcenter.

From this we get

$$\tanh\frac{c}{2} = \tanh R\cos(\pi/2 - (\delta+\gamma)) = \tanh R\sin(\delta+\gamma).$$

From (10) and (11) we get

$$\tanh\frac{c}{2} = \sqrt{\frac{\sin\delta\sin(\delta+\gamma)}{\sin(\delta+\beta)\sin(\delta+\alpha)}}$$

implying

$$\tanh R = \sqrt{\frac{\sin\delta}{\sin(\delta+\alpha)\sin(\delta+\beta)\sin(\delta+\gamma)}}.$$

From this the first equality in (23) immediately follows. Substituting $\alpha' = \alpha$, $\beta' = -\beta+\pi$, $\gamma' = -\gamma+\pi$ into the first equation of (23) and using that $\delta' = (\pi - (\alpha-\beta-\gamma+2\pi))/2 = (-\alpha+\beta+\gamma-\pi)/2 = -(\delta+\alpha)$ we get the formula of (23) on R_A:

$$\tanh R_A = \sqrt{\frac{-\sin(\delta+\alpha)}{\sin(-\delta)\sin(\pi-\delta-\beta-\alpha)\sin(\pi-\delta-\gamma-\alpha)}} = \sqrt{\frac{\sin(\delta+\alpha)}{\sin\delta\sin(\delta+\gamma)\sin(\delta+\beta)}} = \frac{\sin(\delta+\alpha)}{N}.$$

Analogously as of (16) or (17) we have the formulas

$$\sinh\frac{a}{2} = \sqrt{\frac{\sin\delta\sin(\delta+\alpha)}{\sin\gamma\sin\beta}} \text{ and } \sinh\frac{b}{2} = \sqrt{\frac{\sin\delta\sin(\delta+\beta)}{\sin\alpha\sin\gamma}},$$

and

$$\cosh\frac{a}{2} = \sqrt{\frac{\sin(\delta+\beta)\sin(\delta+\gamma)}{\sin\gamma\sin\beta}} \text{ and } \cosh\frac{b}{2} = \sqrt{\frac{\sin(\delta+\gamma)\sin(\delta+\alpha)}{\sin\alpha\sin\gamma}}.$$

Thus we have

$$\frac{\sinh\frac{a}{2}}{\cosh\frac{b}{2}\cosh\frac{c}{2}}=\sqrt{\frac{\sin^2\alpha\sin\delta}{\sin(\delta+\gamma)\sin(\delta+\alpha)\sin(\delta+\beta)}}=\sin\alpha\tanh R$$

giving the formula

$$\tanh R=\frac{\sinh\frac{a}{2}}{\sin\alpha\cosh\frac{b}{2}\cosh\frac{c}{2}}. \tag{26}$$

Similarly we get

$$\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}=\sqrt{\frac{\sin^3\delta\sin(\delta+\alpha)\sin(\delta+\beta)\sin(\delta+\gamma)}{\sin^2\alpha\sin^2\beta\sin^2\gamma}}=\frac{\sin^2\delta\coth R}{\sin\alpha\sin\beta\sin\gamma},$$

and with the same manner we have

$$\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}=\sqrt{\frac{\sin^2(\delta+\alpha)\sin^2(\delta+\beta)\sin^2(\delta+\gamma)}{\sin^2\alpha\sin^2\beta\sin^2\gamma}}=\frac{\sin\delta\coth^2 R}{\sin\alpha\sin\beta\sin\gamma}.$$

Dividing by the two equalities we get the first equality of the first row in (24):

$$\tanh R=\tanh\frac{a}{2}\tanh\frac{b}{2}\tanh\frac{c}{2}\sin\delta.$$

Using (7) and (14) we also have that

$$\sin\alpha\sin\beta\sin\gamma=\frac{8n^3}{\sinh^2 a\sinh^2 b\sinh^2 c}=\frac{8n^3N^2}{4n^4}=\frac{2N^2}{n} \tag{27}$$

giving immediately the second equality of the first row in (24)

$$\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}=\frac{\sin^2\delta\coth R}{\sin\alpha\sin\beta\sin\gamma}=\frac{n\sin^2\delta\coth R}{2N^2}=\frac{n\tanh R}{2}.$$

Substituting the complementary lengths and (the same) angles (if it is necessary) to these equations we get the results of the remaining rows in (24).

By (8) we have that

$$n(AOB)=\frac{1}{2}\sin\frac{\alpha+\beta-\gamma}{2}\sinh R\sinh c$$

and

$$n(BOC)=\frac{1}{2}\sin\frac{-\alpha+\beta+\gamma}{2}\sinh R\sinh a.$$

Hence

$$n_A(0):n_B(O):n_C(O)=\sin\frac{-\alpha+\beta+\gamma}{2}\sinh a:\sin\frac{\alpha-\beta+\gamma}{2}\sinh b:\sin\frac{\alpha+\beta-\gamma}{2}\sinh c,$$

as we stated in (25). □

Remark. The first order Taylor polynomial of the hyperbolic functions of distances leads to a correspondence between the hyperbolic Staudtians and the Euclidean area T leading to further Euclidean formulas. More precisely we have

$$n=T \quad \text{and} \quad N=\frac{T\sin\alpha}{a}=\frac{Ta}{2Ra}=\frac{T}{2R}. \tag{28}$$

Hence we give from (27) the following formula:

$$\sin\alpha\sin\beta\sin\gamma=\frac{2N^2}{n}=\frac{2T^2}{4R^2T}=\frac{T}{2R^2}$$

or equivalently the known Euclidean dependence of these quantities:

$$T=2R^2\sin\alpha\sin\beta\sin\gamma.$$

Remark. Use (21) for the point O. Then we have

$$\sqrt{1+2(1+\cosh a+\cosh b+\cosh c)}\cosh OM=\cosh OA+\cosh OB+\cosh OC=3\cosh R,$$

Implying the approximation of second order (as in the remark before the proof) we get the equation

$$3\left(1+\frac{R^2}{2}\right)=\sqrt{9+a^2+b^2+c^2}\left(1+\frac{OM^2}{2}\right)=3\sqrt{1+\frac{a^2+b^2+c^2}{9}}\left(1+\frac{OM^2}{2}\right).$$

The functions on the right hand side can also be approximated of second order. If we multiply these polynomials and hold only those terms which order at most 2 we can deduce the following equation

$$1+\frac{R^2}{2}=1+\frac{a^2+b^2+c^2}{2\cdot 9}+\frac{OM^2}{2},$$

and hence the Euclidean formula

$$OM^2=R^2-\frac{a^2+b^2+c^2}{9}.$$

Corollary 2. *Applying (24) to a triangle with four ideal circumcenter, we get a formula which determines the common distance of three points of a hypercycle from the basic line of it. In fact, if d means the searched distance that*

$$\frac{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}{n}=\tanh R=\tanh\left(d+\varepsilon\frac{\pi}{2}i\right)=\frac{\sinh\left(d+\varepsilon\frac{\pi}{2}i\right)}{\cosh\left(d+\varepsilon\frac{\pi}{2}i\right)}=\frac{\varepsilon i\cosh d}{\varepsilon i\sinh d}=\coth d,$$

and we get:

$$\tanh d=\frac{n}{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}. \tag{29}$$

For Euclidean analogy of this equation we can use the first order Taylor polynomial of the hyperbolic function. Our formula leads to the following:

$$\frac{1}{R}=d=\frac{4T}{abc}$$

implying a well-known connection among the sides, the circumradius and the area of the triangle.

4.5. **On the center of the inscribed and escribed cycles.** We are aware that the bisectors of the interior angles of a hyperbolic triangle are concurrent at a point I, called the incenter, which is equidistant from the sides of the triangle. The radius of the *incircle* or *inscribed circle*, whose center is at the incenter and touches the sides, shall be designated by r. Similarly the bisector of any interior angle and those of the exterior angles at the other vertices, are concurrent at point outside the triangle; these three points are called *excenters*, and the corresponding tangent cycles *excycles* or *escribed cycles*. The excenter lying on AI is denoted ba I_A, and the radius of the escribed cycle with center at I_A is r_A. We denote by X_A, X_B, X_C the points where the interior bisectors meets BC, AC, AB, respectively. Similarly Y_A, Y_B and Y_C denote the intersection of the exterior bisector at A, B and C with BC, AC and AB, respectively. We

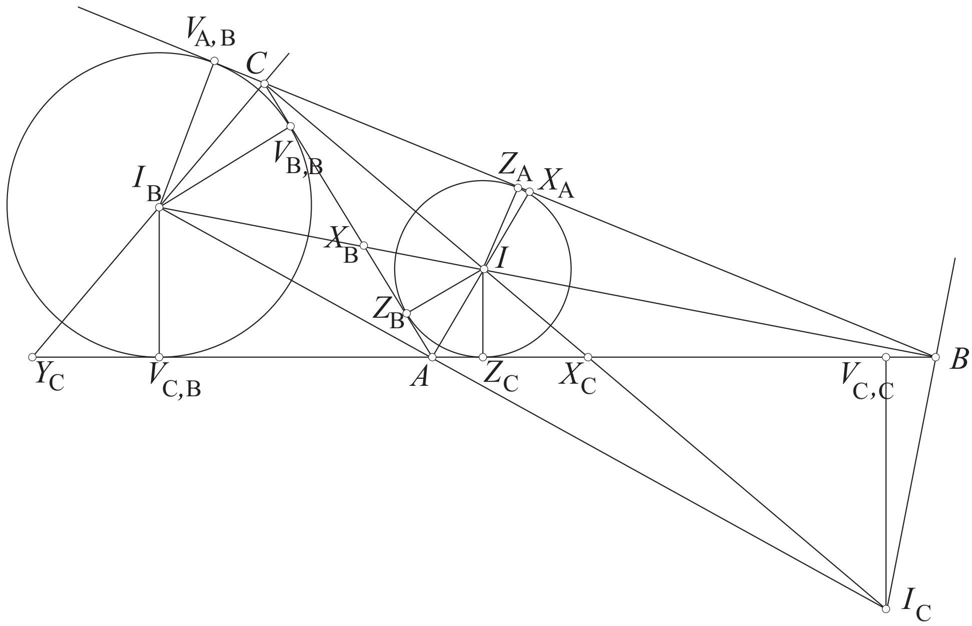

FIGURE 13. Incircles and excycles.

note that the excenters and the points of intersection of the sides with the bisectors of the corresponding exterior angle could be points at infinity or also could be ideal points. Let denote the touching points of the incircle Z_A, Z_B and Z_C on the lines BC, AC and AB, respectively and the touching points of the excycles with center I_A, I_B and I_C are the triples $\{V_{A,A}, V_{B,A}, V_{C,A}\}$, $\{V_{A,B}, V_{B,B}, V_{C,B}\}$ and $\{V_{A,C}, V_{B,C}, V_{C,C}\}$, respectively (see in Fig. 13).

Theorem 4. *On the radiuses r, r_A, r_B or r_C we have the following formulas .*

- **Formulas by Staudtian are:**

$$\tanh r = \frac{n}{\sinh s}, \quad \tanh r_A = \frac{n}{\sinh(s-a)}, \quad \tanh r_B = \frac{n}{\sinh(s-b)}, \quad \tanh r_C = \frac{n}{\sinh(s-c)} \tag{30}$$

- **Formulas by angular Staudtian are**

$$\tanh r = \frac{N}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}, \tag{31}$$

$$\coth r = \frac{\sin(\delta+\alpha)+\sin(\delta+\beta)+\sin(\delta+\gamma)+\sin\delta}{2N} \tag{32}$$

$$\coth r_A = \frac{-\sin(\delta+\alpha)+\sin(\delta+\beta)+\sin(\delta+\gamma)-\sin\delta}{2N} \tag{33}$$

$$\coth r_B = \frac{\sin(\delta+\alpha)-\sin(\delta+\beta)+\sin(\delta+\gamma)-\sin\delta}{2N}$$

$$\coth r_C = \frac{\sin(\delta+\alpha)+\sin(\delta+\beta)-\sin(\delta+\gamma)-\sin\delta}{2N}$$

- **Connections among the circumradiuses and inradiuses are:**

$$\begin{aligned} \tanh R + \tanh R_A &= \coth r_B + \coth r_C \\ \tanh R_B + \tanh R_C &= \coth r + \coth r_A \\ \tanh R + \coth r &= \frac{1}{2}(\tanh R + \tanh R_A + \tanh R_B + \tanh R_C) \end{aligned} \tag{34}$$

- **Triangular coordinates of the incenter and excenters are:**

$$n_A(I) : n_B(I) : n_C(I) = \sinh a : \sinh b : \sinh c \tag{35}$$

$$\begin{aligned} n_A(I_A) : n_B(I_A) : n_C(I_A) &= -\sinh a : \sinh b : \sinh c \\ n_A(I_B) : n_B(I_B) : n_C(I_B) &= \sinh a : -\sinh b : \sinh c \\ n_A(I_C) : n_B(I_C) : n_C(I_C) &= \sinh a : \sinh b : -\sinh c \end{aligned} \tag{36}$$

Proof. The triangular coordinates of I by (8) holds

$$n_A(I) : n_B(I) : n_C(I) = \sinh a : \sinh b : \sinh c$$

proving (35). To (36) we observe that the excircle with center I_B can be considered as the incircle of those triangle of the vertex set $\{A, B, C\}$ which edge-segment AC is equal to that of the corresponding edge-segment of the triangle ABC while the other two edge-segments are complementary to those of ABC. (In spherical geometry the above two triangle is called *colunar* because of their union is a lune.) We also have that the sign of the measure of the radius in one of the cases is the negative as the sign of the corresponding case of the incircle because of the side separates the two centers. Thus

$$n_A(I_B) : n_B(I_B) : n_C(I_C) = \sinh(-a+\pi i) : -\sinh b : \sinh(-c+\pi i) = \sinh a : -\sinh b : \sinh c,$$

implying (36).

The equalities in (30) follows from the observation that we have $CZ_A = CZ_B = s-c$, $BZ_A = BZ_C = s-b$ and $AZ_B = AZ_C = s-a$, respectively, and thus

$$\tan\frac{\gamma}{2} = \frac{\tanh r}{\sinh(s-c)}.$$

In fact, $\sin\frac{\gamma}{2}$ and $\cos\frac{\gamma}{2}$ was calculated before (7) and from these quantities we get that

$$\tan\frac{\gamma}{2} = \sqrt{\frac{\sinh(s-a)\sinh(s-b)}{\sinh s\sinh(s-c)}} \tag{37}$$

Implying the first equality in (30). The other equalities follow from that the circumscribed triangles of the excycles have two sides with the property that its measure is the measure of the corresponding side of ABC subtracting from πi. More precisely the lengths of the sides of the circumscribed triangle of the excycle corresponding to the excenter I_B are $a' = -a + \pi i$, $b' = b$, and $c' = -c + \pi i$, respectively. The corresponding half-perimeter is $s' = (a' + b' + c')/2 = (-a + b - c)/2 + \pi i$. This implies that

$$\tanh r_B = \sqrt{\frac{\sinh(s'-a')\sinh(s'-b')\sinh(s'-c')}{\sinh s'}} =$$
$$= \sqrt{\frac{\sinh(s-c)\sinh(-s+\pi i)\sinh(s-a)}{\sinh(-s+b+\pi i)}} = \frac{n}{\sinh(s-b)},$$

as we stated in (30).

Since we proved before (7) that

$$\cos\frac{\alpha}{2} = \sqrt{\frac{\sinh s\sinh(s-a)}{\sinh c\sinh b}}, \quad \cos\frac{\beta}{2} = \sqrt{\frac{\sinh s\sinh(s-b)}{\sinh a\sinh c}}, \quad \cos\frac{\gamma}{2} = \sqrt{\frac{\sinh s\sinh(s-c)}{\sinh a\sinh b}}, \tag{38}$$

then we have by (15) and (30) that

$$\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} = \sqrt{\frac{\sinh^3 s\sinh(s-a)\sinh(s-b)\sinh(s-c)}{\sinh^2 a\sinh^2 b\sinh^2 c}} =$$
$$= \frac{n\sinh s}{\sinh a\sinh b\sinh c} = \frac{N\sinh a}{2n} = \frac{N}{2\tanh r}$$

and (31) follows, too.

To prove (32) consider the equalities

$$\sin(\delta+\alpha)+\sin(\delta+\beta) = \cos\frac{-(\alpha-\beta)+\gamma}{2} + \cos\frac{(\alpha-\beta)+\gamma}{2} = 2\cos\frac{\alpha-\beta}{2}\cos\frac{\gamma}{2} =$$
$$= 2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2},$$

and

$$\sin(\delta+\gamma)-\sin\delta = \cos\frac{(\alpha+\beta)-\gamma}{2} - \cos\frac{(\alpha+\beta)+\gamma}{2} = 2\cos\frac{\gamma}{2}\cos\frac{\alpha+\beta}{2} =$$
$$= 2\cos\frac{\gamma}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - 2\cos\frac{\gamma}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}.$$

Thus we get the equality

$$4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} = \sin(\delta+\alpha)+\sin(\delta+\beta)+\sin(\delta+\gamma)+\sin(\delta)$$

implying (32). The equations in (33) follow from (32) substituting two times $(\pi - \phi)$ into ϕ ($\phi = \alpha, \beta$ or $\phi = \gamma$).

Finally, (23), (32) and (33)implies the equalities in (34). □

The following formulas connect the radiuses of the circles and the lengths of the edges of the triangle.

Theorem 5. *Let $a, b, c, s, r_A, r_B, r_C, r, R$ be the values defined for a hyperbolic triangle above. Then we have the following formulas:*

$$-\coth r_A - \coth r_B - \coth r_C + \coth r = 2\tanh R \tag{39}$$

$$\begin{aligned}&\coth r_A\coth r_B + \coth r_A\coth r_C + \coth r_B\coth r_C = \\ &= \frac{1}{\sinh s\sinh(s-a)} + \frac{1}{\sinh s\sinh(s-b)} + \frac{1}{\sinh s\sinh(s-c)}\end{aligned} \tag{40}$$

$$\begin{aligned}&\tanh r_A\tanh r_B + \tanh r_A\tanh r_C + \tanh r_B\tanh r_C = \\ &= \frac{1}{2}\left(\cosh(a+b)+\cosh(a+c)+\cosh(b+c)-\cosh a-\cosh b-\cosh c\right)\end{aligned} \tag{41}$$

$$\begin{aligned}&\coth r_A + \coth r_B + \coth r_C = \\ &= \frac{1}{\tanh r}\left(\cosh a+\cosh b+\cosh c-\coth s\left(\sinh a+\sinh b+\sinh c\right)\right)\end{aligned} \tag{42}$$

$$\begin{aligned}&\tanh r_A + \tanh r_B + \tanh r_C = \\ &= \frac{1}{2\tanh r}\left(\cosh a+\cosh b+\cosh c-\cosh(b-a)-\cosh(c-a)-\cosh(c-b)\right)\end{aligned} \tag{43}$$

$$(44)\qquad 2(\sinh a\sinh b+\sinh a\sinh c+\sinh b\sinh c)=$$
$$+\quad \tanh r\left(\tanh r_A+\tanh r_B+\tanh r_C\right)+\tanh r_A\tanh r_B+\tanh r_A\tanh r_C+\tanh r_B\tanh r_C$$

Proof. From (32),(33) and (23) we get that

$$-\coth r_A-\coth r_B-\coth r_C+\coth r=2\frac{\sin\delta}{N}=2\tanh R,$$

as we stated in (39).

To prove (40) consider the equalities in (30) from which

$$\coth r_A\coth r_B+\coth r_A\coth r_C+\coth r_B\coth r_C=$$
$$=\frac{\sinh(s-a)\sinh(s-b)+\sinh(s-a)\sinh(s-c)+\sinh(s-c)\sinh(s-b)}{n^2}=$$
$$\frac{1}{\sinh s\sinh(s-a)}+\frac{1}{\sinh s\sinh(s-b)}+\frac{1}{\sinh s\sinh(s-c)}$$

Similarly we also get (41):

$$\tanh r_A\tanh r_B+\tanh r_A\tanh r_C+\tanh r_B\tanh r_C=\sinh s\sinh(s-a)+\sinh s\sinh(s-b)+$$
$$+\sinh s\sinh(s-c)=\frac{1}{2}\left(\cosh(a+b)+\cosh(a+c)+\cosh(b+c)-\cosh a-\cosh b-\cosh c\right).$$

Since we have

$$-2\tanh R+\coth r=\coth r_A+\coth r_B+\coth r_C=\frac{\sinh(s-a)+\sinh(s-b)+\sinh(s-c)}{n}=$$
$$=\frac{(\sinh(s-a)+\sinh(s-b)+\sinh(s-c))}{\sinh s\tanh r}=\frac{\cosh a+\cosh b+\cosh c-\coth s\left(\sinh a+\sinh b+\sinh c\right)}{\tanh r}$$

(42) is given. Furthermore we also have

$$\tanh r_A+\tanh r_B+\tanh r_C=$$
$$=\frac{n\left(\sinh(s-a)\sinh(s-b)+\sinh(s-a)\sinh(s-c)+\sinh(s-b)\sinh(s-c)\right)}{\sinh(s-a)\sinh(s-b)\sinh(s-c)}=$$
$$=\frac{\sinh s}{n}\left(\sinh(s-a)\sinh(s-b)+\sinh(s-a)\sinh(s-c)+\sinh(s-b)\sinh(s-c)\right)=$$
$$=\frac{(\sinh(s-a)\sinh(s-b)+\sinh(s-a)\sinh(s-c)+\sinh(s-b)\sinh(s-c))}{\tanh r}=$$
$$=\frac{1}{2\tanh r}\left(\cosh a+\cosh b+\cosh c-\cosh(b-a)-\cosh(c-a)-\cosh(c-b)\right)$$

implying (43). From (41) and (43) we get

$$\tanh r\left(\tanh r_A+\tanh r_B+\tanh r_C\right)+\tanh r_A\tanh r_B+\tanh r_A\tanh r_C+\tanh r_B\tanh r_C=$$
$$=\cosh(a+b)+\cosh(a+c)+\cosh(b+c)-\cosh(b-a)-\cosh(c-a)-\cosh(c-b)=$$
$$=2(\sinh a\sinh b+\sinh a\sinh c+\sinh b\sinh c)$$

which implies (44). □

The following theorem gives a connection among the distance of the incenter and circumcenter, the radiuses r,R and the side-lengths a,b,c .

Theorem 6. *Let O and I the center of the circumsrcibed and inscribed circles, respectively. Then we have*

$$(45)\qquad \cosh OI=2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh r\cosh R+\cosh\frac{a+b+c}{2}\cosh(R-r).$$

Proof. Since

$$\cosh(s-a)\cosh r = \cosh AI \text{ and } IAO\measuredangle = \frac{\alpha}{2} - \frac{\alpha+\beta-\gamma}{2} = \frac{-\beta+\gamma}{2}$$

thus from (2) we get that

$$\cosh OI = \cosh AI \cosh R - \sinh AI \sinh R \cos\frac{-\beta+\gamma}{2}.$$

Hence holds the equality

$$\cosh OI = \cosh(s-a)\cosh r\cosh R - \sinh r\sinh R\frac{\cos\frac{-\beta+\gamma}{2}}{\sin\frac{\alpha}{2}}.$$

Analogously to the proof of (6) we get that

$$\frac{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}} = \sqrt{\frac{\frac{\sinh s\sinh(s-b)}{\sinh a\sinh c}\frac{\sinh s\sinh(s-c)}{\sinh a\sinh b}}{\frac{\sinh(s-b)\sinh(s-c)}{\sinh b\sinh c}}} = \frac{\sinh s}{\sinh a}$$

and also we have

$$\frac{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\frac{\alpha}{2}} = \sqrt{\frac{\frac{\sinh(s-a)\sinh(s-c)}{\sinh a\sinh c}\frac{\sinh(s-a)\sinh(s-b)}{\sinh a\sinh b}}{\frac{\sinh(s-b)\sinh(s-c)}{\sinh b\sinh c}}} = \frac{\sinh(s-a)}{\sinh a}.$$

Summing up we get that

$$\cosh OI = \cosh(s-a)\cosh r\cosh R - \sinh r\sinh R\frac{\sinh s + \sinh(s-a)}{\sinh a} =$$

$$= \cosh(s-a)\cosh r\cosh R - 2\sinh r\sinh R\frac{\sinh\frac{b+c}{2}\cosh\frac{a}{2}}{\sinh a} =$$

$$= \cosh\frac{-a+b+c}{2}\cosh r\cosh R - \sinh r\sinh R\frac{\sinh\frac{b+c}{2}}{\sinh\frac{a}{2}},$$

and also the similar formula

$$\cosh OI = \cosh\frac{a-b+c}{2}\cosh r\cosh R - \sinh r\sinh R\frac{\sinh\frac{a+c}{2}}{\sinh\frac{b}{2}}$$

and

$$\cosh OI = \cosh\frac{a+b-c}{2}\cosh r\cosh R - \sinh r\sinh R\frac{\sinh\frac{a+b}{2}}{\sinh\frac{c}{2}}.$$

Adding now the latter three formulas we get that

$$3\cosh OI = \left(\cosh\frac{-a+b+c}{2} + \cosh\frac{a-b+c}{2} + \cosh\frac{a+b-c}{2}\right)\cosh r\cosh R -$$

$$-\sinh r\sinh R\left(\frac{\sinh\frac{b+c}{2}}{\sinh\frac{a}{2}} + \frac{\sinh\frac{a+c}{2}}{\sinh\frac{b}{2}} + \frac{\sinh\frac{a+b}{2}}{\sinh\frac{c}{2}}\right).$$

Since

$$\cosh\frac{-a+b+c}{2} = \left(\cosh\frac{b+c}{2}\cosh\frac{a}{2} - \sinh\frac{b+c}{2}\sinh\frac{a}{2}\right) =$$

$$= \cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2} + \cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2} - \sinh\frac{a}{2}\sinh\frac{b}{2}\cosh\frac{c}{2} - \sinh\frac{a}{2}\cosh\frac{b}{2}\sinh\frac{c}{2},$$

thus

$$\cosh\frac{-a+b+c}{2} + \cosh\frac{a-b+c}{2} + \cosh\frac{a+b-c}{2} =$$

$$= 3\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2} - \cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2} - \sinh\frac{a}{2}\sinh\frac{b}{2}\cosh\frac{c}{2} - \sinh\frac{a}{2}\cosh\frac{b}{2}\sinh\frac{c}{2}.$$

We also have that

$$\frac{\sinh\frac{b+c}{2}}{\sinh\frac{a}{2}} + \frac{\sinh\frac{a+c}{2}}{\sinh\frac{b}{2}} + \frac{\sinh\frac{a+b}{2}}{\sinh\frac{c}{2}} = \frac{\sinh\frac{b+c}{2}\sinh\frac{b}{2}\sinh\frac{c}{2} + \sinh\frac{a+c}{2}\sinh\frac{a}{2}\sinh\frac{c}{2} + \sinh\frac{a+b}{2}\sinh\frac{a}{2}\sinh\frac{b}{2}}{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}$$

and since

$$\sinh\frac{b+c}{2}\sinh\frac{b}{2}\sinh\frac{c}{2} = \sinh\left(s-\frac{a}{2}\right)\sinh\frac{b}{2}\sinh\frac{c}{2} =$$

$$= \sinh s\cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2} - \cosh s\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}$$

we get that

$$\frac{\sinh\frac{b+c}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}+\sinh\frac{a+c}{2}\sinh\frac{a}{2}\sinh\frac{c}{2}+\sinh\frac{a+b}{2}\sinh\frac{a}{2}\sinh\frac{b}{2}}{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}=$$

$$=\Big(\sinh s\cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}+\sinh s\sinh\frac{a}{2}\cosh\frac{b}{2}\sinh\frac{c}{2}+\sinh s\sinh\frac{a}{2}\sinh\frac{b}{2}\cosh\frac{c}{2}-$$

$$-3\cosh s\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}\Big)\frac{1}{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}.$$

Using (46) we get that

$$\frac{\sinh\frac{b+c}{2}}{\sinh\frac{a}{2}}+\frac{\sinh\frac{a+c}{2}}{\sinh\frac{b}{2}}+\frac{\sinh\frac{a+b}{2}}{\sinh\frac{c}{2}}=$$

$$=\frac{2\left(\cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}+\sinh\frac{a}{2}\cosh\frac{b}{2}\sinh\frac{c}{2}+\sinh\frac{a}{2}\sinh\frac{b}{2}\cosh\frac{c}{2}\right)}{\tanh r\tanh R}-3\cosh s.$$

Thus we have

$$3\cosh OI=3\Big(\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}-\cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}-$$

$$-\sinh\frac{a}{2}\sinh\frac{b}{2}\cosh\frac{c}{2}-\sinh\frac{a}{2}\cosh\frac{b}{2}\sinh\frac{c}{2}\Big)\cosh r\cosh R+3\cosh s\sinh r\sinh R$$

implying that

$$\cosh OI=\left(2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}-\cosh s\right)\cosh r\cosh R+\cosh s\sinh r\sinh R=$$

$$=2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh r\cosh R+\cosh s\cosh(R-r)=$$

$$=2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh r\cosh R+\cosh\frac{a+b+c}{2}\cosh(R-r),$$

as we stated in (45). □

Remark. The second order approximation of (45) leads to the equality

$$1+\frac{OI^2}{2}=2\left(1+\frac{r^2}{2}\right)\left(1+\frac{R^2}{2}\right)\left(1+\frac{a^2}{8}\right)\left(1+\frac{b^2}{8}\right)\left(1+\frac{c^2}{8}\right)-\left(1+\frac{(a+b+c)^2}{8}\right)\left(1+\frac{(R-r)^2}{2}\right).$$

From this we get that

$$OI^2=R^2+r^2+\frac{a^2+b^2+c^2}{4}-\frac{ab+bc+ca}{2}+2Rr.$$

But for Euclidean triangles we have (see [2])

$$a^2+b^2+c^2=2s^2-2(4R+r)r \text{ and } ab+bc+ca=s^2+(4R+r)r,$$

the equality above leads to the Euler's formula:

$$OI^2=R^2-2rR.$$

4.6. **On the orthocenter of a triangle.** The most important formulas on the orthocenter are also valid in the hyperbolic plane. We give a collection in which the orthocenter is denoted by H, the feet of the altitudes are denoted by H_A, H_B and H_C, respectively. We also denote by h_a, h_b or h_c the heights of the triangle corresponding to the sides a, b or c, respectively.

Theorem 7. *With the notation above we have the formulas:*

$$\tanh HA\cdot\tanh HH_A=\tanh HB\cdot\tanh HH_B=\tanh HC\cdot\tanh HH_C=\text{const.}=:h \tag{46}$$

$$\begin{aligned}\sinh HA\cdot\sinh HH_A : \sinh HB\cdot\sinh HH_B &: \sinh HC\cdot\sinh HH_C=\\ &=\cosh h_A:\cosh h_B:\cosh h_C\end{aligned} \tag{47}$$

$$n_A(H):n_B(H):n_C(H)=\tan\alpha:\tan\beta:\tan\gamma. \tag{48}$$

Furthermore let P be any point of the plane then we have

$$n_A(H)\cosh PA+n_B(H)\cosh PB+n_C(H)\cosh PC=n\cosh PH \tag{49}$$

and also

$$\cosh c \sinh H_A C + \cosh b \sinh BH_A = \cosh h_A \sinh a. \tag{50}$$

Finally we have also that

$$(h+1)\cosh OH = \left(\frac{\coth h_A}{\sinh HA} + \frac{\coth h_B}{\sinh HB} + \frac{\coth h_C}{\sinh HC}\right)\cosh R. \tag{51}$$

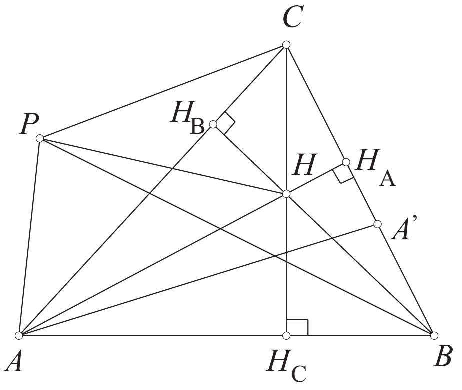

FIGURE 14. Stewart's theorem and the orthocenter.

Before the proof we prove Stewart's Theorem on the hyperbolic plane.

Theorem 8 (**Stewart's theorem**). *Let ABC be a triangle and A' is a point on the side BC. Then we have*

$$\cosh AB \sinh A'C + \cosh AC \sinh BA' = \cosh AA' \sinh BC. \tag{52}$$

Proof. Using (2) to the triangles ABA' and ACA', respectively, we get

$$\cosh AA' \sinh BC = \cosh AA' \sinh(BA' + A'C) = \sinh BA' \cosh A'C \cosh AA' +$$
$$+ \sinh A'C \cosh BA' \cosh AA' = \sinh BA'(\sinh A'C \sinh AA' \cos(AA'C_\measuredangle) + \cosh AC) +$$
$$+ \sinh A'C(\sinh BA' \sinh AA' \cos(\pi - AA'C_\measuredangle) + \cosh AB) =$$
$$= \sinh BA' \cosh AC + \sinh A'C \cosh AB$$

as we stated. □

Remark. Considering third-order approximation of the hyperbolic functions we get the equality:

$$\left(1+\frac{AA'^2}{2}\right)\left(BC + \frac{BC^3}{6}\right) = \left(1+\frac{b^2}{2}\right)\left(BA' + \frac{BA'^3}{6}\right) + \left(1+\frac{c^2}{2}\right)\left(A'C + \frac{A'C^3}{6}\right)$$

or equivalently the equation

$$a + \frac{|AA'|^2}{2}a + \frac{a^3}{6} = BA' + \frac{b^2}{2}BA' + \frac{BA'^3}{6} + A'C + \frac{c^2}{2}A'C + \frac{A'C^3}{6}.$$

Since $a = BA' + A'C$

$$\frac{AA'^2}{2}a + \left(\frac{BA'^3}{6} + \frac{BA'^2A'C}{2} + \frac{BA'A'C^2}{2} + \frac{A'C^3}{6}\right) = \frac{b^2}{2}BA' + \frac{BA'^3}{6} + \frac{c^2}{2}A'C + \frac{A'C^3}{6}$$

implying the well-known Euclidean Stewart's theorem:

$$\left(AA'^2 + BA' \cdot A'C\right)a = b^2 BA' + c^2 A'C.$$

Proof. (Proof of Theorem 7) (51) is the Stewart's theorem for the point H_A.

From the rectangular triangles HCH_A and HH_CA we get that $\tanh HH_A : \tanh HC = \cos H_AHC\measuredangle = \tanh HH_C : \tanh HA$. Similarly we get also that $\tanh HH_B : \tanh HC = \cos H_BHC\measuredangle = \tanh HH_C : \tanh HB$ thus we have (47):

$$\tanh HA \cdot \tanh HH_A = \tanh HB \cdot \tanh HH_B = \tanh HC \cdot \tanh HH_C.$$

From this we get

$$\frac{\sinh HA \cdot \sinh HH_A}{\cosh HA \cdot \cosh HH_A} = \frac{\sinh HB \cdot \sinh HH_B}{\cosh HB \cdot \cosh HH_B}.$$

Thus

$$\frac{\sinh HA \cdot \sinh HH_A}{\sinh HB \cdot \sinh HH_B} = \frac{\cosh HA \cdot \cosh HH_A}{\cosh HB \cdot \cosh HH_B} = \frac{\cosh AH_B}{\cosh BH_A}$$

implying (48). From (9) we get that

$$n_A(H) : n_B(H) = (AH_CB) = \sinh AH_C : \sinh H_CB = \tan\alpha : \tan\beta$$

implying (49). Use now the Stewart's Theorem for the triangle PAB and its secant PH_C (see in Fig.14), where P is arbitrary point of the plane. Then we get

$$\cosh PA \sinh H_CB + \cosh PB \sinh AH_C = \cosh PH_C \sinh c.$$

Applying Stewart's theorem again to the triangle PCH_C and its secant PH, we get

$$\cosh PC \sinh HH_C + \cosh PH_C \sinh CH = \cosh PH \sinh CH_C.$$

Eliminating PH_C from these equations we get

$$\cosh PA \sinh H_CB + \cosh PB \sinh AH_C + \frac{\cosh PC \sinh HH_C \sinh c}{\sinh CH} = \frac{\cosh PH \sinh CH_C \sinh c}{\sinh CH}.$$

On the other hand we have

$$2n_C(H) = \sinh HH_C \sinh c.$$

We also have

$$2n_B(H) = 2\sinh HH_B \sinh b = 2\sinh CH_A \sinh AH = 2\sinh AH_C \sinh CH,$$

and similarly

$$2n_A(H) = 2\sinh H_CB \sinh CH$$

implying the equality

$$n_A(H)\cosh PA + n_B(H)\cosh PB + n_C(H)\cosh PC = \frac{\cosh PH \sinh CH_C \sinh c}{2} = n\cosh PH$$

as we stated in (50).

Use (50) in the case when $P = O$ is the circumcenter of the triangle. Then we have

$$n_A(H)\cosh R + n_B(H)\cosh R + n_C(H)\cosh R = n\cosh OH. \tag{53}$$

Thus we have

$$\cosh OH = \frac{n_A(H) + n_B(H) + n_C(H)}{n}\cosh R = \left(\frac{\sinh HH_A}{\sinh h_A} + \frac{\sinh HH_B}{\sinh h_B} + \frac{\sinh HH_C}{\sinh h_C}\right)\cosh R.$$

From (48) we get

$$\sinh HH_B = \sinh HH_A \frac{\sinh HA}{\sinh HB}\frac{\cosh h_B}{\cosh h_A}$$

and also

$$\sinh HH_C = \sinh HH_A \frac{\sinh HA}{\sinh HC}\frac{\cosh h_C}{\cosh h_A}$$

implying that

$$\cosh OH = \frac{\sinh HH_A \sinh HA}{\cosh h_A}\left(\frac{\cosh h_A}{\sinh HA \sinh h_A} + \frac{\cosh h_B}{\sinh HB \sinh h_B} + \frac{\cosh h_C}{\sinh HC \sinh h_C}\right)\cosh R =$$

$$= \left(\frac{\cosh h_A}{\sinh HA \sinh h_A} + \frac{\cosh h_B}{\sinh HB \sinh h_B} + \frac{\cosh h_C}{\sinh HC \sinh h_C}\right)\frac{\cosh R}{\tanh HH_A \tanh HA + 1}.$$

Now we have

$$(h+1)\cosh OH = \left(\frac{1}{\tanh h_A \sinh HA} + \frac{1}{\tanh h_B \sinh HB} + \frac{1}{\tanh h_C \sinh HC}\right)\cosh R,$$

showing (52).

□

4.7. **Isogonal conjugate of a point.** Let define the *isogonal conjugate* of a point X of the plane in the following way: Reflect the lines through the point X and any of the vertices of the triangle with respect to the bisector of that vertex. Then the getting lines are concurrent at a point X' which we call the isogonal conjugate of X. To prove the concurrence of these lines we have to observe that if the lines AX and AX' intersect the line of the side BC in the points Y and Y' then the ratio of these points with respect to B and C has an inverse connection. In fact, by (1) we have that

$$\frac{\sinh c}{\sinh BY}=\frac{\sin AYB\measuredangle}{\sin BAY\measuredangle} \quad \text{and} \quad \frac{\sinh b}{\sinh YC}=\frac{\sin(\pi-AYB\measuredangle)}{\sin CAY\measuredangle}.$$

This implies that

$$(BYC)=\frac{\sinh BY}{\sinh YC}=\frac{\sinh c}{\sinh b}\frac{\sin BAY\measuredangle}{\sin CAY\measuredangle}.$$

For the point Y' we get similarly that

$$(BY'C)=\frac{\sinh c}{\sinh b}\frac{\sin BAY'\measuredangle}{\sin CAY'\measuredangle}=\frac{\sinh c}{\sinh b}\frac{\sin CAY\measuredangle}{\sin BAY\measuredangle}$$

implying the equation

$$(BYC)(BY'C)=\frac{\sinh^2 c}{\sinh^2 b}. \tag{54}$$

If Z,Z' or V,V' are the intersection points of the examined lines with the corresponding sides CA or AB, respectively, then we get the equation

$$(BYC)(BY'C)(CZA)(CZ'A)(AVB)(AV'B)=1$$

showing that the first three lines are concurrent if and only if the second three lines are. Hence we can prove the following:

Lemma 5. *If X and X' are isogonal conjugate points with respect to the triangle ABC then their triangular coordinates have the following connection:*

$$n_A(X'):n_B(X'):n_C(X')=\frac{\sinh^2 a}{n_A(X)}:\frac{\sinh^2 b}{n_B(X)}:\frac{\sinh^2 c}{n_C(X)}. \tag{55}$$

Proof. Using (55) we have

$$\left(n_C(X):n_B(X)\right)\left(n_C(X'):n_B(X')\right)=(BN_AC)(BN'_AC)=\frac{\sinh^2 c}{\sinh^2 b}$$

implying that

$$n_B(X'):n_C(X')=\frac{\sinh^2 b}{n_B(X)}:\frac{\sinh^2 c}{n_C(X)}$$

as we stated in (56). $\square$

Corollary 3. *As a first consequence we can see immediately (35) again on the triangular coordinates of the incenter. By (56) the triangular coordinates of the isogonal conjugate H' of the orthocenter is*

$$n_A(H'):n_B(H'):n_C(H')=\frac{\sinh^2 a}{\tan\alpha}:\frac{\sinh^2 b}{\tan\beta}:\frac{\sinh^2 c}{\tan\gamma}.$$

Thus

$$n_A(H'):n_B(H')=\frac{\sinh^2 a}{\tan\alpha}\frac{\tan\beta}{\sinh^2 b}=\frac{\sin\alpha\cos\alpha}{\sin\beta\cos\beta}=\frac{\sin 2\alpha}{\sin 2\beta}$$

implying that

$$n_A(H'):n_B(H'):n_C(H')=\sin 2\alpha:\sin 2\beta:\sin 2\gamma. \tag{56}$$

Compare the coordinates of H' with the triangular coordinates of the circumcenter (see (25)) we can see that the isogonal conjugate of the orthocenter is the circumcenter if and only if the defect of the triangle is zero implying that the geometry of the plane is Euclidean.

A minimality property of the incenter follows from a generalization of the equality (50). Similarly as in the proof of (50) (see Theorem 8 and the equality (53)) we can prove that for any triangle ABC with any fixed point Q and any various point P of the plane the following equality holds:

$$n_A(Q)\cosh PA+n_B(Q)\cosh PB+n_C(Q)\cosh PC=n(ABC)\cosh PQ. \tag{57}$$

Theorem 9. *The sum of the triangular coordinates of a point P of the plane is minimal if and only if P is the center of the inscribed circle of the triangle ABC.*

Proof. Assume that the vertices of the triangle ABC are real points and the edges of it are those real segments which are connecting these real vertices, respectively. Let A', B' and C' be the respective poles of the lines BC, AC and AB. These poles are ideal points and the corresponding lines $A'B'$, $A'C'$ and $B'C'$ are also ideal lines, respectively. If P is any point of the plane let $d(P,BC)$, ε_A and α' be the distance of P and the line BC the sign of this distance and the angle of the polar triangle at the vertex A', respectively. We choose the sign to positive if P and A are the same (real) half-plane determined by the line BC. Then the investigated quantity is

$$n_A(P)+n_B(P)+n_C(P)=$$
$$=\frac{1}{2}\left(\varepsilon_A\sinh d(P,BC)\sinh a+\varepsilon_B\sinh d(P,AC)\sinh b+\varepsilon_C\sinh d(P,AB)\sinh c\right)=$$
$$=\frac{1}{2i}\left(\cosh\left(d(P,BC)+\varepsilon_A\frac{\pi}{2}i\right)\sinh a+\cosh\left(d(P,AC)+\varepsilon_B\frac{\pi}{2}i\right)\sinh b+\right.$$
$$\left.+\cosh\left(d(P,AB)+\varepsilon_C\frac{\pi}{2}i\right)\sinh c\right)=\frac{1}{2i}\left(\cosh PA'\sinh a+\cosh PB'\sinh b+\cosh PC'\sinh c\right).$$

Hence using (58) we have that

$$\frac{1}{2i}\left(\cosh PA'\sinh a+\cosh PB'\sinh b+\cosh PC'\sinh c\right)=\frac{1}{2i}n(A'B'C')\cosh PQ$$

where the triangular coordinates of the point Q with respect to the polar triangle are

$$n_{A'}(Q)=\sinh a,\quad n_{B'}(Q)=\sinh b,\quad \text{and}\quad n_{C'}(Q)=\sinh c.$$

It follows from (8) that the Staudtian of the triangle $A'B'C'$ is

$$n(A'B'C')=\frac{1}{2}\sin\alpha'\sinh b'\sinh c'=\frac{1}{2}\sin\frac{a}{i}\sinh i\beta\sinh i\gamma=\frac{i}{2}\sinh a\sin\beta\sin\gamma$$

implying that.

$$n_A(P)+n_B(P)+n_C(P)=\frac{1}{4}\sinh a\sin\beta\sin\gamma\cosh PQ=\frac{N}{2}\cosh PQ,$$

where the triangular coordinates of Q are $\sinh a$, $\sinh b$ and $\sinh c$, respectively. Thus from (35) we get that $Q=I$ and the sum in the question is minimal if and only if P is equal to $Q=I$. This proves the statement. □

4.7.1. *Symmedian point.* We recall that the isogonal conjugate of the centroid is the so-called *symmedian point* of the triangle. The triangular coordinates of the symmedian point are

$$n_A(M'):n_B(M'):n_C(M')=\sinh^2 a:\sinh^2 b:\sinh^2 c. \tag{58}$$

From (8) immediately follows that the hyperbolic sine of the distances of the symmedian point to the sides are proportional to the hyperbolic sines of the corresponding sides:

$$\sinh d(M',BC):\sinh d(M',AC):\sinh d(M',AB)=\sinh a:\sinh b:\sinh c \tag{59}$$

showing the validity of the analogous Euclidean theorem in the hyperbolic geometry, too.

We note that the symmedian point of a hyperbolic triangle does not coincides with the *Lemoine point* L of the triangle. This center can be defined on the following way: If tangents be drawn at A,B,C to the circumcircle of the triangle ABC, forming a triangle $A'B'C'$, the lines AA', BB' and CC', are concurrent. The point of concurrence, is the Lemoine point of the triangle. The concurrency follows from Menelaos-theorem applying it to the triangle $A'B'C'$. We note that L is also (by definition) the so-called *Gergonne point* of the triangle $A'B'C'$. To prove that the symmedian point does not coincides with the Lemoine point we determine the triangular coordinates of the latter, too. Let L_A, L_B or L_C be the intersection point of $AA'\cap BC$, $BB'\cap AC$ or $CC'\cap AB$ (see in Fig. 15),respectively. Then we have

$$n_B(L):n_A(L)=(AL_CB)=\frac{\sinh AL_C}{\sinh L_CB}=\frac{\sinh C'B}{\sinh C'A}\frac{\sin AC'L_C\measuredangle}{\sin BC'L_C\measuredangle}=\frac{\sin AC'L_C\measuredangle}{\sin BC'L_C\measuredangle}.$$

On the other hand we have by (1)

$$\frac{\sin AC'L_C\measuredangle}{\sin CAC'\measuredangle}=\frac{\sinh CA}{\sinh CC'}\quad \text{and}\quad \frac{\sin BC'L_C\measuredangle}{\sin CBC'\measuredangle}=\frac{\sinh CB}{\sinh CC'}$$

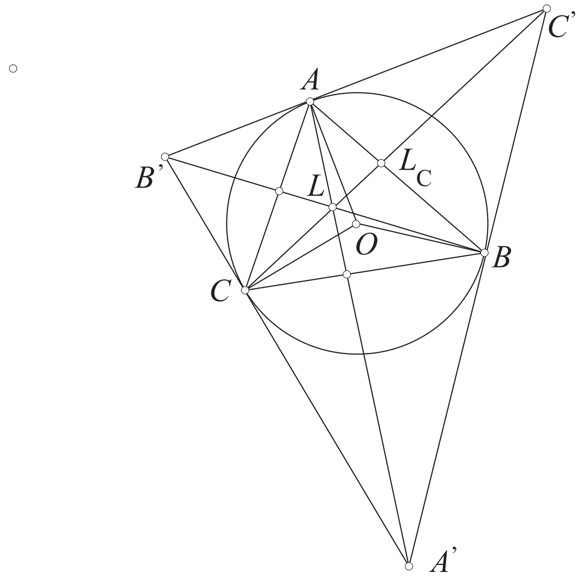

FIGURE 15. The Lemoine point of the triangle.

implying that

$$\frac{\sin AC'L_C\measuredangle}{\sin BC'L_C\measuredangle}=\frac{\sinh CA\sin CAC'\measuredangle}{\sinh CB\sin CBC'\measuredangle}=\frac{\sinh CA\cos CAO\measuredangle}{\sinh CB\sin CBO\measuredangle}=\frac{2\sinh\frac{CA}{2}\cosh\frac{CA}{2}\cos CAO\measuredangle}{2\sinh\frac{CB}{2}\cosh\frac{CB}{2}\sin CBO\measuredangle}=$$

$$=\frac{\sinh\frac{CA}{2}\cosh\frac{CA}{2}\frac{\tanh\frac{CA}{2}}{\tanh R}}{\sinh\frac{CB}{2}\cosh\frac{CB}{2}\frac{\tanh\frac{CB}{2}}{\tanh R}}=\frac{\sinh^2\frac{b}{2}}{\sinh^2\frac{a}{2}}=(\cosh b-1):(\cosh a-1)\,.$$

Thus the triangular coordinates of the Lemoine point are:

$$n_A(L):n_B(L):n_C(L)=(\cosh a-1):(\cosh b-1):(\cosh a-1)\,. \tag{60}$$

Now the symmedian point and the Lemoine point coincides for a triangle if and only if the equation array

$$\begin{aligned}(\cosh a-1)\sinh^2 b&=(\cosh b-1)\sinh^2 a\\(\cosh a-1)\sinh^2 c&=(\cosh c-1)\sinh^2 a\end{aligned} \tag{61}$$

gives an identity. Since

$$(\cosh a-1)(\cosh^2 b-1)=(\cosh a-1)(\cosh b-1)(\cosh b+1)=(\cosh b-1)(\cosh a-1)(\cosh a+1)=$$

$$=(\cosh b-1)\sinh^2 a$$

implies $a=b$, the only solution is when $a=b=c$ and the triangle is an equilateral (regular) one.

4.8. **On the "Euler line".** An interesting question in elementary hyperbolic geometry is the existence of the Euler line. Known fact (see e.g. in [16]) that the circumcenter, the centroid and the orthocenter of a triangle having in a common line if and only if the triangle is isoscale. In this sense Euler line does not exist for each triangle. A nice result from the recent investigations on the triangle centers is the paper of A.V. Akopyan [1] in which the author defined the concepts of "pseudomedians" and "pseudoaltitudes" giving two new centers of the hyperbolic triangle holding a deterministic Euclidean property of Euclidean centroid and orthocenter, respectively. He proved that the circumcenter, the intersection points of the pseudomedians (pseudo-centroid), the intersection points of the pseudoaltitudes (pseudo-orthocenter) and the circumcenter of the circle through the footpoints of the bisectors (the center of the Feuerbach circle) are on a hyperbolic line. A line through a vertex is called by *pseudomedian* if divides the area of the triangle in half. (We note that in spherical geometry Steiner proved the statement that the great circles through angular points of a spherical triangle, and which bisect its area, are concurrent (see [5]). Of course the pseudomedians are not medians and their point of concurrency is not the centroid of the triangle. We call it *pseudo-centroid.* He called *pseudoaltitude* a cevian (AZ_A) with the property that with its foot Z_A on BC holds the equality

$$AZ_AB\measuredangle-Z_ABA\measuredangle-BAZ_A\measuredangle=CZ_AA\measuredangle-Z_AAC\measuredangle-ACZ_a\measuredangle$$

where the angles above are directed, respectively. Throughout on his paper Akopyan assume that *"any two lines intersects and that three points determine a circle"*. He note in the introduction also that *"Consideration of all possible cases would not only complicate the proof, but would contain no fundamentally new ideas. To complete our arguments, we could always say that other cases follow from a theorem by analytic continuation, since the cases considered by us are sufficiently general (they include an interior point in the configuration space). Nevertheless, in the course of our argument we shall try to avoid major errors and show that the statements can be demonstrated without resorting to more powerful tools"*. We note that in our paper the reader can find this required extraction of the real elements by the ideal elements and the elements at infinity. We also defined all concepts using by Akopyan with respect to general points and lines, furthermore his lemmas and theorem can be extracted from circles onto cycles with our method. This prove the truth of Akopyan's note, post factum. To see the equivalence of the

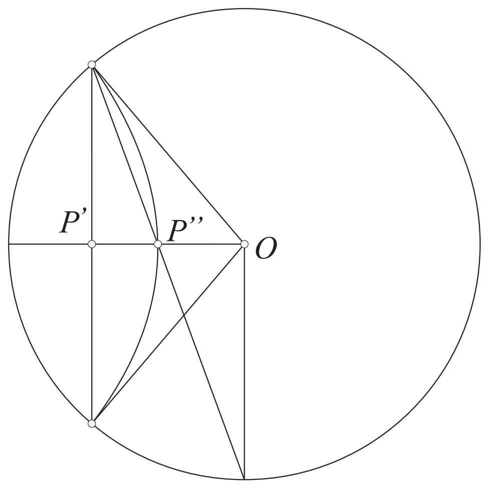

FIGURE 16. The connection between the projective and conformal models

two theory on real elements we recall that between the projective (Cayley-Klein-Beltrami) and Poincare models of the unit disk there is a natural correspondence, when we map to a line of the projective model to the line of the Poincare model with the same ends (points at infinity). On Fig. 16 we can see the corresponding mapping. A point P can be realized in the first model as the point P' and in the second one as the point P''. It is easy to see that if the hyperbolic distance of the points P and O is a then the Euclidean distances $P'O$ or $P''O$ are equals to $\tanh a$ or $\tanh(a/2)$, respectively. Thus our analytic definitions on similarity or inversion are model independent (end extracted) variations of the definitions of Akopyan, respectively. Thus we have

Theorem 10 ([1]). *The center O of the cycle around the triangle, the center of the cycle F around the feet of the pseudomedians, the pseudo-centroid S and the pseudo-orthocenter Z are on the same line.*

By Akopyan's opinion this is the *Euler line* of the triangle and thus he avoided the problem is to determination of the connection among the three important classical centers of the triangle. Our aim to give some analytic determination for the pseudo-centers introduced by Akopyan.

Theorem 11. *Let S_A, S_B, S_C be the feet of the pseudo-medians. Then we have the following formulas:*

$$\begin{aligned} \sinh\frac{AN_C}{2} : \sinh\frac{N_CB}{2} &= \cosh\frac{b}{2} : \cosh\frac{a}{2} \\ \sinh\frac{BN_A}{2} : \sinh\frac{N_AC}{2} &= \cosh\frac{c}{2} : \cosh\frac{b}{2} \\ \sinh\frac{CN_B}{2} : \sinh\frac{N_BA}{2} &= \cosh\frac{a}{2} : \cosh\frac{c}{2} \end{aligned} \tag{62}$$

implying that they are concurrent in a point S. We call S the pseudo-centroid *of the triangle. The triangular coordinates of the pseudo-centroid hold:*

$$n_A(R):n_B(R):n_C(R) = \frac{1}{\left(\cosh^2\frac{b}{2}\cosh^2\frac{c}{2}+\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\right)}: \tag{63}$$
$$:\frac{1}{\left(\cosh^2\frac{a}{2}\cosh^2\frac{c}{2}+\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\right)}:$$
$$:\frac{1}{\left(\cosh^2\frac{b}{2}\cosh^2\frac{a}{2}+\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\right)}.$$

Proof. From (12) we know that

$$\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}=\frac{N^2}{\sin\alpha\sin\beta\sin\gamma\sin\delta}.$$

(15) says that

$$2n^2=N\sinh a\sinh b\sinh c,$$

and we also have

$$\sin\alpha\sin\beta\sin\gamma\sinh a\sinh b\sinh c=4nN.$$

From these equalities we get the analogous of the spherical Cagnoli's theorem:

$$\sin\delta=\frac{N^2}{\sin\alpha\sin\beta\sin\gamma\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}}=\frac{N^2\sinh a\sinh b\sinh c}{4nN\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}}=\frac{n}{2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}}. \tag{64}$$

Using the formulas before (26) we get that

$$\cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}=\sqrt{\frac{\sin(\delta+\beta)\sin(\delta+\gamma)}{\sin\gamma\sin\beta}}\sqrt{\frac{\sin\delta\sin(\delta+\beta)}{\sin\gamma\sin\alpha}}\sqrt{\frac{\sin\delta\sin(\delta+\gamma)}{\sin\alpha\sin\beta}}=$$
$$=\frac{N^2}{\sin(\delta+\alpha)\sin\alpha\sin\beta\sin\gamma},$$

implying (with the above manner) the equality

$$\sin(\delta+\alpha)=\frac{n}{2\cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}. \tag{65}$$

From these equalities we get that

$$\frac{\sin(\delta+\alpha)}{\sin\delta}=\cos\alpha+\cot\delta\sin\alpha=\coth\frac{b}{2}\coth\frac{c}{2}. \tag{66}$$

Thus if the area of a triangle and one of its angles be given, the product of the semi hyperbolic tangents of the containing sides is given. Since the area of the examined triangles are equals to each other we get that

$$\frac{n}{2\cosh\frac{a}{2}\cosh\frac{BN_C}{2}\cosh\frac{CN_C}{2}}=\frac{\sinh a\sinh BN_C\sin\beta}{4\cosh\frac{a}{2}\cosh\frac{BN_C}{2}\cosh\frac{CN_C}{2}}=\frac{\sinh\frac{a}{2}\sinh\frac{BN_C}{2}\sin\beta}{\cosh\frac{CN_C}{2}}$$

and similarly

$$\frac{n}{2\cosh\frac{b}{2}\cosh\frac{N_CA}{2}\cosh\frac{CN_C}{2}}=\frac{\sinh\frac{b}{2}\sinh\frac{N_CA}{2}\sin\alpha}{\cosh\frac{CN_C}{2}}$$

implying that

$$\sinh\frac{a}{2}\sinh\frac{BN_C}{2}\sin\beta=\sinh\frac{b}{2}\sinh\frac{N_CA}{2}\sin\alpha.$$

From this we get that

$$\frac{\sinh\frac{AN_C}{2}}{\sinh\frac{N_CB}{2}}=\frac{\sinh\frac{a}{2}\sin\beta}{\sinh\frac{b}{2}\sin\alpha}=\frac{\cosh\frac{b}{2}}{\cosh\frac{a}{2}}$$

as we stated in (63). The production of the equalities in (63) gives the equality

$$\sinh\frac{AN_C}{2}\sinh\frac{BN_A}{2}\sinh\frac{CN_B}{2}=\sinh\frac{N_CB}{2}\sinh\frac{N_AC}{2}\sinh\frac{N_BA}{2}. \tag{67}$$

On the other hand the triangles CAN_C, N_BAB having equal areas and also have a common angle, in virtue of (67) we get that

$$\tanh\frac{b}{2}\tanh\frac{AN_C}{2}=\tanh\frac{c}{2}\tanh\frac{N_BC}{2},$$

implying that

$$\tanh\frac{AN_C}{2}\tanh\frac{BN_A}{2}\tanh\frac{CN_B}{2}=\tanh\frac{N_BC}{2}\tanh\frac{N_CB}{2}\tanh\frac{N_AC}{2}.$$

So we also have

$$\cosh\frac{AN_C}{2}\cosh\frac{BN_A}{2}\cosh\frac{CN_B}{2}=\cosh\frac{N_BC}{2}\cosh\frac{N_CB}{2}\cosh\frac{N_AC}{2},$$

and as a consequence the equality

$$\sinh AN_C\sinh BN_A\sinh CN_B=\sinh N_CB\sinh N_AC\sinh N_BA.$$

Menelaos theorem now gives the existence of the pseudo-centroid.

From (63) we get that

$$\frac{\cosh\frac{a}{2}}{\cosh\frac{b}{2}}=\frac{\sinh\left(\frac{c}{2}-\frac{AN_C}{2}\right)}{\sinh\frac{AN_C}{2}}=\sinh\frac{c}{2}\coth\frac{AN_C}{2}-\cosh\frac{c}{2},$$

hence

$$\coth\frac{AN_C}{2}=\frac{\cosh\frac{b}{2}\cosh\frac{c}{2}+\cosh\frac{a}{2}}{\sinh\frac{c}{2}\cosh\frac{b}{2}}$$

or equivalently

$$\cosh\frac{AN_C}{2}=\frac{\cosh\frac{b}{2}\cosh\frac{c}{2}+\cosh\frac{a}{2}}{\sinh\frac{c}{2}\cosh\frac{b}{2}}\sinh\frac{AN_C}{2}.$$

From this we get

$$1=\sinh^2\frac{AN_C}{2}\left(-1+\left(\frac{\cosh\frac{b}{2}\cosh\frac{c}{2}+\cosh\frac{a}{2}}{\sinh\frac{c}{2}\cosh\frac{b}{2}}\right)^2\right)=$$

$$=\frac{-\sinh^2\frac{c}{2}\cosh^2\frac{b}{2}+\left(\cosh\frac{b}{2}\cosh\frac{c}{2}+\cosh\frac{a}{2}\right)^2}{\sinh^2\frac{c}{2}\cosh^2\frac{b}{2}}\sinh^2\frac{AN_C}{2}=$$

$$\frac{\cosh^2\frac{b}{2}+2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}+\cosh^2\frac{a}{2}}{\sinh^2\frac{c}{2}\cosh^2\frac{b}{2}}\sinh^2\frac{AN_C}{2}$$

Thus

$$\sinh AN_C=2\sinh\frac{AN_C}{2}\cosh\frac{AN_C}{2}=2\sinh^2\frac{AN_C}{2}\frac{\cosh\frac{b}{2}\cosh\frac{c}{2}+\cosh\frac{a}{2}}{\sinh\frac{c}{2}\cosh\frac{b}{2}}=$$

$$=2\frac{\sinh\frac{c}{2}\cosh\frac{b}{2}\left(\cosh\frac{b}{2}\cosh\frac{c}{2}+\cosh\frac{a}{2}\right)}{\cosh^2\frac{b}{2}+2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}+\cosh^2\frac{a}{2}}.$$

Hence we also have

$$\sinh N_CB=2\frac{\sinh\frac{c}{2}\cosh\frac{a}{2}\left(\cosh\frac{a}{2}\cosh\frac{c}{2}+\cosh\frac{b}{2}\right)}{\cosh^2\frac{a}{2}+2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}+\cosh^2\frac{b}{2}}$$

implying that

$$n_B(N):n_A(N)=(AN_CB)=\left(\cosh^2\frac{b}{2}\cosh\frac{c}{2}+\cosh\frac{b}{2}\cosh\frac{a}{2}\right):\left(\cosh^2\frac{a}{2}\cosh\frac{c}{2}+\cosh\frac{a}{2}\cosh\frac{b}{2}\right)=$$

$$=\left(\cosh^2\frac{b}{2}\cosh^2\frac{c}{2}+\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\right):\left(\cosh^2\frac{a}{2}\cosh^2\frac{c}{2}+\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\right).$$

From this we get that

$$n_A(N):n_B(N)=\frac{1}{\left(\cosh^2\frac{b}{2}\cosh^2\frac{c}{2}+\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\right)}:\frac{1}{\left(\cosh^2\frac{a}{2}\cosh^2\frac{c}{2}+\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\right)}.$$

Similarly we get

$$n_B(N):n_C(N)=\frac{1}{\left(\cosh^2\frac{c}{2}\cosh^2\frac{a}{2}+\cosh\frac{c}{2}\cosh\frac{b}{2}\cosh\frac{a}{2}\right)}:\frac{1}{\left(\cosh^2\frac{b}{2}\cosh^2\frac{a}{2}+\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\right)}$$

as we stated in (64). $\square$

Remark. We note that there are many Euclidean theorems can be investigated on the hyperbolic plane by our more-less trigonometric way. We note that on the hyperbolic plane the usual isoptic property of the circle lost (see [6]) and thus all the Euclidean statements using this property can be investigated only the way of [1]. To that we can use trigonometry in this method we can concentrate on the introduced concept of angle sums which in a trigonometric calculation can be handed well. Thus the isoptic property of a cycle (or which is the same the cyclical property of a set of points) can lead for new hyperbolic theorems suggested by known Euclidean analogy.

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