Malfatti's problem on the hyperbolic plane*

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Abstract

More than two centuries ago Malfatti (see [10]) raised and solved the following problem (the so-called Malfatti's construction problem):Construct three circles into a triangle so that each of them touches the two others from outside moreover touches two sides of the triangle too. It is an interesting fact that nobody investigated this problem on the hyperbolic plane, while the case of the sphere was solved simultaneously with the Euclidean case. In order to compensate this shortage we solve the following exercise: *Determine three cycles of the hyperbolic plane so that each of them touches the two others moreover touches two of three given cycles of the hyperbolic plane.* We also give a proof of the hyperbolic version of Malfatti's marble problem.

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1 Malfatti's construction problem

1.1 The history of the problem

Malfatti (see [10]) raised and solved the following problem: *construct three circles into a triangle so that each of them touches the two others from outside moreover touches two sides of the triangle too.* Malfatti determined the radii of the required circles giving an analytic-geometric solution. This formula for the radii was only the beginning of a research motivated by the original problem. We would like to mention here some nice further investigations.

The first nice moment was Steiner's construction. He gave an elegant method (without proof) to construct the given circles. He also extended the problem and his construction to the case of three given circles instead of the sides of a triangle (see in [14], [15]). Cayley referred to this problem in [3] as *Steiner's extension of Malfatti's problem.* We note that Cayley investigated and solved its generalization in [3], he called it also by Steiner's extension of Malfatti's problem. His problem is *to determine three conic sections so that each of them touches the two others, and also touches two of three more given conic sections.* Since the case of circles on the sphere is a generalization of the case of circles of the plane (as it can be seen easily by stereographic projection) Cayley indirectly proved

*Dedicated to the memory of my colleague, patron and friend István Reiman.

Steiner's second construction. We also have to mention Hart's nice geometric proof for Steiner's construction which was published in [9]. (It can be found in various textbooks e.g. [6] and also on the web.)

The second moment which I would like to mention here is two short papers [12] and [13] written by Dr.Schellbach. He solved the original construction problem and its "spherical variation", too. The elegant goniometric solution determines the touching points of the circles on the sides of the triangle (spherical triangle). These nice and surprising results inspired Cayley to suggest a little bit more elegant way to get Schellbach's formulas by a change of notation [4]. Then he supposed that the sides are indefinitely small and he reduced the problem to of plane triangle. In this way he got some formulas leading to the formulas of own previous paper [5]. Finally he determined the distances between the points of contact from the adjacent angles of the triangle by the side-lengthes of the triangle. He noted also that the equations are very similar in form to those given in the same paper for the determination of the radii of the inscribed Malfatti's circles. At this point it seemed that there is not possible to give any new observation for this problem. However Bottema (in [2]) observed a simple solution by inversion for the case when the determining figures are touching circles on the euclidean plane. The idea of the construction is that the center of the reference circle of the inversion must be the point of contact of two given circles. Then the inversion leads to a configuration in which the circles can be got in a simple way, and their radii can also be calculated from the radii of the given circles.

It is an interesting fact that nobody has investigated the hyperbolic case of this – more than two centuries ago raised – problem before, while the case of the sphere was solved simultaneously with the Euclidean case. Our aim is to compensate for this shortage.

Additionally we give a proof for the hyperbolic case of the so-called "marble problem", which also corresponds to the Malfatti's original paper. In agreement with Remark 2 in [1] we prove that: *On the hyperbolic plane the greedy arrangement has the largest total area among the non-overlapping circles in a triangle.*

1.2 The case of hyperbolic triangle

Note that the case of the hyperbolic triangle can be solved immediately by the method of Steiner and also by the method of Schellbach. The problem now is the following: *Construct three circles into a triangle so that each of them touches the two others from outside, and moreover touches two sides of the triangle.*

1.2.1 Steiner's construction for hyperbolic triangle

To solve Malfatti's problem Steiner proposed the following construction [15]:

1. Draw three angle bisectors OA, OB and OC. In the triangles $\triangle_{OAB}$, $\triangle_{OBC}$, $\triangle_{OCA}$ inscribe circles c_C, c_A, c_B, respectively.

2. For each pair of the circles consider the second (distinct to the corresponding angle bisector) internal tangents. The latter concur in a point K and cross the sides in points H, I; D, E; and F, G; respectively.

3. The three quadrilaterals $KHCI$, $KGBH$, and $KEAG$ are inscriptible. Their incircles solve Malfatti's problem.

The proof of Hart [9] works on the hyperbolic plane, too. It is based on two elementary but absolute lemmas, implying that their corollaries are also valid on the hyperbolic plane. These are:

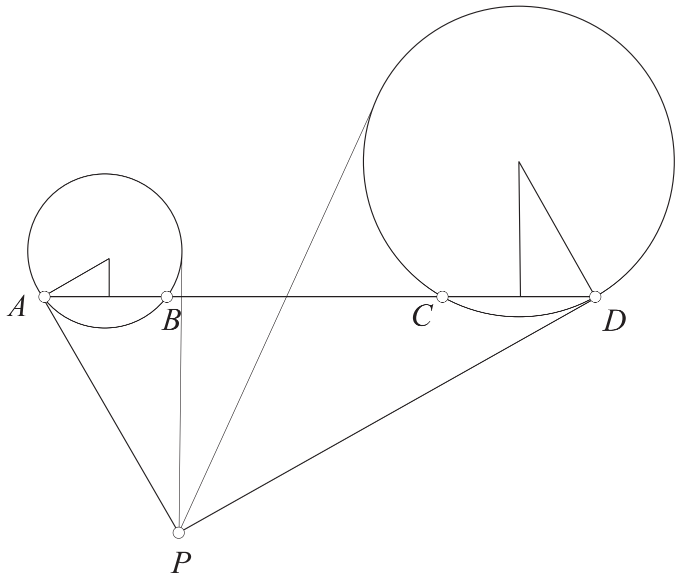

Figure 1: Hart's Lemma 2

Lemma 1 ([9]) *If the sum or difference of two tangents drawn from a point P to two circles is equal to a common tangent of the circles, the point P is on a common tangent.*

Hence it is evident that if the common tangents to each pair of three circles pass through the same point, one of these common tangents must be equal to the sum of the other two, and that therefore the other three common tangents will also pass through a point.

Lemma 2 ([9]) *If two circles cut off equal parts AB and CD (Fig. 1.) from a given line and if tangents at the extreme points A, D intersect at P, the circles will subtend equal angles at P, and also that if tangents be drawn from each point A and D to the other circle, they will be equal.*

The proof of the first lemma is based on two facts; on the triangle inequality and on the tangents drawing from a point to a circle are equals. Both of them are valid on the hyperbolic plane, too. The proof of the second one use trigonometry and can be done in the hyperbolic plane, too. Hart himself observed that his proof works on the sphere. Hart's proof of the construction (which we are citing word-by-word from the original paper with respect to its elegancy and clarity) is the following (see Fig. 2.):

Let L, M, N, be the points of contact of three circles which touch one another, and each touch two sides of the given triangle $\triangle_{ABC}$. Draw DE, FG,

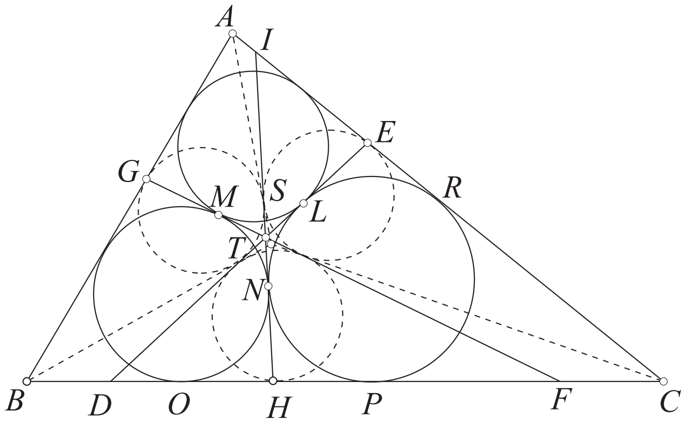

Figure 2: Hart's proof of Steiner's construction.

HI, touching these circles at L, M, N and meeting one another at K. Then since $FH - HD = FO - DP = FM - DL = FK - DK$, H is the point of contact of the circle inscribed in the triangle $\triangle_{DKF}$; in like manner E and G are the points of contact of circles which touch IK, KF, and AC; IK, KD, and AB, respectively; but $HN = HP = QL$, and $NS = ER = EL$, therefore $HS = EQ$, and therefore the circles HQ and ES subtend equal angles at C (Lemma 2). Also three common tangents QL, SN, and KF of the circles HQ, ES, PNR, pass through K; therefore C must be a point on the other common tangent to HQ and ES (Cor. Lemma 1). In like manner it is proved that the bisectors of A and B are common tangents to ES and GT; HQ and GT respectively. Whence Steiner's construction is evident for plane and spherical triangles.

By inversion Hart extends his proof to all configurations of the plane or the sphere in which the sides of the given triangle are arcs of circles. We note that inversion could not change intersecting circles into non-intersecting one thus this method inappropriate to prove Steiner's extension of Malfatti's problem in the case when the given circles are pairwise disjoints.

1.2.2 Schellbach's solution

Schellbach's solution is a goniometric construction. Historically, this solution preceded Hart's proof (see [12] and [13]). To use this for the hyperbolic case we have to change the Schellbach's equalities on spherical triangles onto the corresponding equalities of hyperbolic triangles. Practically we can use imaginary values to the measure of the lengthes and so the trigonometric functions of these quantities give hyperbolic functions depending on their imaginary part. Formally we follow Cayley's simplified terminology. Let the sides of the triangle be a, b, c and let x, y, z be the distances of the touching points of the circles from the adjacent vertices of the triangle. Then writing

$$a + b + c = 2s,$$

$$a - \frac{1}{2}s = l, \quad b - \frac{1}{2}s = m, \quad c - \frac{1}{2}s = n$$

whence $l+m+n=\frac{1}{2}s$, and putting also

$$\frac{1}{2}s-x=\xi,\quad \frac{1}{2}s-y=\eta,\quad \frac{1}{2}s-z=\zeta,$$

then we have

$$\begin{aligned}
\frac{\cosh l\cosh\eta\cosh\zeta}{\cosh\frac{1}{2}s}+\frac{\sinh l\sinh\eta\sinh\zeta}{\sinh\frac{1}{2}s} &= 1,\\
\frac{\cosh m\cosh\zeta\cosh\xi}{\cosh\frac{1}{2}s}+\frac{\sinh m\sinh\zeta\sinh\xi}{\sinh\frac{1}{2}s} &= 1,\\
\frac{\cosh n\cosh\xi\cosh\eta}{\cosh\frac{1}{2}s}+\frac{\sinh n\sinh\xi\sinh\eta}{\sinh\frac{1}{2}s} &= 1,
\end{aligned}$$

from which equations the unknown quantities ξ,η,ζ are to be determined. To solve the equations, let the subsidiary angles λ,μ,ν be determined by the conditions

$$\begin{aligned}
\frac{\cosh\lambda\cosh m\cosh n}{\cosh\frac{1}{2}s}-\frac{\sinh\lambda\sinh m\sinh n}{\sinh\frac{1}{2}s} &= 1,\\
\frac{\cosh\mu\cosh n\cosh l}{\cosh\frac{1}{2}s}-\frac{\sinh\mu\sinh n\sinh l}{\sinh\frac{1}{2}s} &= 1,\\
\frac{\cosh\nu\cosh l\cosh m}{\cosh\frac{1}{2}s}-\frac{\sinh\nu\sinh l\sinh m}{\sinh\frac{1}{2}s} &= 1.
\end{aligned}$$

Then it may be shown that

$$\cosh(\eta+\zeta)=\frac{\cosh\left(\frac{s+\lambda-l}{2}\right)}{\cosh\left(\frac{\lambda+l}{2}\right)},\quad \cosh(\eta-\zeta)=\frac{\cosh\left(\frac{s-\lambda+l}{2}\right)}{\cosh\left(\frac{\lambda+l}{2}\right)},$$

$$\cosh(\zeta+\xi)=\frac{\cosh\left(\frac{s+\mu-m}{2}\right)}{\cosh\left(\frac{\mu+m}{2}\right)},\quad \cosh(\zeta-\xi)=\frac{\cosh\left(\frac{s-\mu+m}{2}\right)}{\cosh\left(\frac{\mu+m}{2}\right)},$$

$$\cosh(\xi+\eta)=\frac{\cosh\left(\frac{s+\nu-n}{2}\right)}{\cosh\left(\frac{\nu+n}{2}\right)},\quad \cosh(\xi-\eta)=\frac{\cosh\left(\frac{s-\nu+n}{2}\right)}{\cosh\left(\frac{\nu+n}{2}\right)}.$$

If we write

$$\begin{aligned}
\tanh\phi &= \tanh m\tanh n\coth\frac{1}{2}s,\\
\tanh\chi &= \tanh n\tanh l\coth\frac{1}{2}s,\\
\tanh\psi &= \tanh l\tanh m\coth\frac{1}{2}s,
\end{aligned}$$

then

$$\begin{aligned}
\cosh(\lambda-\phi) &= \frac{\cosh\frac{1}{2}s\cosh\phi}{\cosh m}\cosh n,\\
\cosh(\mu-\chi) &= \frac{\cosh\frac{1}{2}s\cosh\chi}{\cosh n}\cosh l,\\
\cosh(\nu-\psi) &= \frac{\cosh\frac{1}{2}s\cosh\psi}{\cosh l}\cosh m,
\end{aligned}$$

which give the values of λ,μ,ν by the help of hyperbolic rectangular triangles. From these quantities ξ,η,ζ and thus x,y,z can be constructed, too.

1.3 The case of the cycles

Our aim is to investigate touching cycles of the hyperbolic plane. For simplicity, we assume that the domains of the given cycles are pairwise non-overlapping. We formulate the exercise as follows: *Determine three cycles of the hyperbolic plane so that each of them touches the two others moreover touches two of three given cycles of the hyperbolic plane.*

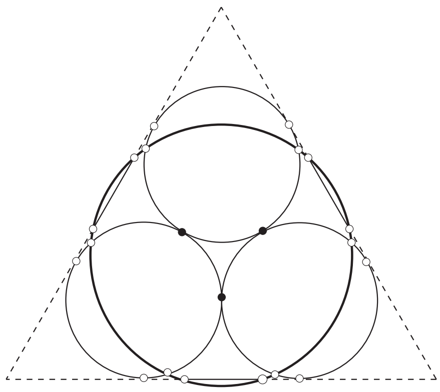

Figure 3: The euclidean solution does not work on the hyperbolic plane.

Using Poincare's conformal disk model the given cycles represented by the arcs of certain circles. By Steiner's method we can construct the corresponding Malfatti's circles solving the Steiner's extension of Malfatti's problem. But it is possible that the corresponding arcs of circles have no common point in the model-circle. This means that there is no solution of the problem with respect to the hyperbolic plane. In fact, consider the situation of Fig.3. The edges of the regular triangle with respect to the basic circle of the model represent arcs of hypercycles of the hyperbolic plane. On the other hand the uniquely determined Malfatti's circles with respect to this regular triangle touche the corresponding edges in external points of the model implying that the corresponding cycles cannot be touching cycles to the given hypercycles.

The trouble of this example can be solved easily if we consider the whole cycle without an arc of a cycle. We will use the following concepts:

Definition 1 *The* hypercycle *is the locus of points of the plane with distances are equal to a constant. (Its domain is convex bounded by a curve with two connected components.)* Two cycles are touching *if they have a common point with a common tangent line.*

We can state the following theorem on existence.

Theorem 1 *For three given cycles have a Malfatti's system of cycles, so there are three cycles that each of them touches the two others moreover touches two from the three given cycles.*

Remark that in this theorem the domains of the corresponding six cycles need not to be disjoint. On the other hand the tools of a concrete construction will require some further geometric conditions thus in the rest of this paper we assume that the domains of the given cycles are pairwise non-overlapping.

Proof: Consider the Poincare's disk model of the hyperbolic plane. In the case, when there is no hypercycle among the given cycles the statement is trivial. On the other hand by our definition a hypercycle is the union of two arcs of circles. Represent now a hypercycle with one of the whole circle containing one of the original arcs. The inner arc (which lies in the interior of the model) of the chosen circle is the inverse image (with respect to the model circle) of the outer arc of the other possible circle. Now Steiner's construction can be done on the embedding Euclidean plane and we get a Malfatti's system of circles for the representing circles. If we take the inverse of those points of the six circles which are external to the model, we get either whole hypercycle; paracycle; or a circle of the hyperbolic plane accordingly to the cases where the getting circle intersects; touched externally; or lies in the complement of the model circle, respectively.

So with this transformation from a Malfatti's system of circles of the embedding Euclidean plane, we get a Malfatti's system of cycles of the hyperbolic plane corresponding to the given cycles as we stated. □

Unfortunately the using condition for touching cycles cannot exclude the possibility of existences of other common points of the two cycles (contrary to the case of circles). Two possible points of intersection can lie on the other connected components of the hypercycle. (On the example of Fig.3 we can find this situation after the inversion of the external parts of the hypercycles.)

Thus with more rigorous definitions of touching we have new (typically harder) problems on existence. Two possibilities for the concepts of touching are:

- two cycles are touching if they have exactly one common (real) point on the hyperbolic plane (in that point they tangents are also the same), or
- two cycles are touching if they domains are touching externally.

In this paper we do not deal with these questions which can give the base of a foregoing paper. We consider here the most simple case of touching as we fixed in our definition Definition 1. Our question is:

How can we construct the Malfatti's system of cycles of a given system of cycles?

The question is interesting if we cannot use model representations. Our purpose is to give such a construction. To this we have to build the hyperbolic analogue of some Euclidean concepts.

1.3.1 Hyperbolic power, inversion and centres of similitude on the hyperbolic plane

It is not clear who investigated first the concept of inversion with respect to hyperbolic geometry. A synthetic approach can be found in [11] using reflections in Bachmann's metric plane. To our purpose it is more convenient an analytic approach in which the concepts of centres of similitude and axe of similitude can be defined. We consider – as an analogy – the following spherical approach of these concepts.

It can be proved that if an arc of a great circle (line) passing through a fixed point O cut a small circle in the points A,B,

$$\tan\frac{1}{2}OA\cdot\tan\frac{1}{2}OB$$

is constant. This product is called the *spherical power* of O with respect to circle. It is positive or negative, according as O is exterior or interior to the circle. If from any point O outside a small circle two arcs be drawn to it, of which one, OD, is a tangent, and the other a secant, meeting it in the points A, B; then

$$\tan^2\frac{1}{2}OD=\tan\frac{1}{2}AO\cdot\tan\frac{1}{2}OB.$$

If we have two small circles on the sphere then the locus of points P for which the tangents to these circles are equal is a great circle called the *radical circle (axe of power)* of them. The radical circles of three small circle taken in pairs are concurrent. The common point is the *power point* of the three small circle. This is the centre of the circle orthogonal to each of them.

For two small circles there are two *centres of similitude.* These are the points on the line connecting their centres which divide the segment joining the centers of the two circles externally and internally in the spherical ratio of the sine of the radii. Common tangents to the circles pass through the centres of similitude, viz., the direct common tangents through the external centre and the inverse common tangent through the internal centre. If through a centre of similitude we draw a secant cutting the circles, then the pairs of points M, M';N, N' of Fig.4 are said to be homothetic and M, N'; M', N are inverse.

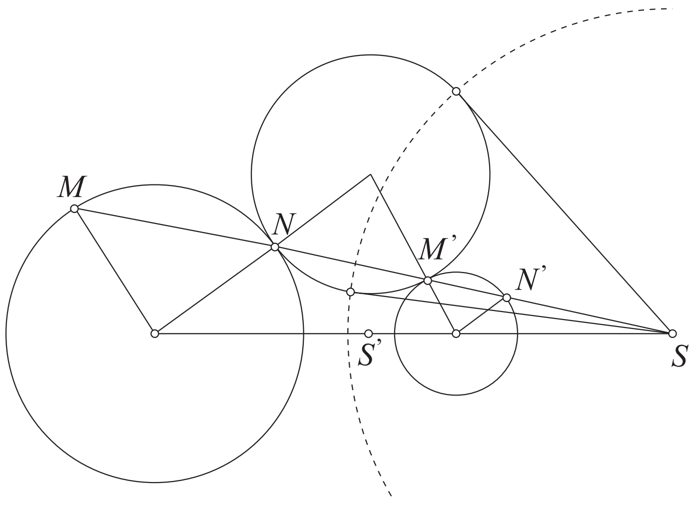

Figure 4: Centres of similitude

Then for the homothetic points M, M'

$$\tan\frac{SM}{2}:\tan\frac{SM'}{2}$$

are in a given ratio. Thus

$$\tan\frac{SM}{2}\tan\frac{SN'}{2}=\tan\frac{SM'}{2}\cdot\tan\frac{SN}{2}$$

is a constant. The six centres of similitude of three small circles taken in pairs lie three by three on four lines, called *axes of similitude* of the circles. Consequently if a variable circle touch two fixed circles, the line passing through the points of contact passes through a fixed point, namely, a centre of similitude of the two circles; for the points of contact are centres of similitude. Moreover if a variable circle touch two fixed circles, the tangent drawn to it from centre of similitude, through which the chord of contact passes, is constant. Thus if being given a fixed point S and any curve whatever γ, on the sphere, if upon the segment of a line joining S to any point M of γ a point N' be taken, such that $\tan \frac{SM}{2} \tan \frac{SM'}{2}$ is constant, the locus of N' is called the inverse of γ.

Revert to the hyperbolic case we have a new situation, namely two lines do not intersect in every case. For example, if we consider three points A, B, C on a line (with this order) then the ratio defined by

$$\frac{\sinh |AC|}{\sinh |BC|}$$

is equal to

$$\frac{\sinh |AC|}{\sinh |BC|} = \frac{\sinh (|AB| + |BC|)}{\sinh |BC|} = \cosh |AB| + \coth |BC| \sinh |AB|,$$

and by the assumption $\coth |BC| > 1$ it is greater then $e^{|AB|}$. Therefore a ratio can be attached by a real point C only if it is greater than $e^{|AB|}$. (Obviously, this quantity depends on the distance of the points A, B). On the other hand every number greater or equal to 1 could be the ratio of hyperbolic sines of the radius of circles with origin A and B, respectively. To solve this problem we follows the method of the book of Cyrill Vörös ([16]). We extract the concepts of measure of real elements. We expand the plane with two types of points, one type of the points at infinity and the other one the type of ideal points. In a projective model these are the boundary and external points of a model with respect to the embedding real projective plane. Two parallel lines determine a point at infinity and two ultraparallel lines an ideal point which is the pole of their common transversal. Now the concept of the line can be expanded; a line is real if it has real points (in this case it also has two points at infinity and the other points on it are ideal points being the poles of the real lines orthogonal to the mentioned one). The expanded real line is a closed compact set with finite length. We also distinguish the line at infinity containing precisely one point at infinity and the so-called ideal line which contains only ideal points. By definition of the common lengthes of the lines are $\pi k i$, where k is a constant of the hyperbolic plane and i is the imaginary unit. In this paper we assume that $k = 1$. The distance of a real and an ideal point is a complex number. Its real part is the distance of the real point to the polar of the ideal point with a sign, this sign is positive in the case when the polar line intersects the investigated segment and is negative otherwise. The imaginary part of the length is $\frac{\pi}{2} i$ implying that the sum of the lengthes of the two disjoint segments of this projective line is the total length πi. By definition the distance from a point at infinity is infinite. The distance of two ideal points of a real line is the negative of the distance of their polars. The distance of two points of an ideal line is the angle of their polars multiplied by i. Finally, every distances from a point at infinity of a line at infinity are undeterminable and we can define such manner that the trigonometric relationships could be valid. Thus let this distance be $\frac{\pi}{2}i$. Finally, let the distance of two ideal points of a line at infinity be 0 or πi, accordingly that the point at infinity of the line does not belong or belongs to the corresponding segment, respectively. Now we are ready to the definition.

Definition 2 *The* power *of a point P with respect to a given cycle is the value*

$$\tanh \frac{1}{2}PA \cdot \tanh \frac{1}{2}PB,$$

where the points A, B are on the cycle, such that their lines passes through the point P. The axe of power *of two cycles is the locus of points having the same powers with respect to the cycles. The* centres of similitude *of two cycles with non-overlapping interiors are the common points of their pairs of tangents touching direct or inverse, respectively. The first point is the* external centre of similitude *the second one is* the internal centre of similitude*.*

The usual statements on the Euclidean or spherical power is valid also in the hyperbolic plane. The power of a point could be positive, negative or complex. (For example, in the case when the meeting points of the secant are real we have the following possibilities: positive if P is a real point and it is in the exterior of the cycle; negative if P is real and it is in the interior of the cycle, infinity if P is a point at infinity, or complex if P is an ideal point.)

Hence we have the following six possibilities for the centres of similitude:

1. *The two cycles are circles.* To get the centres of similitude we have to solve an equation in x. Here d means the distance of the centers of the circles and $r \leq R$ denotes the respective radii.

   $$\sinh(d \pm x) : \sinh x = \sinh R : \sinh r$$

   from which we get that

   $$\coth x = \frac{\sinh R \mp \cosh d \sinh r}{\sinh r \sinh d}$$

   or equivalently

   $$e^x = \sqrt{\frac{\sinh R \mp \cosh d \sinh r + \sinh r \sinh d}{\sinh R \mp \cosh d \sinh r - \sinh r \sinh d}} = \sqrt{\frac{\frac{\sinh R}{\sinh r} \mp e^d}{\frac{\sinh R}{\sinh r} + e^d}}.$$

   Hence in every cases the internal centre of similitude is a real point and for the external centre we have three cases. It is an ideal point, point at infinity or a real point accordingly to the cases $\frac{\sinh R}{\sinh r} < e^d$, $\frac{\sinh R}{\sinh r} = e^d$ or $\frac{\sinh R}{\sinh r} > e^d$, respectively.

2. *One of the cycles is a circle and the other one is a paracycle.* The line joining their centers (which we call axe of symmetry) is a real line, but the respective ratio is zero or infinity. To determine the centres we have to decide the common tangents and their points of intersections, respectively. The external centre is an ideal point and the internal centre is a real point.

3. *One of the cycles is a circle and the other one is a hypercycle.* The axe of symmetry is a real line that the ratio of the sine of the radii is complex. The external centre is an ideal point the internal one is always real point. Each of them can be determined as in the case of two circles.

4. *Each of them is a paracycle.* The axe of symmetry is a real line and the internal centre is a real point. The external centre is an ideal point.

5. *One of them paracycle and the other one is a hypercycle.* The axe of symmetry is a real line or a line at infinity. The internal centre could be in the first case a real point, a point at infinity or an ideal point; and in the second case it is an ideal point. The external centre is always ideal point.

6. *Both of them are hypercycles.* The axe of symmetry could be real line, ideal line or a line at infinity. For the internal centre we have three possibilities as above. On the other hand the external centre is always an ideal point.

Since using the expanded concepts two points always determine a line and two lines always determine a point, all concepts defined on the sphere also can be used on the hyperbolic plane. Thus we use the concepts of "axe of similitude", "inverse and homothetic pair of points", "homothetic to" and "inverse of" a curve γ with respect to a generalized fix point S as in the case of the sphere. Thus being given a fixed point S (which is the center of the cycle for which we would like to invert) and any curve whatever γ, on the plane, if upon the segment of a line joining S to any point M of γ a point N' be taken, such that

$$\tanh \frac{SM}{2} \tanh \frac{SN'}{2}$$

is constant, the locus of M' is *called the inverse* of γ. We also use the name *cycle of inversion* for the locus of the points whose squared distance from S is

$$\tanh \frac{SM}{2} \cdot \tanh \frac{SN'}{2}.$$

Among the projective elements of the pole and its polar either one of always real or both of them are at infinity. Thus in a construction the common point of two lines is well-defined, and in every situation it can be joined with another point; for example, if both of them are ideal points they given by their polars (which are constructible real lines) and the required line is the polar of the intersection point of these two real lines. Thus the lengthes in the definition of the inverse can be constructed. This implies that the inverse of a point can be constructed on the hyperbolic plane, too.

Finally we remark that all of the concepts and results of inversion with respect to a sphere of the Euclidean space can be defined also in the hyperbolic space, the "basic sphere" could be hypersphere, parasphere or sphere, respectively. We can use also the concept of ideal elements and the concept of elements at infinity, if it is necessary. It can be proved (using Poincare's ball-model) that every hyperbolic plane of the hyperbolic space can be inverted to a sphere by such a general inversion. This map sends the cycles of the plane to circles of the sphere.

1.3.2 Steiner's construction on the hyperbolic plane

Theorem 2 *Steiner's construction can be done also in the hyperbolic plane. Precisely, for three given non-overlapping cycles can be constructed three other, each of them touches the two others and also touches two of the three given one.*

Proof: Denote by c_i the given cycles. Now the steps of Steiner's construction are the following:

1. Construct the cycle of inversion $c_{i,j}$, for the given cycles c_i and c_j, where the center of inversion is the external centre of similitude of them.
2. Construct cycle k_j touching two cycles $c_{i,j}, c_{j,k}$ and the given cycle c_j.
3. Construct the cycle $l_{i,j}$ touching k_i and k_j through the point $P_k = k_k \cap c_k$.
4. Construct the Malfatti's cycle m_j as the common touching cycle of the four cycles $l_{i,j}$, $l_{j,k}$, c_i, c_k.

From the definitions of the preceding paragraph we know that the first step is constructible.

To the second step we follow Gergonne's construction (see e.g. [8]). This construction is:

Draw the point P of power of the given circles and an axe of similitude of certain three centres of similitude. The poles of the axe of similitude with respect to the given circles are points of the rays from P passes through the touching points of a sought circle and one of the three given ones.

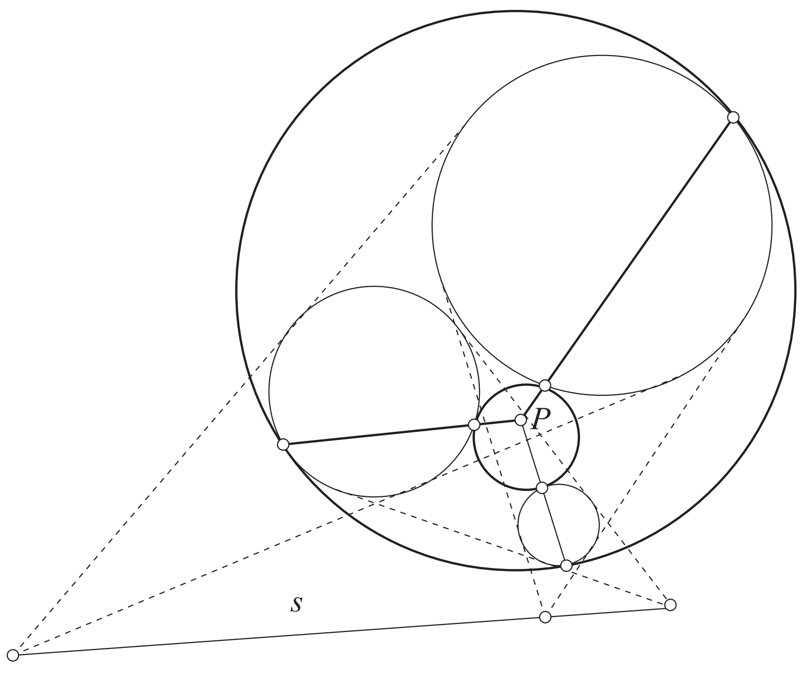

Figure 5: The construction of Gergonne

We note that by this method we find the common touching circles pairwise, correspondingly to the choose of the axe of similitude. In our problem we have to take into consideration only one axe of similitude and the corresponding touching cycles. As we saw, all using concepts work on the hyperbolic plane, too. A hyperbolic proof of the construction is the following (see on Fig.5). Consider a pair of sought cycles touching the three given ones then the line joining the touching points passes through one of their centre of similitude. Thus this centre of similitude of the sought cycles is the point of power of the given triplet. On the other hand, two of the three given cycles give a touching pair with respect to the sought cycles, hence its centre of similitude has the same power with respect to the sought cycles. So the three centres of similitude are on the axe of power of the sought cycles hence it is also the axe of similitude with respect to the given cycles, say s. Since the pole (with respect to a given cycle) of the line joining the corresponding two touching points is also on s, by the theorem of pole-polar we get that the pole of s lies on the mentioned secant. This proves the construction.

The third step is a trivialization of the second one. (A given cycle is a point now.) Obviously the general construction can be done in this case, too.

The fourth step is again the second one choosing three arbitrary cycles from the four ones if the quadrangles determined by the cycles are inscriptible.

Finally we have to prove that this construction gives the Malfatti's cycles. As we saw the Malfatti's cycles are exist (see Theorem 1). We also know that in an embedding hyperbolic space the examined plane can be inverted to a sphere. The trigonometry of the sphere is absolute implying that the possibility of a construction which can be checked by trigonometric calculations, independent from the fact that the embedding space was hyperbolic space or Euclidean one. Of course, the Steiner's construction just such a like construction, the touching position of circles on the sphere can be checked by spherical trigonometry. So we may assume that the examined sphere is a sphere of the Euclidean space and we can apply Cayley's analytical research (see in [3]) in which he proved that Steiner's construction works on a surface of second order. Hence the above construction produces the required touches. □

2 A note on Malfatti's marble problem

As we can read in the paper [1] there is a long history of Malfatti's marble problem which is: *"Given a triangle find three non-overlapping circles inside it of total maximum area."* Also in this paper we can find an elegant proof for the theorem: *"The greedy arrangement has the largest total area among of non-overlapping circles in a triangle."*

The apropos that we recall to the marble problem the fact that the hint for Remark 2 in [1] is insufficient. For clarity, here we give a short proof for the mentioned statement.

Theorem 3 *On the hyperbolic plane the greedy arrangement has the largest total area among of non-overlapping circles in a triangle.*

Proof:

$$|F_1F_2| \leq \frac{|A_1B_1| + |A_2B_2|}{2} = \frac{(r_1 + r_2) + (R(r_1) + R(r_2))}{2}$$

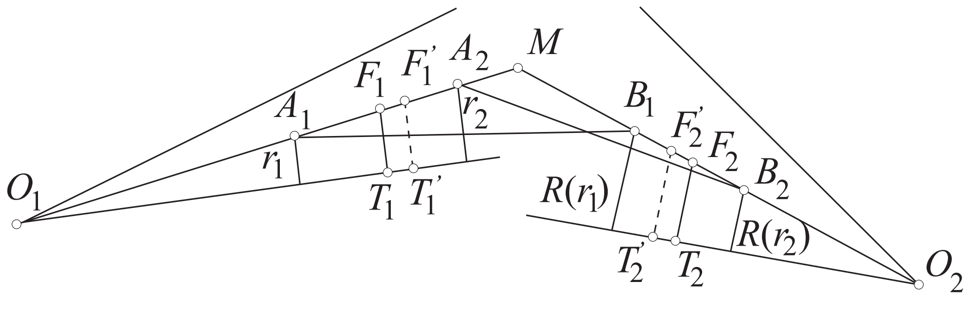

Figure 6: Comparison of two circles rigid arrangements on the hyperbolic plane

On the other hand on the hyperbolic plain we have

$$|F_1T_1| \le \frac{r_1 + r_2}{2} \text{ and } |F_2T_2| \le \frac{R(r_1) + R(r_2)}{2}.$$

Let now F_i' be such that for the segments $F_i'T_i'$ the equalities hold

$$|F_1'T_1'| = \frac{r_1 + r_2}{2} \text{ and } |F_2'T_2'| = \frac{R(r_1) + R(r_2)}{2},$$

respectively. If the bisectors O_1A_1, O_2A_2 intersect in M, then the respective orders of the points on their lines are

$$(O_1A_iM), (O_2B_iM) \text{ for } i = 1, 2.$$

So the orders

$$(F_1F_1'M), (F_2F_2'M)$$

are also valid, hence the inequality

$$|F_1'F_2'| \le |F_1F_2| \le \frac{r_1 + r_2 + R(r_1) + R(r_2)}{2} = |F_1'T_1'| + |F_2'T_2'|$$

is true. This implies that

$$R\left(\frac{r_1 + r_2}{2}\right) = R(F_1'T_1') \le |F_2'T_2'| = \frac{R(r_1) + R(r_2)}{2}$$

showing that R is a convex function. Since the area of a hyperbolic circle as the function of its radius is $2\pi(\cosh r - 1)$ an increasing convex function the proof of Bezdek and at all is valid in the hyperbolic plane (as they stated in [1]). $\square$

We note that this proof is valid also in the case of asymptotic triangle's. The statements on concave curvilinear triangles are also correct. Finally, the statement is true for asymptotic concave curvilinear triangles, too.

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