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On the number of the minima of N-lattices

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1. Definition of the lattice Λ

Let V be an n-dimensional vector space over the Galois field $GF(2)$. In terms of a basis $\varepsilon_1, \dots, \varepsilon_n$, we may write the elements as $\alpha = \sum \alpha_i \varepsilon_i$ with coordinates α_i which are integers taken modulo 2. The additive group of V, which we shall also denote by V, is the elementary Abelian group of order $N = 2^n$. Subgroups and cosets of dimension r will be denoted generically by V_r and C_r, respectively. In N-dimensional Euclidean space E. S. Barnes and G. E. Wall [1] consider integral vectors $\underline{x} = (x_\alpha)$ with coordinates x_α indexed by the N elements α of V. If W is any subset of V, $[W]$ will denote the characteristic vector $\underline{x}$ defined by

$$x_\alpha = \begin{cases} 1, & \text{if } \alpha \in W \\ 0 & \text{if } \alpha \notin W. \end{cases}$$

Barnes and Wall denoted by Λ the sublattices of Z^N generated by all vectors $2^{[\frac{n-R}{2}]}[C_r]$, where C_r runs over all cosets in V. They proved the following theorems:

Theorem 1.1. *Let $\varepsilon_1, \dots, \varepsilon_n$ be any basis of V. Then a basis of Λ is given by the N vectors $2^{[\frac{n-r}{2}]}[C_r]$, where V_r runs through the subgroups of V which have a subset of $\varepsilon_1, \dots, \varepsilon_n$ as basis. (see* [1] T.3.1*)*

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Theorem 1.2. *Λ is invariant under the following orthogonal transformations:*
i. *the permutation of the coordinates x_α induced by the transformation $\alpha \longmapsto \tau\alpha + \gamma$ of V, where τ is a non-singular matrix over $GF(2)$ and γ is any fixed element of V,*
ii. *the involution*

$$y_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in W \\ -x_\alpha & \text{if } \alpha \notin W, \end{cases}$$

where W is any fixed subgroup of V of dimension $n-1$.

Barnes and Wall defined the rank of a point $\underline{x} \neq 0$ of Λ as the largest integer r $(0 \leq r \leq n)$ for which all coordinates x_α are divisible by $2^{[\frac{r}{2}]}$ and proved

Theorem 1.3. *A point $\underline{x} \neq 0$ of Λ is a minimal vector if and only if it is of rank R, where $n - R + 2\left[\frac{R}{2}\right] = \min(n - r + 2[\frac{r}{2}])$, $0 \leq r \leq n$ and for some coset C_{n-R} of dimension $n - R$*

$$|x_\alpha| = \begin{cases} 2^{[\frac{R}{2}]} & \text{if } \alpha \in C_{n-R} \\ 0 & \text{if } \alpha \notin C_{n-R} \end{cases}$$

(see T.3.2 *and* (5.2) *in* [1]*).*

Theorem 1.4. *For the lattice Λ the number of minima is*

$$s(\Lambda) = 2^{n+1} \sum_{R \text{ odd}} 2^{\binom{n-R}{2}} K_{n,R} \tag{1.1}$$

where

$$K_{n,R} = \frac{(2^n - 1)\cdots(2^{n-R+1} - 1)}{(2^R - 1)\cdots(2-1)}.$$

J. Leech [4] *determined this sum: he showed that*

$$s(\Lambda) = (2+2)(2+2^2)\cdots(2+2^n) \sim l \cdot N^{\frac{1}{2}(\log_2 N + 1)} \tag{1.2}$$

where $l = 4.7684\ldots$ is a constant.

In the special case when $n = 3$ the element $\alpha = \sum_{i \in \mathcal{I}} \varepsilon_i$ of V where $\mathcal{I} \subset \{1,2,3\}$ can be regarded as a vertex $\mathbf{P}$ of the 3-dimensional unit cube, $\overrightarrow{OP} = \sum_{i \in \mathcal{J}} \underline{e}_i$ where $\mathcal{J}$ is the complement of the set $\mathcal{I}$. Arrange these column-vectors $\overrightarrow{OP_i}$ as the columns of a matrix as follows

$$\begin{bmatrix} 1\,1\,1\,1\,0\,0\,0\,0 \\ 1\,1\,0\,0\,1\,1\,0\,0 \\ 1\,0\,1\,0\,1\,0\,1\,0 \end{bmatrix}.$$

Then the rows of this matrix are the characteristic vectors of the three 2-dimensional subspaces of V generated by the vectors $\{\varepsilon_i, \varepsilon_j\}$, where $i \neq j$. The characteristic vectors of the subspaces of dimension 1 (resp. 0) are the collection of vectors formed by component-wise multiplying these vectors two at a time (resp. three at a time):

$$[1\,0\,0\,0\,1\,0\,0\,0] \longleftrightarrow \{0, \varepsilon_3\}$$

$$[1\,0\,1\,0\,0\,0\,0\,0] \longleftrightarrow \{0, \varepsilon_2\}$$

$$[1\,1\,0\,0\,0\,0\,0\,0] \longleftrightarrow \{0, \varepsilon_1\}$$

$$[1\,0\,0\,0\,0\,0\,0\,0] \longleftrightarrow \{0\}.$$

The lattice Λ is generated by the vectors:

$$[V] = [1\,1\,1\,1\,1\,1\,1\,1]$$

$$[V_2] = [1\,1\,1\,1\,0\,0\,0\,0]$$

$$[V_2'] = [1\,1\,0\,0\,1\,1\,0\,0]$$

$$[V_2''] = [1\,0\,1\,0\,1\,0\,1\,0]$$

$$2[V_1] = 2[V_2] * [V_2'] = [2\,2\,0\,0\,0\,0\,0\,0]$$

$$2[V_1'] = 2[V_2] * [V_2''] = [2\,0\,2\,0\,0\,0\,0\,0]$$

$$2[V_1''] = 2[V_2'] * [V_2''] = [2\,0\,0\,0\,2\,0\,0\,0]$$

$$2[V_0] = 2[V_2] * [V_2'] * [V_2''] = [2\,0\,0\,0\,0\,0\,0\,0]$$

We remark that this lattice is the extremal lattice E_8. From Theorem 1.3 we get that the rank of a minimal vector is 1 or 3 and the absolute value of the non-zero coordinates of this vector is 1 or 2, respectively. So the number of the minima in this lattice is equal to

$$s(\Lambda) = 2^4 \sum_{R=1,3} 2^{\binom{3-R}{2}} K_{3,R} = 240.$$

2. On the $2^N - 1$ sublattices of the lattice Λ

Let H be an arbitrary subgroup of V. Let Λ_H denote the set of those vectors of Λ for which $\sum_{\alpha \in H} x_\alpha = 0$. Then Λ_H is a sublattice of Λ. We prove the following basic theorem:

Theorem 2.1. *Let* $\dim H$ *be the dimension of the subspace* H.
1. If $\dim H = \dim G$ *where* H *and* G *are subgroups of* V *then the lattices* Λ_H *and* Λ_G *are congruent.*
2. For every subgroup H *of* V *the minimal value of the sublattice* Λ_H *is equal to the minimal value of the original lattice* Λ.

Proof. 1. This statement follows from Theorem 1.2 part i.

2. Since $\Lambda_H \subseteq \Lambda$, it is enough to prove that every sublattice Λ_H contains a minimal vector of the lattice Λ. We shall prove that there are such minimal vectors of rank R in the lattice Λ which are in the sublattice Λ_H, too. If we assume that $n > \dim H \geq n - R$ then there is such a coset C_{n-R} for which $C_{n-R} \cap H$ is empty. But in this case the vector $2^{[\frac{R}{2}]}[C_{n-R}]$ is a minimal vector in Λ (see [1] (4.1)) and it is in the sublattice Λ_H. If $\dim H = n$, then the vectors of the form $2^{[\frac{R}{2}]}[C_{n-R-1}] - 2^{[\frac{R}{2}]}[C'_{n-R-1}]$ are minimal ones ([1] (4.1)) and they are in the original sublattice $\Lambda_H = \Lambda_V$. So we have to examine the case of $0 \leq \dim H < n - R$. Since the set H is a subgroup of V it can be written in the form $H = H' \cup C'$ where H' is a subgroup of dimension $r = \dim H - 1 < n - R - 1$. Let H'' be a subgroup of dimension $n - R - 1$ for which $H \cap H'' = H'$ and denote by C'' that coset of H'' which contains the coset C'. Then the vector of the form $2^{[\frac{R}{2}]}[H''] - 2^{[\frac{R}{2}]}[C'']$ is minimal and it is in the sublattice Λ_H, so we have proved the statement 2 too. ■

If $n = 3$ then by the first statement of Theorem 2.1 we have four different possibility for the choice of the subspace H. For example if H is the trivial subgroup $< \{0\} >$, then the 7-lattice $\Lambda_{<\{0\}>}$ is generated by the linearly independent lattice-vectors $[V]-[V_2]$, $[V]-[V_2']$, $[V]-[V_2'']$, $2([V]-[V_1])$, $2([V]-[V_1'])$, $2([V]-[V_1''])$, $2([V]-[V_0])$. We shall see in the paragraph 4 that this lattice is the well-known lattice E_7. An interesting consequence of this theorem is the following.

Corollary 2.1. *The minimal vectors of the sublattice Λ_H are minima of the original lattice Λ, too. For this reason, if a sublattice of Λ can be established in the above mentioned way, then the number of minima of it is not greater than the number of minima of the lattice Λ.*

3. On the number of the minima of the lattices Λ_H

In this paragraph let the dimension r of the subgroup H be fixed ($0 \leq r \leq n$). In accordance with the notations of the paper [1] let N_R be the number of minimal vectors of the lattice Λ with support C_{n-R}, where C_{n-R} is a coset of dimension $n-R$. (It can be seen from Theorem 1.2 that this number is the same for all cosets of dimension $n-R$.) Denote by $N_{R,k}$ the number of those minima of Λ_H with support C_{n-R} for which $\dim(C_{n-R}\cap H)=k$. It is easy to see that $N_{R,k}$ is also independent of the choice of the coset C_{n-R}. Finally, let us introduce the number $K_{n,R,k}(H)$ which is the number of such subgroups of dimension $(n-R)$ of V for which $\dim(V_{n-R}\cap H)=k$. We now prove the main result of this paper:

Theorem 3.1. *The number of the minimal vectors of the lattice Λ_H is:*

$$s(\Lambda_H)=\sum_{R\in\mathcal{I}}\sum_{k=\max\{r-R,0\}}^{\min\{r,n-R\}}[(2^R-2^{r-k})N_R+2^{r-k}N_{R,k}]K_{n,R,k}(H), \tag{3.1}$$

where

$$\mathcal{I}=\left\{R \mid n-R+2\left[\frac{R}{2}\right]=\min\left\{n-r+2\left[\frac{r}{2}\right] \mid 0\leq r\leq n\right\}\right\}=$$
$$=\{R \mid 1\leq R\leq n \ R \text{ is odd}\}.$$

Proof. Let V_{n-R} be a subgroup of dimension $(n-R)$ of V for which $\dim(V_{n-R}\cap H)=k$. Then we have:

$$\dim([V_{n-R},H])+\dim(V_{n-R}\cap H)=\dim V_{n-R}+\dim H,$$

and taking into consideration the trivial inequality

$$\dim([V_{n-R}, H]) \leq n,$$

we get the inequality

$$n + k \geq n - R + r \text{ so } k \geq r - R$$

for the parameters k, r and R. But $k \geq 0$ so

$$k \geq \max\{r - R, 0\}.$$

Fixing a coset C_{n-R} of the group V_{n-R} we shall distinguish two cases :

$$\text{a) } C_{n-R} \cap H = 0, \qquad \text{b) } C_{n-R} \cap H \neq 0.$$

It is clear that in case *a)* the number of minima of the lattice Λ_H with support C_{n-R} is equal to N_R and in case *b)* this number is $N_{R,k}$, because if $\alpha \in C_{n-R} \cap H$, we have the identity

$$C_{n-R} \cap H = (V_{n-R} + \alpha) \cap H = (V_{n-R} \cap H) + \alpha, \tag{3.2}$$

so $\dim(C_{n-R} \cap H) = k$, too. From (3.2) we see that the number of those cosets C_{n-R} for which the intersection $C_{n-R} \cap H$ is a coset of dimension k is equal to the index of the subgroup $V_{n-R} \cap H$ related by the subgroup H, so this number is 2^{r-k}. Since the other cosets of V_{n-R} are disjoint from H, we get that the number of those minima which is connected with the fixed subgroup V_{n-R} is equal to

$$(2^R - 2^{r-k})N_R + 2^{r-k}N_{R,k}.$$

On the base of the definition of the number $K_{n,R,k}(H)$ the number of minima of rank R is

$$\sum_{k=\max\{r-R,0\}}^{\min\{r,n-R\}} [(2^R - 2^{r-k})N_R + 2^{r-k}N_{R,k}]K_{n,R,k}(H).$$

After taking the sum for the admissible indices R from Theorem 1.3 we have the formula (3.1). ■

We determine now the number $K_{n,R,k}(H)$. First we prove the following formula:

$$K_{n,R,k}(H) = K_{n-k,R,0}(H/(H \cap V_{n-R}))K_{r,r-k}, \tag{3.3}$$

where V contains $K_{n,R}$ subgroups $V_{n,R}$ of dimension $n-R$. It can be seen that the number of those subgroups V' of V for which $H \cap V' = H \cap V_{n-R}$ (V_{n-R} is a fixed subgroup of dimension $(n-R)$) is equal to the number of those subgroups V'' of $V/H \cap V_{n-R}$ for which the intersection $V'' \cap H/H \cap V_{n-R}$ is trivial. Furthermore the number of the subgroups of dimension k of the group H is equal to $K_{r,r-k}$. (Every k-dimensional subgroup of H is an intersection of the form $H \cap V_{n-R}$.)So we have the formula (3.3). Since

$$H/H \cap V_{n-R} \cong V_{r-k},$$

we get the equality

$$K_{n,R,k}(H) = K_{n-k,R,0}(V_{r-k})K_{r,r-k}.$$

Now assume that we know the number of those subgroups V' of dimension $n-R-1$ for which the intersection $V' \cap V_r$ is trivial. (This is the number $K_{n,R+1,0}(V_r)$.) Hence, there are $2^{R+1}-2^r$ cosets C' of V' in V for which $C' \cap V_r = 0$, and each of these may be combined with the given subgroup V' to form a subgroup $V_{n-R} = V' \cup C'$ of order $n-R$. But, since every such subgroup of order $n-R$ will be formed like that, and each of them the same number $K_{n-R,1} = 2^{n-R}-1$ of times, we get the equality

$$K_{n,R,0}(V_r) = \frac{2^{R+1}-2^r}{2^{n\text{-}R}-1}K_{n,R+1,0}(V_r).$$

Hence

$$K_{n,R,0}(V_r) = \frac{(2^{R+1}-2^r)\cdots(2^n-2^r)}{(2^{n\text{-}R}-1)\cdots(2-1)}K_{n,n,0}(V_r),$$

where $K_{n,n,0}(V_r) = 1$. For this reason we have the formula

$$K_{n,R,k}(H) = \prod_{i=1}^{n-R-k}\frac{(2^{R+i}-2^{r-k})}{(2^i-1)}\prod_{i=1}^{R\text{-}k}\frac{(2^{k+i}-1)}{(2^i-1)}.$$

4. The number of minima of the special lattices $\Lambda_{<\{0\}>}$ and $\Lambda_{V_{n-1}}$

In this paragraph let H denote the trivial subgroup $<\{0\}>$. In this case r is equal to 0 so $K_{n,R,k}(H) = 0$ if $k \geq 1$, and if $k = 0$, then $K_{n,R,0}(H)$ equals to $K_{n,R}$. Taking into account that

$$n - r + 2\left[\frac{r}{2}\right] = \begin{cases} n & \text{if } r \text{ is even} \\ n-1 & \text{if } r \text{ is odd} \end{cases}$$

we get from Theorem 3.1 the formula

$$s(\Lambda_{<\{0\}>}) = \sum_{R\in\mathcal{I}}\sum_{k=0}[(2^R - 2^{r-k})N_R + 2^{r-k}N_{R,k}]K_{n,R,k}(H) =$$

$$(4.1) \qquad = \sum_{R\in\mathcal{I}}[(2^R - 1)N_R + N_{R,0}]K_{n,R} = \sum_{R\in\mathcal{I}}(2^R - 1)N_R K_{n,R}$$

Barnes and Wall proved in [1] that in this case the number N_R equals to $2^{n-R+1+\binom{n-R}{2}}$. (See also [2].) Let $T_{n,n-R}$ be the following number

$$T_{n,n-R} = \begin{cases} N_R K_{n,R} & \text{if } 0 \leq R \leq n \\ 0 & \text{if } n-R < 0 \text{ or } n-R > n, \end{cases}$$

where

$$N_R K_{n,R} = 2^{n-R+1+\binom{n-R}{2}} \frac{(2^n - 2^0)\cdots(2^n - 2^{n-R-1})}{(2^{n-R} - 2^0)\cdots(2^{n-R} - 2^{n-R-1})}.$$

With this notation we find that the recurrence relation

$$T_{n+1,n-R+1} = 2^{n-R+1}(T_{n,n-R} + T_{n,n-R+1})$$

is valid for all $n - R$. From this we get for the generating function

$$g_n(x) = \sum_{n-R} T_{n,n-R}x^{n-R}$$

the relation

$$\begin{aligned} g_{n+1}(x) &= \sum_{n-R} T_{n+1,n-R}x^{n-R+1} = \\ &= \sum_{n-R}(2^{n-R+1}T_{n,n-R} + 2^{n-R+1}T_{n,n-R+1})x^{n-R+1} = \\ &= 2x\sum_{n-R} 2^{n-R}T_{n,n-R}x^{n-R} + \sum_{n-R} 2^{n-R+1}T_{n,n-R+1}x^{n-R+1} = \\ &= (2x+1)g_n(2x). \end{aligned}$$

From this

$$g_n(x) = (1+2x)\cdots(1+2^n x)g_1(2^{n-1}x), \tag{4.2}$$

where

$$g_1(x) = T_{1,1}x + T_{1,0} = 4x + 2.$$

Denote by A the sum of those terms for which the value $n - R$ is even and denote by B the sum of the other terms. Then

$$g_n(1) = A + B \text{ and } g_n(-1) = A - B,$$

so we have:

$$A = \prod_{l=1}^{n}(1+2^l) + \prod_{l=1}^{n}(1-2^l)$$

and

$$B = \prod_{l=1}^{n}(1+2^l) - \prod_{l=1}^{n}(1-2^l).$$

If n is even then the sum $\sum_{n-R} T_{n,n-R}$ is equal to B if n is odd then this sum equals to A. (R is odd.) For this reason, from the formulas (1.1) and (1.2) we get

$$\Lambda_{<\{0\}>} = (2^n - 1)\left[\prod_{l=1}^{n-1}(1+2^l) + \prod_{l=1}^{n-1}|1-2^l|\right] =$$

$$= (2^n-1)2^{\frac{n(n-1)}{2}}\left[\prod_{l=1}^{n-1}(1+\frac{1}{2^l}) + \prod_{l=1}^{n-1}(1-\frac{1}{2^l})\right].$$

It is clear that the products in the bracket are rapidly convergent, so we have the asymptotic form for the number of minima

$$\Lambda_{<\{0\}>} = cN(N+1)^{\frac{1}{2}(\log_2(N+1)-1} = O(N^{\frac{1}{2}(\log_2 N+1)}),$$

where

$$N = 2^n - 1 \text{ and } c = \lim_{n\to\infty}\left[\prod_{l=1}^{n-1}(1+\frac{1}{2^l}) + \prod_{l=1}^{n-1}(1-\frac{1}{2^l})\right].$$

In the case of $n = 3$ and so $N = 7$ we have a lattice with 126 minima. Watson proved in [6] the unicity of the lattice E_7; if there are 126 minima in a 7- lattice, then this lattice is congruent to E_7. From this reason, the lattice constructed above in the case of $n = 3$ is the well-known lattice E_7. Second, we shall examine the case when $H = V_{n\text{-}1}$. From the general formula we get that

$$(4.10)\qquad s(\Lambda_{V_{n\text{-}1}}) = \sum_{\mathcal{I}}[2^{R+R}N_{R,n-R-1}K_{n-1,R}+ \\ +2^{R-1}(N_R + N_{R,n-R})K_{n-1,R-1}].$$

The general formula for the value of $N_{R,k}$ was determined by the author in [3]. For the cases when $k = N - R$ and $k = N - R - 1$ we get the sums

$$N_{R,N\text{-}R} = \sum_{\sigma=1}^{[\frac{N\text{-}R+1}{2}]} 2^{\left(\frac{n\text{-}R\text{-}2\sigma+1}{2}\right)+1}(2^{n\text{-}R}-1)(2^{n\text{-}R\text{-}1}-1)\cdots(2^{n\text{-}R\text{-}2\sigma+2}-1)$$

and

$$N_{R,N\text{-}R\text{-}1} = \sum_{\sigma=1}^{[\frac{N\text{-}R+1}{2}]} 2^{\left(\frac{n\text{-}R\text{-}2\sigma+1}{2}\right)+2\sigma+1}(2^{n\text{-}R\text{-}1}-1)(2^{n\text{-}R\text{-}2}-1)\cdots(2^{n\text{-}R\text{-}2\sigma+1}-1),$$

respectively. If now n is equal to three, then $R = 1$ or $R = 3$. The values $N_{1,2}$, $N_{1,1}$ and $N_{3,0} = N_{3,\text{-}1}$ are equal to 6, 8 and 0, respectively. So we get $s(\Lambda_{V_{3\text{-}1}}) = 126$. Therefore this lattice is also congruent to E_7.

5. New lattice in the dimension $2^n - 2$

From the lattices $\Lambda_{V_{n\text{-}1}}$ and $\Lambda_{<\{0\}>}$ we make a lattice of dimension $2^n - 2$ which has a lot of minima, too. Denote by $\Lambda_{H,G}$ the (2^n-2) -lattice $\Lambda_H \cap \Lambda_G$, where H, G are subgroups of the group V. In the case when $H = V_{n-1}$ and $G =< \{0\} >$ we have the lattice $\Lambda_{V_{n\text{-}1},<\{0\}>}$. The number of minima of this lattice can be determined from the formula (4.3). Regard now the formula (4.3):

$$s(\Lambda_{V_{n\text{-}1}}) = \sum_{R\in\mathcal{I}}\{2^{(n-1)-(n-1-R)}N_{R,n-1-R}K_{n,R,n-1-R}(V_{n\text{-}1})+$$

$$+[(2^R - 2^{(n-1)-(n-R)})N_R + 2^{(n-1)-(n-R)}N_{R,n-R}]K_{n,R,n-R}(V_{n\text{-}1})\} =$$

$$= \sum_{R \in \mathcal{I}} \{2^R N_{R,n-1-R} K_{n,R,n-1-R}(V_{n\text{-}1}) +$$

$$+ [(2^R - 2^{R-1}) N_R + 2^{R-1} N_{R,n-R}] K_{n,R,n-R}(V_{n\text{-}1})\}.$$

It is clear that a minimum of $\Lambda_{V_{n\text{-}1}}$ is in the lattice $\Lambda_{V_{n\text{-}1},<\{0\}>}$ if and only if its support C_{n-R} doesn't contain the zero element of the space V. So we have to omit those minima of $\Lambda_{V_{n\text{-}1}}$ only the support of which is a subspace of V. Therefore the number of the minima of the lattice $\Lambda_{V_{n\text{-}1},<\{0\}>}$ is equal to the sum

$$s(\Lambda_{V_{n\text{-}1},<\{0\}>}) = \sum_{R \in \mathcal{I}} \{(2^R - 1) N_{R,n-1-R} K_{n,R,n-1-R}(V_{n\text{-}1}) +$$

$$+[(2^R - 2^{R-1}) N_R + (2^{R-1} - 1) N_{R,n-R}] K_{n,R,n-R}(V_{n\text{-}1})\} =$$

$$= \sum_{R \in \mathcal{I}} \{(2^R - 1) N_{R,n-R-1} 2^R K_{n-1,R} +$$

$$+ (2^{R-1} N_R + (2^{R-1} - 1) N_{R,n-R}) K_{n-1,R-1}\}.$$

If $n = 3$ the number of minima is

$$s(\Lambda_{V_{n\text{-}1},<\{0\}>}) = 72,$$

so, in view of Watson's theorem [6] $s(\Lambda_{V_2,<\{0\}>})$ is nothing else than E_6.

Remark. It can be proved that $s(\Lambda_{V_{n\text{-}1},<\{0\}>}) \sim O(N^{\frac{1}{2}(\log_2 N+1)})$, too. So these lattices give good lower bounds for the maximum number of minima of the lattices of dimensions $2^n - 1$ and $2^n - 2$, respectively. Since $N^{\frac{1}{2}(\log_2 N+1)} = 2^{\frac{1}{2}[(log_2 N)^2 + log_2 N]}$ we have in these dimensions that

$$c \cdot 2^{\frac{1}{2}[(log_2 N)^2 + log_2 N]} < s_N < 2(2^N - 1),$$

where c is a constant. The second inequality is the famous Voronoi's result. (See for example [5].)

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