Malfatti's problem on the hyperbolic plane

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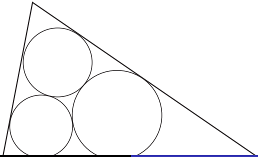
December 7, 2012

Gianfrancesco Malfatti

The problem

Construct three circles into a triangle so that each of them touches the two others from outside moreover touches two sides of the triangle too.



Malfatti's solution.

Malfatti, G., *Memoria sopra un problema sterotomico* Memorie dimatematica e di Fisica della Societa Italiano delle Scienze, **10** (1803), 235-244.

$$\begin{aligned} r_1 &= \frac{r}{2(s-a)}(s+d-r-e-f), \\ r_2 &= \frac{r}{2(s-b)}(s+e-r-d-f), \\ r_3 &= \frac{r}{2(s-c)}(s+f-r-d-e). \end{aligned}$$

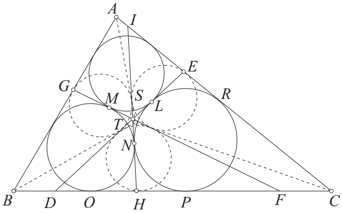
Jacob Steiner

Steiner's extension of Malfatti problem

Determine three circles so that each of them touches the two others, and also touches two of three more given circles.

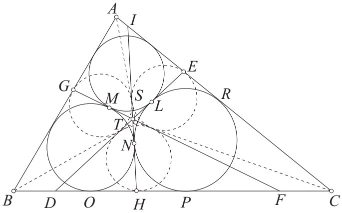
- Steiner's gesammelte werke, (Herausgegeben K. Weierstrass) Berlin, 1881.
- Steiner, J., *Einige geometrische Betrachtungen.* Journal für die reine und angewandte Mathematik **1/2** (1826) 161-184, **1/3** (1826)

Steiner's construction on triangle



1. Step 1: Draw three angle bisectors OA, OB and OC. In the triangles $\triangle_{OAB}$, $\triangle_{OBC}$, $\triangle_{OCA}$ inscribe circles c_C, c_A, c_B, respectively.

Steiner's construction on triangle



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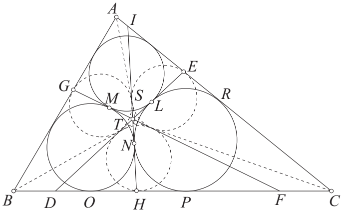
2. Step 2: For each pair of the circles consider the second internal tangents. The latter concur in a point K and cross the sides in points H, I; D, E; and F, G; respectively.

Steiner's construction on triangle



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2. Step 2: For each pair of the circles consider the second internal tangents. The latter concur in a point K and cross the sides in points H, I; D, E; and F, G; respectively.
3. Step 3: The three quadrilaterals $KHCI$, $KGBH$, and $KEAG$ are inscriptible. Their incircles solve Malfatti's problem.

Steiner's extension of Malfatti's problem

1. Construct the circle of inversion $c_{i,j}$, for the given circles c_i and c_j, where the center of inversion is the external centre of similitude of them.
2. Construct circle k_j touching two circles $c_{i,j}, c_{j,k}$ and the given circle c_j.
3. Construct the circle $l_{i,j}$ touching k_i and k_j through the point $P_k = k_k \cap c_k$.
4. Construct the Malfatti's circle m_j as the common touching circle of the four circles $l_{i,j}$, $l_{j,k}$, c_i, c_k.

Arthur Cayley

Cayley's problem

Determine three conic sections so that each of them touches the two others, and also touches two of three more given conic sections.

Cayley's work connected with the problem

- Cayley, A.,*Analytical Researches Connected with Steiner's Extension of Malfatti's Problem* Phil. Trans. of the Roy. Soc. of London, **142** (1852), 253–278.
- Cayley, A.,*On a System of Equations connected with Malfatti's Problem and on another Algebraic System* Camb. and Dubl. Math. Journ. **4** (1849) 270 [79].
- Cayley, A.,*On Schellbach's solution of Malfatti's problem* Quart. J. Pure Appl. Math. **1** (1857) 222226; CMP III (1890) 4447.

Carl Heinrich Schellbach

- Schellbach, C.H. *Eine Lösung der Malfattischen Aufgabe* Journal für die reine und angewandte Mathematik **45/1** (1853) 91-92.
- Schellbach, C.H. *Eine Erweiterung der Malfattischen Aufgabe* Journal für die reine und angewandte Mathematik **45/2** (1853) 186-187.

Schellbach's solution (simplified by Cayley)

$$a+b+c=2s, \quad a-\frac{1}{2}s=l, \quad b-\frac{1}{2}s=m, \quad c-\frac{1}{2}s=n$$

whence $l+m+n=\frac{1}{2}s$, and putting also

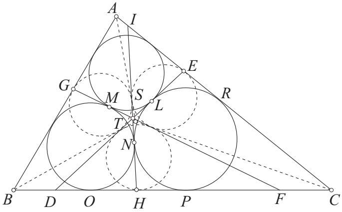
$$\frac{1}{2}s-x=\xi, \quad \frac{1}{2}s-y=\eta, \quad \frac{1}{2}s-z=\zeta,$$

then we have

$$\begin{aligned}
\frac{\cos l\cos\eta\cos\zeta}{\cos\frac{1}{2}s}-\frac{\sin l\sin\eta\sin\zeta}{\sin\frac{1}{2}s} &= 1,\\
\frac{\cos m\cos\zeta\cos\xi}{\cos\frac{1}{2}s}-\frac{\sin m\sin\zeta\sin\xi}{\sin\frac{1}{2}s} &= 1,\\
\frac{\cos n\cos\xi\cos\eta}{\cos\frac{1}{2}s}-\frac{\sin n\sin\xi\sin\eta}{\sin\frac{1}{2}s} &= 1,
\end{aligned}$$

Andrew Searle Hart

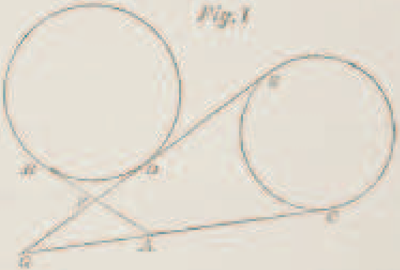
- Hart, A. S., *Geometric investigations of Steiner's construction for Malfatti's problem.* Quart. J. Pure Appl. Math. **1** (1857) 219–221.



Hart's first lemma

Lemma 1. If the sum or difference of the tangents AB, AC drawn from a point A (*Fig.* 1.) to two circles be equal to a common tangent DE of the circles, the point A is on a common tangent.

Fig. 1

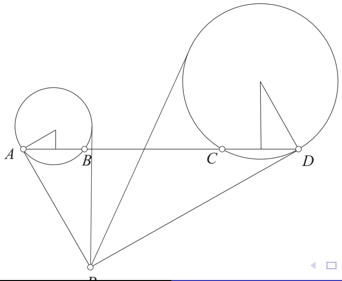


For if not, let $AC = AB + DE = AF + FE$, then $GC = AG + AF + FE$, therefore $FG = AG + AF$, which is absurd. This lemma is obviously true of circles of the

Hart's second lemma

Lemma

If two circles cut off equal parts AB and CD from a given line and if tangents at the extreme points A, D intersect at P, the circles will subtend equal angles at P, and also that if tangents be drawn from each point A and D to the other circle, they will be equal.



The case of triangle

Theorem

The case of the hyperbolic triangle can be solved immediately by the method of Steiner and also by the method of Schellbach.

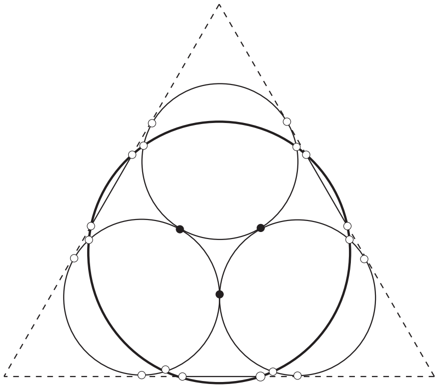
$$
\begin{aligned}
\frac{\cosh l \cosh \eta \cosh \zeta}{\cosh \frac{1}{2}s} + \frac{\sinh l \sinh \eta \sinh \zeta}{\sinh \frac{1}{2}s} &= 1, \\
\frac{\cosh m \cosh \zeta \cosh \xi}{\cosh \frac{1}{2}s} + \frac{\sinh m \sinh \zeta \sinh \xi}{\sinh \frac{1}{2}s} &= 1, \\
\frac{\cosh n \cosh \xi \cosh \eta}{\cosh \frac{1}{2}s} + \frac{\sinh n \sinh \xi \sinh \eta}{\sinh \frac{1}{2}s} &= 1,
\end{aligned}
$$

The problem on cycles

Extension for cycles

Determine three cycles of the hyperbolic plane so that each of them touches the two others moreover touches two of three given cycles.

Counterexample



Definitions

The definitions of hypercycle and touching

The hypercycle is the locus of points of the plane with distances are equal to a constant. (Its domain is convex bounded by a curve with two connected components.) Two cycles are touching if they have a common point with a common tangent line.

Existance

Theorem

For three given cycles have a Malfatti's system of cycles, so there are three cycles that each of them touches the two others moreover touches two from the three given cycles.

The problem of construction

How can we construct the Malfatti's system of cycles of a given system of cycles?

Theorem on construction.

Theorem

Steiner's construction can be done also in the hyperbolic plane. Precisely, for three given non-overlapping cycles can be constructed three other, each of them touches the two others and also touches two of the three given one.

On the proof

1. Construct the cycle of inversion $c_{i,j}$, for the given cycles c_i and c_j, where the center of inversion is the external centre of similitude of them.

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4. Construct the Malfatti's cycle m_j as the common touching cycle of the four cycles $l_{i,j}$, $l_{j,k}$, c_i, c_k.

First step

- Dr. Vörös, C., Analytic Bolyai's geometry, Budapest, 1909 (in hungarian)

Power

The power of a point P with respect to a given cycle is the value $\tanh\frac{1}{2}PA \cdot \tanh\frac{1}{2}PB$, where the points A, B are on the cycle, such that their lines passes through the point P. The axe of power of two cycles is the locus of points having the same powers with respect to the cycles.

Further definitions

Centres of similitude

The centres of similitude of two cycles with non-overlapping interiors are the common points of their pairs of tangents touching direct or inverse, respectively. The first point is the external centre of similitude the second one is the internal centre of similitude.

Further definitions 2

Pole and polar

Among the projective elements of the pole and its polar either one of always real or both of them are at infinity. Thus in a construction the common point of two lines is well-defined, and in every situation it can be joined with another point; for example, if both of them are ideal points they given by their polars (which are constructible real lines) and the required line is the polar of the intersection point of these two real lines. Thus the lengthes in the definition of the inverse can be constructed. This implies that the inverse of a point can be constructed on the hyperbolic plane, too.

Second step

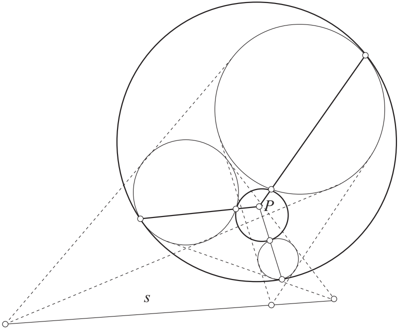

Figure : The construction of Gergonne