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Premanifolds

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Abstract. The tangent hyperplanes of the "manifolds" of this paper equipped a so-called Minkowski product. It is neither symmetric nor bilinear. We give a method to handing such an object as a locally hypersurface of a generalized space-time model and define the main tools of its differential geometry: its fundamental forms, its curvatures and so on. In the case, when the fixed space-time component of the embedding structure is a continuously differentiable semi-inner product space, we get a natural generalization of some important semi-Riemann manifolds as the hyperbolic space, the de Sitter sphere and the light cone of a Minkowski-Lorenz space, respectively.

Keywords: arc-length, curvature, generalized space-time model, generalized Minkowski space, Minkowski product, indefinite-inner product, Riemann manifold, semi-inner product, semi-indefinite inner product, semi-Riemann manifold

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1 Introduction

There is no and we will not give a formal definition of an object calling in this paper *premanifold* . We use this word for a set if it has a manifold-like structure with high freedom in the choosing of the distance function of its tangent hyperplanes. For example we get premanifolds if we investigate the hypersurfaces of a generalized space-time model. The most important types of manifolds as Riemannian, Finslerian or semi-Riemannian can be investigated in this way. The structure of our embedding space was introduced in [6] and in this paper we shall continue our investigations by the build up of differential geometry of hypersurfaces. We will give the pre-version of the usual semi-Riemannian or Finslerian spaces, the hyperbolic space, the de Sitter sphere, the light cone and the unit sphere of the rounding semi-inner product space, respectively. In the case, when the space-like component of the generalized space-time model is a continuously differentiable semi-inner product space then we will get back the known and usable geometrical informations on the corresponding hypersurfaces of a pseudo-Euclidean space, e.g. we will show that a pre-hyperbolic space has constant negative curvature.

1.1 Notation

$\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^n$, S^n: The complex line, the real line, the n-dimensional Euclidean space and the n-dimensional unit sphere, respectively.

$\langle\cdot,\cdot\rangle$: The notion of scalar product and all its suitable generalizations.

$[\cdot,\cdot]^-$: The notion of s.i.p. corresponding to a generalized Minkowski space.

$[\cdot,\cdot]^+$: The notion of Minkowski product of a generalized Minkowski space.

f': The derivative of a real-valued function f with domain in $\mathbb{R}$.

Df: The Frechet derivative of a map between two normed spaces.

f'_e: The directional derivative of a real-valued function of a normed space into the direction of e.

$[x,\cdot]'_z(y)$: The derivative map of an s.i.p. in its second argument, into the direction of z at the point (x,y). See Definition 3.

$\|\cdot\|'_x(y)$, $\|\cdot\|''_{x,z}(y)$: The derivative of the norm in the direction of x at the point y, and the second derivative of the norm in the directions x and z at the point y.

$\Re\{\cdot\}$,$\Im\{\cdot\}$: The real and imaginary part of a complex number, respectively.

T_v: The tangent space of a Minkowskian hypersurface at its point v.

$\mathcal{S},\mathcal{T},\mathcal{L}$: The set of space-like, time-like and light-like vectors respectively.

S,T: The space-like and time-like orthogonal direct components of a generalized Minkowski space, respectively.

$\{e_1,\ldots,e_k,e_{k+1},\ldots,e_n\}$: An Auerbach basis of a generalized Minkowski space with $\{e_1,\ldots,e_k\}\subset S$ and $\{e_{k+1},\ldots,e_n\}\subset T$, respectively. All of the e'_i orthogonal to the another ones with respect to the Minkowski product.

G,G^+: The unit sphere of a generalized space-time model and its upper sheet, respectively.

H,H^+: The sphere of radius i and its upper sheet, respectively.

K, K^+: The unit sphere of the embedding semi-inner product space and its upper sheet, respectively.

L,L^+: The light cone of a generalized space-time model and its upper sheet, respectively.

g**:** The function $g(s) = s + \mathfrak{g}(s)e_n$ with $\mathfrak{g}(s) = \sqrt{-1+[s,s]}$ defines the points of $G := \{s + g(s) | s \in S\}$.

h**:** The function $h(s) = s + \mathfrak{h}(s)e_n$ with $\mathfrak{h}(s) = \sqrt{1+[s,s]}$ defines the points of $H^+ := \{s + h(s) | s \in S\}$.

k**:** The function $k(s) = s + \mathfrak{k}(s)e_n$ with $\mathfrak{k}(s) = \sqrt{1-[s,s]}$ defines the points of $K^+ := \{s + k(s) | s \in S\}$.

l**:** The function $l(s) = s + \mathfrak{l}(s)e_n$ with $\mathfrak{l}(s) = \sqrt{[s,s]}$ defines the points of $L^+ := \{s + l(s) | s \in S\}$.

1.2 History, basic definitions with completion of the preliminaries

A generalization of the inner product and the inner product spaces (briefly i.p spaces) was raised by G. Lumer in [10].

Definition 1 ([10])**.** The *semi-inner product (s.i.p)* on a complex vector space V is a complex function $[x,y] : V \times V \longrightarrow \mathbb{C}$ with the following properties:

s1 : $[x+y,z] = [x,z] + [y,z]$,

s2 : $[\lambda x, y] = \lambda[x,y]$ for every $\lambda \in \mathbb{C}$,

s3 : $[x,x] > 0$ when $x \neq 0$,

s4 : $|[x,y]|^2 \leq [x,x][y,y]$.

A vector space V with a s.i.p. is an *s.i.p. space*.

G. Lumer proved that an s.i.p space is a normed vector space with norm $\|x\| = \sqrt{[x,x]}$ and, on the other hand, that every normed vector space can be represented as an s.i.p. space. In [7] J. R. Giles showed that the following homogeneity property holds:

s5 : $[x, \lambda y] = \bar{\lambda}[x,y]$ for all complex λ.

This can be imposed, and all normed vector spaces can be represented as s.i.p. spaces with this property. Giles also introduced the concept of **continuous s.i.p. space** as an s.i.p. space having the additional property

s6 : For any unit vectors $x, y \in S$, $\Re\{[y, x+\lambda y]\} \to \Re\{[y,x]\}$ for all real $\lambda \to 0$.

The space is uniformly continuous if the above limit is reached uniformly for all points x, y of the unit sphere S. A characterization of the continuous s.i.p. space is based on the differentiability property of the space.

Giles proved in [7] that

Theorem 1 ([7])**.** *An s.i.p. space is a continuous (uniformly continuous) s.i.p. space if and only if the norm is Gâteaux (uniformly Frèchet) differentiable.*

In [6] Á.G.Horváth defined the differentiable s.i.p. as follows:

Definition 2. A *differentiable s.i.p. space* is an continuous s.i.p. space where the s.i.p. has the additional property
s6': For every three vectors x,y,z and real λ

$$[x,\cdot]'_z(y) := \lim_{\lambda\to 0} \frac{\Re\{[x,y+\lambda z]\} - \Re\{[x,y]\}}{\lambda}$$

does exist. We say that the s.i.p. space is *continuously differentiable*, if the above limit, as a function of y, is continuous.

First we note that the equality $\Im\{[x,y]\} = \Re\{[-ix,y]\}$ together with the above property guarantees the existence and continuity of the complex limit:

$$\lim_{\lambda\to 0} \frac{[x,y+\lambda z] - [x,y]}{\lambda}.$$

The following theorem was mentioned without proof in [6]:

Theorem 2 ([6])**.** *An s.i.p. space is a (continuously) differentiable s.i.p. space if and only if the norm is two times (continuously) Gâteaux differentiable. The connection between the derivatives is*

$$\|y\|(\|\cdot\|''_{x,z}(y)) = [x,\cdot]'_z(y) - \frac{\Re[x,y]\Re[z,y]}{\|y\|^2}.$$

Since the present paper often use this statement, we give a proof for it. We need the following useful lemma going back, with different notation, to McShane [15] or Lumer [11].

Lemma 1 ([11])**.** *If E is any s.i.p. space with $x,y\in E$, then*

$$\|y\|(\|\cdot\|'_x(y))^- \le \Re\{[x,y]\} \le \|y\|(\|\cdot\|'_x(y))^+$$

holds, where $(\|\cdot\|'_x(y))^-$ and $(\|\cdot\|'_x(y))^+$ denotes the left hand and right hand derivatives with respect to the real variable λ. In particular, if the norm is differentiable, then

$$[x,y] = \|y\|\{(\|\cdot\|'_x(y)) + \|\cdot\|'_{-ix}(y)\}.$$

Now we prove Theorem 2.

Proof. To determine the derivative of the s.i.p., assume that the norm is twice differentiable. Then, by Lemma 1 above, we have

$$\frac{\Re\{[x,y+\lambda z]\} - \Re\{[x,y]\}}{\lambda} = \frac{\|y+\lambda z\|(\|\cdot\|'_x(y+\lambda z)) - \|y\|(\|\cdot\|'_x(y))}{\lambda} =$$

$$= \frac{\|y\|\|y+\lambda z\|(\|\cdot\|'_x(y+\lambda z)) - \|y\|^2(\|\cdot\|'_x(y))}{\lambda\|y\|} \geq$$

$$\geq \frac{|[y+\lambda z, y]|(\|\cdot\|'_x(y+\lambda z)) - \|y\|^2(\|\cdot\|'_x(y))}{\lambda\|y\|},$$

where we have assumed that the sign of $\frac{\|\cdot\|'_x(y+\lambda z)}{\lambda}$ is positive. Since the derivative of the norm is continuous, this follows from the assumption that $\frac{\|\cdot\|'_x(y)}{\lambda}$ is positive. Considering the latter condition, we get

$$\frac{\Re\{[x, y+\lambda z]\} - \Re\{[x,y]\}}{\lambda} \geq \|y\|^2 \frac{\|\cdot\|'_x(y+\lambda z) - (\|\cdot\|'_x(y))}{\lambda\|y\|} + \frac{\Re[z,y]}{\|y\|}\|\cdot\|'_x(y+\lambda z).$$

On the other hand,

$$\frac{\|y+\lambda z\|(\|\cdot\|'_x(y+\lambda z)) - \|y\|(\|\cdot\|'_x(y))}{\lambda} \leq$$

$$\leq \frac{\|y+\lambda z\|^2(\|\cdot\|'_x(y+\lambda z)) - |[y, y+\lambda z]|(\|\cdot\|'_x(y))}{\lambda\|y+\lambda z\|} =$$

$$= \frac{\|y+\lambda z\|^2(\|\cdot\|'_x(y+\lambda z)) - (\|\cdot\|'_x(y))}{\lambda\|y+\lambda z\|} + \lambda\Re[z, y+\lambda z]\frac{(\|\cdot\|'_x(y))}{\lambda\|y+\lambda z\|}.$$

Analogously, if $\frac{\|\cdot\|'_x(y)}{\lambda}$ is negative, then both of the above inequalities are reversed, and we get that the limit $\lim_{\lambda\mapsto 0} \frac{\Re\{[x,y+\lambda z]\} - \Re\{[x,y]\}}{\lambda}$ exists, and equals to

$$\|y\|(\|\cdot\|''_{x,z}(y)) + \frac{\Re[x,y]\Re[z,y]}{\|y\|^2}.$$

Here we note that also in the case $\frac{\|\cdot\|'_x(y)}{\lambda} = 0$ there exists a neighborhood in which the sign of the function $\frac{\|\cdot\|'_x(y+\lambda z)}{\lambda}$ is constant. Thus we, need not investigate this case by itself. Conversely, consider the fraction

$$\|y\|\frac{\|\cdot\|'_x(y+\lambda z) - (\|\cdot\|'_x(y))}{\lambda}.$$

We assume now that the s.i.p. is differentiable, implying that it is continuous, too. The norm is differentiable by the theorem of Giles. Using again Lemma 1 and assuming that $\frac{\Re[x,y]}{\lambda} > 0$, we have

$$\|y\|\frac{\|\cdot\|'_x(y+\lambda z) - (\|\cdot\|'_x(y))}{\lambda} = \frac{\Re[x, y+\lambda z]\|y\| - \Re[x,y]\|y+\lambda z\|}{\lambda\|y+\lambda z\|} =$$

$$= \frac{\Re[x,y+\lambda z]\|y\|^2 - \Re[x,y]\|y+\lambda z\|\|y\|}{\lambda\|y\|\|y+\lambda z\|} \leq$$

$$\frac{\Re[x,y+\lambda z]\|y\|^2 - \Re[x,y]|[y+\lambda z,y]|}{\lambda\|y\|\|y+\lambda z\|} =$$

$$= \frac{\Re\{[x,y+\lambda z]\} - \Re\{[x,y]\}}{\lambda}\frac{\|y\|}{\|y+\lambda z\|} - \frac{\Re[x,y]\Re[z,y]}{\|y\|\|y+\lambda z\|}.$$

On the other hand, using the continuity of the s.i.p. and our assumption $\frac{\Re[x,y]}{\lambda} > 0$ similarly as above, we also get an inequality:

$$\|y\|\frac{\|\cdot\|'_x(y+\lambda z) - (\|\cdot\|'_x(y))}{\lambda} \geq$$

$$\geq \frac{\Re\{[x,y+\lambda z]\} - \Re\{[x,y]\}}{\lambda} - \frac{\Re[x,y+\lambda z]\Re[z,y+\lambda z]}{\|y+\lambda z\|^2}.$$

If we reverse the assumption of signs, then the direction of the inequalities will also change. Again a limit argument shows that the first differential function is differentiable, and the connection between the two derivatives is

$$\|y\|(\|\cdot\|''_{x,z}(y)) = [x,\cdot]'_z(y) - \frac{\Re[x,y]\Re[z,y]}{\|y\|^2}.$$

QED

From geometric point of view we know that if C is a 0-symmetric, bounded, convex body in the Euclidean n-space $\mathbb{R}^n$ (with fixed origin O), then it defines a norm whose unit ball is C itself (see [9]). Such a space is called (Minkowski or) normed linear space. Basic results on such spaces are collected in the surveys [13], [14], and [12]. In fact, the norm is a continuous function which is considered (in geometric terminology, as in [9]) as a gauge function. Combining this with the result of Lumer and Giles we get that a normed linear space can be represented as an s.i.p space. The metric of such a space (called Minkowski metric), i.e., the distance of two points induced by this norm, is invariant with respect to translations.

Another concept of Minkowski space was also raised by H. Minkowski and used in Theoretical Physics and Differential Geometry, based on the concept of indefinite inner product. (See, e.g., [8].)

Definition 3 ([8])**.** The *indefinite inner product (i.i.p.)* on a complex vector space V is a complex function $[x,y] : V \times V \longrightarrow \mathbb{C}$ with the following properties:

i1 : $[x+y,z] = [x,z] + [y,z]$,

i2 : $[\lambda x,y] = \lambda[x,y]$ for every $\lambda \in \mathbb{C}$,

i3 : $[x,y]=\overline{[y,x]}$ for every $x,y\in V$,

i4 : $[x,y]=0$ for every $y\in V$ then $x=0$.

A vector space V with an i.i.p. is called an *indefinite inner product space.*

The standard mathematical model of space-time is a four dimensional i.i.p. space with signature $(+,+,+,-)$, also called Minkowski space in the literature. Thus we have a well known homonymism with the notion of Minkowski space!

In [6] the concepts of s.i.p. and i.i.p. was combined in the following one:

Definition 4 ([6])**.** The *semi-indefinite inner product (s.i.i.p.)* on a complex vector space V is a complex function $[x,y]:V\times V\longrightarrow\mathbb{C}$ with the following properties:

1 $[x+y,z]=[x,z]+[y,z]$ (additivity in the first argument),

2 $[\lambda x,y]=\lambda[x,y]$ for every $\lambda\in\mathbb{C}$ (homogeneity in the first argument),

3 $[x,\lambda y]=\overline{\lambda}[x,y]$ for every $\lambda\in\mathbb{C}$ (homogeneity in the second argument),

4 $[x,x]\in\mathbb{R}$ for every $x\in V$ (the corresponding quadratic form is real-valued),

5 if either $[x,y]=0$ for every $y\in V$ or $[y,x]=0$ for all $y\in V$, then $x=0$ (nondegeneracy),

6 $|[x,y]|^2\leq[x,x][y,y]$ holds on non-positive and non-negative subspaces of V, respectively (the Cauchy-Schwartz inequality is valid on positive and negative subspaces, respectively).

A vector space V with an s.i.i.p. is called an *s.i.i.p. space.*

It was conclude that an s.i.i.p. space is a homogeneous s.i.p. space if and only if the property **s3** holds, too. An s.i.i.p. space is an i.i.p. space if and only if the s.i.i.p. is an antisymmetric product. It is clear that both of the classical "Minkowski spaces" can be represented either by an s.i.p or by an i.i.p., so automatically they can also be represented as an s.i.i.p. space.

The following fundamental lemma was proved in [6]:

Lemma 2 ([6])**.** *Let* $(S,[\cdot,\cdot]_S)$ *and* $(T,-[\cdot,\cdot]_T)$ *be two s.i.p. spaces. Then the function* $[\cdot,\cdot]^-:(S+T)\times(S+T)\longrightarrow\mathbb{C}$ *defined by*

$$[s_1+t_1,s_2+t_2]^-:=[s_1,s_2]-[t_1,t_2]$$

is an s.i.p. on the vector space $S+T$.

It is possible that the s.i.i.p. space V is a direct sum of its two subspaces where one of them is positive and the other one is negative. Then there are two more structures on V, an s.i.p. structure (by Lemma 2) and a natural third one, which was called by Minkowskian structure.

Definition 5 ([6])**.** Let $(V, [\cdot, \cdot])$ be an s.i.i.p. space. Let $S, T \leq V$ be positive and negative subspaces, where T is a direct complement of S with respect to V. Define a product on V by the equality

$$[u, v]^+ = [s_1 + t_1, s_2 + t_2]^+ = [s_1, s_2] + [t_1, t_2]$$

, where $s_i \in S$ and $t_i \in T$, respectively. Then we say that the pair $(V, [\cdot, \cdot]^+)$ is a *generalized Minkowski space with Minkowski product* $[\cdot, \cdot]^+$. We also say that V is a *real generalized Minkowski space* if it is a real vector space and the s.i.i.p. is a real valued function.

The Minkowski product defined by the above equality satisfies properties **1-5** of the s.i.i.p.. But in general, property **6** does not hold. (See an example in [6].)

If now we consider the theory of s.i.p in the sense of Lumer-Giles, we have a natural concept of orthogonality. For the unified terminology we change the original notation of Giles and use instead

Definition 6 ([7])**.** The vector *x is orthogonal to the vector y* if $[x, y] = 0$.

Since s.i.p. is neither antisymmetric in the complex case nor symmetric in the real one, this definition of orthogonality is not symmetric in general.

Let $(V, [\cdot, \cdot])$ be an s.i.i.p. space, where V is a complex (real) vector space. The orthogonality of such a space can be defined an analogous way to the definition of the orthogonality of an i.i.p. or s.i.p. space.

Definition 7 ([6])**.** The vector *v is orthogonal to the vector u* if $[v, u] = 0$. If U is a subspace of V, define the orthogonal companion of U in V by

$$U^\perp = \{v \in V | [v, u] = 0 \text{ for all } u \in U\}.$$

We note that, as in the i.i.p. case, the orthogonal companion is always a subspace of V. The following theorem is important one:

Theorem 3 ([6])**.** *Let V be an n-dimensional s.i.i.p. space. Then the orthogonal companion of a non-neutral vector u is a subspace having a direct complement of the linear hull of u in V. The orthogonal companion of a neutral vector v is a degenerate subspace of dimension $n - 1$ containing v.*

Observe that the statements of Theorem 3 are true for any concepts of product satisfying properties **1-5**. As we saw, the Minkowski product is also such a product.

Let V be a generalized Minkowski space. Then we call a vector **space-like, light-like, or time-like** if its scalar square is positive, zero, or negative, respectively. Let $\mathcal{S}, \mathcal{L}$ and $\mathcal{T}$ denote the sets of the space-like, light-like, and time-like vectors, respectively.

In a finite dimensional, real generalized Minkowski space with $\dim T = 1$ these sets of vectors can be characterized in a geometric way. Such a space is called by a **generalized space-time model**. In this case $\mathcal{T}$ is a union of its two parts, namely

$$\mathcal{T} = \mathcal{T}^+ \cup \mathcal{T}^-,$$

where

$$\mathcal{T}^+ = \{s + t \in \mathcal{T} | \text{ where } t = \lambda e_n \text{ for } \lambda \geq 0\} \text{ and}$$

$$\mathcal{T}^- = \{s + t \in \mathcal{T} | \text{ where } t = \lambda e_n \text{ for } \lambda \leq 0\}.$$

It has special interest, the imaginary unit sphere of a finite dimensional, real, generalized space-time model. (See Def.8 in [6].) It was given a metric on it, and thus got a structure similar to the hyperboloid model of the hyperbolic space embedded in a space-time model. In the case when the space S is an Euclidean space this hypersurface is a model of the n-dimensional hyperbolic space thus it is such-like generalization of it.

In [6] was proved the following theorem:

Theorem 4 ([6])**.** *Let V be a generalized space-time model. Then $\mathcal{T}$ is an open double cone with boundary $\mathcal{L}$, and the positive part $\mathcal{T}^+$ (resp. negative part $\mathcal{T}^-$) of $\mathcal{T}$ is convex.*

We note that if $\dim T > 1$ or the space is complex, then the set of time-like vectors cannot be divided into two convex components. So we have to consider that our space is a generalized space-time model.

Definition 8 ([6])**.** The set $H := \{v \in V | [v, v]^+ = -1\}$ is called the *imaginary unit sphere* of the generalized space-time model.

With respect to the embedding real normed linear space $(V, [\cdot, \cdot]^-)$ (see Lemma 2) H is a generalized two sheets hyperboloid corresponding to the two pieces of $\mathcal{T}$, respectively. Usually we deal only with one sheet of the hyperboloid, or identify the two sheets projectively. In this case the space-time component $s \in S$ of v determines uniquely the time-like one, namely $t \in T$. Let $v \in H$ be arbitrary. Let T_v denote the set $v + v^\perp$, where $v^\perp$ is the orthogonal complement of v with respect to the s.i.i.p., thus a subspace.

The set T_v corresponding to the point $v = s + t \in H$ is a positive, (n-1)-dimensional affine subspace of the generalized Minkowski space $(V, [\cdot, \cdot]^+)$. Each of the affine spaces T_v of H can be considered as a semi-metric space, where the semi-metric arises from the Minkowski product restricted to this positive subspace of V. We recall that the Minkowski product does not satisfy the Cauchy-Schwartz inequality. Thus the corresponding distance function does not satisfy the triangle inequality. Such a distance function is called in the literature semi-metric (see [17]). Thus, if the set H is sufficiently smooth, then a metric can be adopted for it, which arises from the restriction of the Minkowski product to the tangent spaces of H. Let us discuss this more precisely.

The directional derivatives of a function $\mathfrak{f} : S \longmapsto \mathbb{R}$ with respect to a unit vector e of S can be defined in the usual way, by the existence of the limits for real λ:

$$\mathfrak{f}_e'(s) = \lim_{\lambda \mapsto 0} \frac{\mathfrak{f}(s + \lambda e) - \mathfrak{f}(s)}{\lambda}.$$

Let now the generalized Minkowski space be a generalized space-time model, and consider a mapping f on S to $\mathbb{R}$. Denote by e_n a basis vector of T with length i as in the definition of $\mathcal{T}^+$ before Theorem 4. The set of points

$$F := \{(s + \mathfrak{f}(s)e_n) \in V \text{ for } s \in S\}$$

is a so-called **hypersurface** of this space. Tangent vectors of a hypersurface F in a point p are the vectors associated to the directional derivatives of the coordinate functions in the usual way. So u is a **tangent vector** of the hypersurface F in its point $v = (s + \mathfrak{f}(s)e_n)$, if it is of the form

$$u = \alpha(e + \mathfrak{f}_e'(s)e_n) \text{ for real } \alpha \text{ and unit vector } e \in S.$$

The linear hull of the tangent vectors translated into the point s is the tangent space of F in s. If the tangent space has dimension $n-1$, we call it **tangent hyperplane**.

We now reformulate Lemma 3 of [6]:

Lemma 3 (See also in [6] as Lemma 3)**.** *Let S be a continuous (complex) s.i.p. space. (So the property* **s6** *holds.) Then the directional derivatives of the real valued function*

$$\mathfrak{h} : s \longmapsto \sqrt{1 + [s, s]}$$

are

$$\mathfrak{h}_e'(s) = \frac{\Re[e, s]}{\sqrt{1 + [s, s]}}.$$

The following theorem is a consequence of this result.

Theorem 5. *Let assume that the s.i.p. $[\cdot, \cdot]$ of S is differentiable. (So the property* **s6'** *holds.) Then for every two vectors x and z in S we have:*

$$[x, \cdot]_z'(x) = 2\Re[z, x] - [z, x],$$

and

$$\|\cdot\|''_{x,z}(x) = \frac{\Re[z,x] - [z,x]}{\|x\|}.$$

If we also assume that the s.i.p. is continuously differentiable (so the norm is a C^2 function), then we also have

$$[x,\cdot]'_x(y) = [x,x],$$

and thus

$$\|\cdot\|''_{x,x}(y) = \|x\|^2 - \frac{\Re[x,y]^2}{\|y\|^2}.$$

Proof. Since

$$\frac{1}{\lambda}\left([x+\lambda z, x+\lambda z] - [x,x]\right) = \frac{1}{\lambda}\left([x, x+\lambda z] - [x,x]\right) + \frac{1}{\lambda}[\lambda z, x+\lambda z],$$

if λ tends to zero then the right hand side tends to $[x,\cdot]'_z(x) + [z,x]$. The left hand side is equal to

$$\frac{\left(\sqrt{1+[x+\lambda z, x+\lambda z]} - \sqrt{1+[x,x]}\right)\left(\sqrt{1+[x+\lambda z, x+\lambda z]} + \sqrt{1+[x,x]}\right)}{\lambda}$$

thus by Lemma 3 it tends to

$$\frac{\Re[z,x]}{\sqrt{1+[x,x]}} 2\sqrt{1+[x,x]}.$$

This implies the first equality

$$[x,\cdot]'_z(x) = 2\Re[z,x] - [z,x].$$

Using Theorem 2 in [6] we also get that

$$\|x\|(\|\cdot\|''_{x,z}(x)) = [x,\cdot]'_z(x) - \frac{\Re[x,x]\Re[z,x]}{\|x\|^2},$$

proving the second statement, too.

If we assume that the norm is a C^2 function of its argument then the first derivative of the second argument of the product is a continuous function of its arguments. So the function $A(y) : S \longrightarrow \mathbb{R}$ defined by the formula

$$A(y) = [x,\cdot]'_x(y) = \lim_{\lambda \mapsto 0} \frac{1}{\lambda}\left([x, y+\lambda x] - [x,y]\right)$$

continuous in $y = 0$. On the other hand for non-zero $t \in \mathbb{R}$ we use the notation $t\lambda' = \lambda$ and we get that

$$A(ty) = \lim_{\lambda \mapsto 0} \frac{1}{\lambda} \left([x, ty + \lambda x] - [x, y]\right) = \lim_{\lambda' \mapsto 0} \frac{t}{t\lambda'} \left([x, y + \lambda' x] - [x, y]\right) = A(y).$$

From this we can see immediately that

$$[x, \cdot]'_x(y) = A(y) = A(0) = [x, x]$$

holds for every y. Applying again the formula connected the derivative of the product and the norm we get the last statement of the theorem, too. QED

A connection between the differentiability properties and the orthogonality one was given also in [6]. The tangent vectors of the hypersurface H^+ in its point $v = s + \sqrt{1 + [s, s]}e_n$ form the orthogonal complement $v^\perp$ of v with respect to the Minkowski product. We now recall the definition of the Minkowski-Finsler space.

Definition 9 ([6])**.** Let F be a hypersurface of a generalized space-time model for which the following properties hold:

i, In every point v of F, there is a (unique) tangent hyperplane T_v for which the restriction of the Minkowski product $[\cdot, \cdot]_v^+$ is positive, and

ii, the function $ds_v^2 := [\cdot, \cdot]_v^+ : F \times T_v \times T_v \longrightarrow \mathbb{R}^+$

$$ds_v^2 : (v, u_1, u_2) \longmapsto [u_1, u_2]_v^+$$

varies differentiably with the vectors $v \in F$ and $u_1, u_2 \in T_v$.

Then we say that the pair (F, ds^2) *is a Minkowski-Finsler space* with semi-metric ds^2 embedding into the generalized space-time model V.

One of the important results on the imaginary unit sphere is the following:

Theorem 6 ([6])**.** *Let V be a generalized space-time model. Let S be a continuously differentiable s.i.p. space, then (H^+, ds^2) is a Minkowski-Finsler space.*

In present paper we will prefer the name "pre-hyperbolic space" for this structure.

2 Hypersurfaces as premanifolds

2.1 Convexity, fundamental forms

Let S be a continuously differentiable s.i.p. space, V be a generalized space-time model and F a hypersurface. We shall say that F is a **space-like hypersurface** if the Minkowski product is positive on its all tangent hyperplanes.

The objects of our examination are the convexity, the fundamental forms, the concepts of curvature, the arc-length and the geodesics. In this section we define these concepts with respect to a generalized space-time model. With respect to a pseudo-Euclidean or a semi-Riemann space these definitions can be found e.g. in the notes [4] and the book [5], respectively.

Definition 10 ([4])**.** We say that a hypersurface is *convex* if it lies on one side of its each tangent hyperplanes. It is *strictly convex* if it is convex and its tangent hyperplanes contain precisely one points of the hypersurface, respectively.

In an Euclidean space the **first fundamental form** is a positive definite quadratic form induced by the inner product of the tangent space.

In our generalized space-time model the first fundamental form is giving by the scalar square of the tangent vectors with respect to the Minkowski product restricted to the tangent hyperplane. If we have a map $f : S \longrightarrow V$ then it can be decomposed to a sum of its space-like and time-like components. We have

$$f = f_S + f_T$$

where $f_S : S \longrightarrow S$ and $f_T : S \longrightarrow T$, respectively. With respect to the embedding normed space we can compute its Frechet derivative by the low

$$Df = \begin{bmatrix} Df_S \\ Df_T \end{bmatrix}$$

implying that

$$Df(s) = Df_S(s) + Df_T(s).$$

Introduce the following notation

$$[f_1(c(t)), \cdot]^{+\prime}{}_{D(f_2 \circ c)(t)}(f_2(c(t))) :=$$

$$:= \left([(f_1)_S(c(t)), \cdot]'_{D((f_2)_S \circ c)(t)}((f_2)_S(c(t))) - (f_1)_T(c(t))((f_2)_T \circ c)'(t) \right).$$

Now we state:

Lemma 4. *If $f_1, f_2 : S \longrightarrow V$ are two C^2 maps and $c : \mathbb{R} \longrightarrow S$ is an arbitrary C^2 curve then*

$$([(f_1 \circ c)(t)), (f_2 \circ c)(t))]^+)' =$$

$$= [D(f_1 \circ c)(t), (f_2 \circ c)(t))]^+ + [(f_1 \circ c)(t)), \cdot]^{+\prime}{}_{D(f_2 \circ c)(t)}((f_2 \circ c)(t)).$$

Proof. By definition

$$([f_1 \circ c, f_2 \circ c)]^+)'|_t := \lim_{\lambda \to 0} \frac{1}{\lambda} \left([f_1(c(t+\lambda)), f_2(c(t+\lambda))]^+ - [f_1(c(t)), f_2(c(t))]^+\right)$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda} \left([(f_1)_S(c(t+\lambda)), (f_2)_S(c(t+\lambda))] - [(f_1)_S(c(t)), (f_2)_S(c(t))]\right) +$$

$$+ \lim_{\lambda \to 0} \frac{1}{\lambda} \left([(f_1)_T(c(t+\lambda)), (f_2)_T(c(t+\lambda))] - [(f_1)_T(c(t)), (f_2)_T(c(t))]\right).$$

We prove that the first part is

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \left([(f_1)_S(c(t+\lambda)) - (f_1)_S(c(t)), (f_2)_S(c(t+\lambda))] + \right.$$

$$\left. +[(f_1)_S(c(t)), (f_2)_S(c(t+\lambda))] - [(f_1)_S(c(t)), (f_2)_S(c(t))]\right) =$$

$$= [D((f_1)_S \circ c)|_t, (f_2)_S(c(t))] + [(f_1)_S(c(t)), \cdot]'_{D((f_2)_S \circ c)(t)}((f_2)_S(c(t))).$$

To this take a coordinate system $\{e_1, \cdots, e_{n-1}\}$ in S and consider the coordinate-wise representation

$$(f_2)_S \circ c = \sum_{i=1}^{n-1} ((f_2)_S \circ c)_i e_i$$

of $(f_2)_S \circ c$. Using Taylor's theorem for the coordinate functions we have that there are real parameters $t_i \in (t, t+\lambda)$, for which

$$((f_2)_S \circ c)(t+\lambda) = ((f_2)_S \circ c)(t) + \lambda D((f_2)_S \circ c)(t) + \frac{1}{2}\lambda^2 \sum_{i=1}^{n-1} ((f_2)_S \circ c)''_i(t_i) e_i.$$

Thus we can get

$$[(f_1)_S(c(t)), (f_2)_S(c(t+\lambda))] - [(f_1)_S(c(t)), (f_2)_S(c(t))] =$$

$$= [(f_1)_S(c(t)), (f_2)_S(c(t)) + D((f_2)_S \circ c)(t)\lambda +$$

$$+ \frac{1}{2}\lambda^2 \sum_{i=1}^{n-1} ((f_2)_S \circ c)''_i(t_i) e_i] - [(f_1)_S(c(t)), (f_2)_S(c(t))] =$$

$$= \left([(f_1)_S(c(t)), (f_2)_S(c(t)) + D((f_2)_S \circ c)(t)\lambda] - [(f_1)_S(c(t)), (f_2)_S(c(t))]\right) +$$

$$+ [(f_1)_S(c(t)), (f_2)_S(c(t)) + D((f_2)_S \circ c)(t)\lambda + \frac{1}{2}\lambda^2 \sum_{i=1}^{n-1} ((f_2)_S \circ c)''_i(t_i) e_i] -$$

$$- [(f_1)_S(c(t)), (f_2)_S(c(t)) + D((f_2)_S \circ c)(t)\lambda].$$

In the second argument of this product, the Lipschwitz condition holds with a real constant K for enough small λ's, so we have that the absolute value of the substraction of the last two terms is less or equal to

$$K\left[(f_1)_S(c(t)),\frac{1}{2}\lambda^2\sum_{i=1}^{n-1}((f_2)_S\circ c)_i''(t_i)e_i\right].$$

Applying now the limit procedure at $\lambda\to 0$ we get the required equality.

In the second part $(f_1)_T$ and $(f_2)_T$ are real-real functions, respectively so

$$\lim_{\lambda\to 0}\frac{1}{\lambda}([(f_1)_T(c(t+\lambda)),(f_2)_T(c(t+\lambda))]-[(f_1)_T(c(t)),(f_2)_T(c(t))])=$$

$$=-((f_1)_T\circ c)'(t)(f_2)_T(c(t))-(f_1)_T(c(t))((f_2)_T\circ c)'(t).$$

Hence we have

$$([(f_1\circ c)(t)),(f_2\circ c)(t))]^+)'=$$

$$=[D((f_1)_S\circ c)(t),((f_2)_S\circ c)(t))]+[(f_1)_S(c(t)),\cdot]'_{D((f_2)_S\circ c)(t)}(((f_2)_S\circ c)(t)))-$$

$$-((f_1)_T\circ c)'(t)(f_2)_T(c(t))-(f_1)_T(c(t))((f_2)_T\circ c)'(t)=$$

$$=[D(f_1\circ c)(t),f_2(c(t))]^+ +$$

$$+\left([(f_1)_S(c(t)),\cdot]'_{D((f_2)_S\circ c)(t)}((f_2)_S(c(t)))-(f_1)_T(c(t))((f_2)_T\circ c)'(t)\right),$$

and the statement is proved. QED

Let F be a hypersurface defined by the function $f: S\longrightarrow V$. Here $f(s)=s+\mathfrak{f}(s)e_n$ denotes the point of F. The curve $c:\mathbb{R}\longrightarrow S$ define a curve on F. We assume that c is a C^2-curve. The following definition is very important one.

Definition 11. The *first fundamental form* in a point $(f(c(t))$ of the hypersurface F is the product

$$\mathrm{I}_{f(c(t)}:=[D(f\circ c)(t),D(f\circ c)(t)]^+.$$

The variable of it is a tangent vector, a tangent vector of a variable curve c lying on F through the point $(f(c(t))$. We can see that the first fundamental form is homogeneous of the second order but (in general) it has no a bilinear representation. In fact, by the definition of f, (if $\{e_i: i=1\cdots n-1\}$ is a basis in S) the computation

$$\mathrm{I}_{f(c(t))}=[\dot{c}(t)+(\mathfrak{f}\circ c)'(t)e_n,\dot{c}(t)+(\mathfrak{f}\circ c)'(t)e_n]^+=$$

$$=[\dot{c}(t),\dot{c}(t)]-[(\mathfrak{f}\circ c)'(t)]^2=[\dot{c}(t),\dot{c}(t)]-\sum_{i,j=1}^{n-1}\dot{c}_i(t)\dot{c}_j(t)\mathfrak{f}'_{e_i}(c(t))\mathfrak{f}'_{e_j}(c(t))=$$

$$= [\dot{c}(t), \dot{c}(t)] - \dot{c}(t)^T \left[\mathfrak{f}'_{e_i}(c(t))\mathfrak{f}'_{e_j}(c(t))\right]_{i,j=1}^{n-1} \dot{c}(t)$$

shows that it is not a quadratic form. It would be a quadratic form if and only if the quantity

$$[\dot{c}(t), \dot{c}(t)] - \dot{c}(t)^T \dot{c}(t) = [\dot{c}(t), \dot{c}(t)] - \sum_{i=1}^{n-1} \dot{c}_i^2(t)$$

vanishes. Thus if the Minkowski product is an i.p. than we can assume that the basis $\{e_1, \ldots, e_{n-1}\}$ in S is orthonormal and we have that the mentioned difference is vanishing, furtermore $c_i(t) = \langle e_i, c(t)\rangle = \langle c(t), e_i\rangle$ and $\dot{c}(t) = \sum\limits_{i=1}^{n-1} \dot{c}_i(t) e_i$. So

$$\mathrm{I}_{f(c(t))} = \dot{c}(t)^T \left(\mathrm{Id} - \left[\mathfrak{f}'_{e_i}(c(t))\mathfrak{f}'_{e_j}(c(t))\right]_{i,j=1}^{n-1}\right) \dot{c}(t),$$

and we get back the classical local quadratic representation of the first fundamental form. Now if $c_i(t) = 0$ for $i \geq 3$ then

$$\det I = 1 - (\mathfrak{f}'_{e_1}(c(t)))^2 - (\mathfrak{f}'_{e_2}(c(t)))^2.$$

We now extend the definition of the second fundamental form take into consideration that the product has neither symmetry nor bilinearity properties. If v is a tangent vector and n is a normal vector of the hypersurface at its point $f(c(t))$ then we have

$$0 = [v, n]^+ = [D(f \circ c)(t), (f \circ c)(t)]^+.$$

Using Lemma 4 and the notation follows it, we get

$$0 = ([D(f \circ c)(t), (n \circ c)(t)]^+)' =$$

$$= [D^2(f \circ c), n(c(t))]^+ + [D(f \circ c)(t), \cdot]^{+\prime}_{D(n \circ c)(t)}(n(c(t))).$$

We introduce the unit normal vector fields n^0 by the definition

$$n^0(c(t)) := \begin{cases} n(c(t)) & \text{if } n \text{ light-like vector} \\ \frac{n(c(t))}{\sqrt{|[n(c(t)), n(c(t))]^+|}} & \text{otherwise.} \end{cases}$$

Definition 12. The *second fundamental form* at the point $f(c(t))$ defined by one of the equivalent formulas:

$$\mathrm{II} := [D^2(f \circ c)(t), (n^0 \circ c)(t)]^+_{(f \circ c)(t)} = -[D(f \circ c)(t), \cdot]^{+\prime}_{D(n^0 \circ c)(t)}((n^0 \circ c)(t)).$$

By the structure of the generalized space-time model assuming that $n(s) = s + \mathfrak{n}(s)e_n$ we get that

$$\mathrm{II} = [D^2(f\circ c)(t), (n^0\circ c)(t)]^+_{(f\circ c)(t)} =$$

$$= \left[D(\dot{c}(t) + D(\mathfrak{f}\circ c)(t)e_n), \frac{c(t)+(\mathfrak{n}\circ c)(t)e_n}{\sqrt{|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}}\right]^+ =$$

$$= \frac{\left[\ddot{c}(t) + \left(\dot{c}(t)^T\left[\mathfrak{f}''_{e_i,e_j}|_{c(t)}\right]\dot{c}(t) + \left[\mathfrak{f}'_{e_i}|_{c(t)}\right]\ddot{c}(t)\right)e_n, c(t)+\mathfrak{n}(c(t))e_n\right]^+}{\sqrt{|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}} =$$

$$= \frac{\left[\ddot{c}(t) + [\mathfrak{f}'_{e_i}|_{c(t)}]\ddot{c}(t)e_n, (n\circ c)(t)\right]^+ - \left(\dot{c}(t)^T\left[\mathfrak{f}''_{e_i,e_j}|_{c(t)}\right]\dot{c}(t)\right)(\mathfrak{n}(c(t))}{\sqrt{|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}} =$$

$$= \frac{\left[D(f)|_{c(t)}\ddot{c}(t), (n\circ c)(t)\right]^+ - \left(\dot{c}(t)^T\left[\mathfrak{f}''_{e_i,e_j}|_{c(t)}\right]\dot{c}(t)\right)(\mathfrak{n}(c(t))}{\sqrt{|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}} =$$

$$= -\left(\dot{c}(t)^T\left[\frac{\mathfrak{f}''_{e_i,e_j}|_{c(t)}\mathfrak{n}(c(t))}{\sqrt{|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}}\right]_{i,j=1}^{n-1}\dot{c}(t)\right).$$

We now can adopt a determinant of this fundamental form. It is the determinant of its quadratic form:

$$\det \mathrm{II} := \det\left(\left[\frac{\mathfrak{f}''_{e_i,e_j}|_{c(t)}\mathfrak{n}(c(t))}{\sqrt{|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}}\right]_{i,j=1}^{n-1}\right).$$

If we consider a two-plane in the tangent hyperplane then it has a two dimensional pre-image in S by the regular linear mapping Df. The getting plane is a normed one and we can consider an Auerbach basis $\{e_1, e_2\}$ in it.

Definition 13. The *sectional principal curvature* of a 2-section of the tangent hyperplane in the direction of the 2-plane spanned by $\{u = Df(e_1)$ and $v = Df(e_2)\}$ are the extremal values of the function

$$\rho(D(f\circ c)) := \frac{\mathrm{II}_{f\circ c(t)}}{\mathrm{I}_{f\circ c(t)}},$$

of the variable $D(f\circ c)$. We denote them by $\rho(u,v)_{\max}$ and $\rho(u,v)_{\min}$, respectively. The *sectional (Gauss) curvature* $\kappa(u,v)$ (at the examined point $c(t)$) is the product

$$\kappa(u,v) := [n^0(c(t)), n^0(c(t))]^+\rho(u,v)_{\max}\rho(u,v)_{\min}.$$

In the case of a symmetric and bilinear product, both of the fundamental forms are quadratic and the sectional principal curvatures attained in orthogonal directions. They are the eigenvalues of the pair of quadratic forms $\mathrm{II}_{f\circ c(t)}$ and $\mathrm{I}_{f\circ c(t)}$. This implies that $\rho(u,v)_{\max}$ and $\rho(u,v)_{\min}$ are the solutions of the equality:

$$0=\det\left(\mathrm{II}_{f\circ c(t)}-\lambda\mathrm{I}_{f\circ c(t)}\right)=\det\left(\mathrm{I}_{f\circ c(t)}\right)\det\left((\mathrm{I}_{f\circ c(t)})^{-1}\mathrm{II}_{f\circ c(t)}-\lambda\mathrm{Id}\right),$$

showing that

$$\kappa(u,v):=[n^0(c(t)),n^0(c(t))]^+\rho(u,v)_{\max}\rho(u,v)_{\min}=$$

$$=[n^0(c(t)),n^0(c(t))]^+\det\left(\mathrm{I}^{-1}_{f\circ c(t)}\mathrm{II}_{f\circ c(t)}\right)=[n(c(t)),n(c(t))]^+\frac{\det\mathrm{II}_{f\circ c(t)}}{\det\mathrm{I}_{f\circ c(t)}}=$$

$$=[n^0(c(t)),n^0(c(t))]^+\frac{\left(\mathfrak{f}''_{e_1,e_1}|_{c(t)}\mathfrak{f}''_{e_2,e_2}|_{c(t)}-\left(\mathfrak{f}''_{e_1,e_2}|_{c(t)}\right)^2\right)(\mathfrak{n}(c(t)))^2}{\left(1-(\mathfrak{f}'_{e_1}(c(t)))^2-(\mathfrak{f}'_{e_2}(c(t)))^2\right)|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}.$$

But we can choose for the function n

$$n(c(t)):=\mathfrak{f}'_{e_1}(c(t))e_1+\mathfrak{f}'_{e_2}(c(t))e_2+e_n$$

with $\mathfrak{n}(c(t))=1$ and for a 2-plane of the tangent hyperplane which contains only space-like vectors and has time-like normal vector with absolute value

$$[n(c(t)),n(c(t))]^+=\sqrt{1-(\mathfrak{f}'_{e_1}(c(t)))^2-(\mathfrak{f}'_{e_2}(c(t)))^2}$$

getting the well-known formula

$$\kappa(u,v)=\frac{-\mathfrak{f}''_{e_1,e_1}|_{c(t)}\mathfrak{f}''_{e_2,e_2}|_{c(t)}+\left(\mathfrak{f}''_{e_1,e_2}|_{c(t)}\right)^2}{\left(1-(\mathfrak{f}'_{e_1}(c(t)))^2-(\mathfrak{f}'_{e_2}(c(t)))^2\right)^2}$$

(see in [5] p.95.).

The Ricci curvature of a Riemannian hypersurface at a point $p=(f\circ c)(t)$ in the direction of the tangent vector $v=D(f\circ c)$ is the sum of the sectional curvatures in the directions of the planes spanned by the tangent vectors v and u_i, where u_i are the vectors of an orthonormal basis of the orthogonal complement of v. This value is independent from the choosing of the basis. Choose random (by uniform distribution) the orthonormal basis! ([3]) The corresponding sectional curvatures $\kappa(u_i,v)$ will be random variables with the same expected values. The sum of them is again a random variable which expected value corresponding to the Ricci curvature at p with respect to v. Hence it is equal to $n-2$-times the expected value of the random sectional curvature determined by all of the two planes through v. Similarly the scalar curvature of the hypersurface at a point is the sum of the sectional curvatures defined by any two vectors of an orthonormal basis of the tangent space, it is also can be considered as an expected value. This motivates the following definition:

Definition 14. The *Ricci curvature* $\mathrm{Ric}(v)$ in the direction of the tangent vector v at the point $f(c(t))$ is

$$\mathrm{Ric}(v)_{f(c(t))} := (n-2) \cdot E(\kappa_{f(c(t))}(u,v))$$

where $\kappa_{f(c(t))}(u,v)$ is the random variable of the sectional curvatures of the two planes spanned by v and a random u of the tangent hyperplane holding the equality $[u,v]^+ = 0$. We also say that the scalar curvature of the hypersurface f at its point $f(c(t))$ is

$$\Gamma_{f(c(t))} := \binom{n-1}{2} \cdot E(\kappa_{f(c(t))}(u,v)).$$

2.2 Arc-length

In this section we also assume that the s.i.p. of S is continuously differentiable. If the first fundamental form is positive we can adopt for the curves well-defined arc-lengthes and we can define a metric on the hypersurface.

The following definition was used in [6] for the metric of the imaginary unit sphere. We now adopt it for an arbitrary hypersurface.

Definition 15. Denote by p, q a pair of points in F where F is a hypersurface of the generalized space-time model. Consider the set $\Gamma_{p,q}$ of equally oriented piecewise differentiable curves $(f \circ c)(t)$, $a \leq t \leq b$, of F emanating from p and terminating at q. Then the *pre-distance* of these points is

$$\rho(p,q) = \inf \left\{ \int\limits_a^b \sqrt{|\mathrm{I}_{(f \circ c)(x)}|} dx \text{ for } f \circ c \in \Gamma_{p,q} \right\}.$$

It is easy to see that pre-distance satisfies the triangle inequality; thus it gives a metric on F (see [17]). On a hypersurface which contains only space-like tangent vectors it is the usual definition of the Minkowski-Finsler distance. Such hypersurfaces were called by space-like ones, we mention for an example the imaginary unit sphere. Introduce the **arc-length function** $l_a(\tau)$ of a curve $f \circ c$ for which the light-like points gives a closed, zero measured set by the function

$$l_a(\tau) = \int\limits_a^\tau \sqrt{|[D(f \circ c)(x), D(f \circ c)(x)]^+_{f(c(x))}|} dx = \int\limits_a^\tau \sqrt{|\mathrm{I}_{f(c(x))}|} dx.$$

Give parameters only those points of the curve in which the tangent vector of the curve is a non-light-like one. Thus a corresponding reparametrization could be well-defined and we get a pair of inverse formulas which are almost all valid; we have that

$$(l_a(\tau))' = \sqrt{|I_{f(c(\tau))}|},$$

and for the inverse function $\tau(l_a) : [0, \varepsilon) \longrightarrow [a, l_a^{-1}(\varepsilon))$ holds

$$(\tau(l_a))' = \left(l_a^{-1}(\tau)\right)' = \frac{1}{\sqrt{|\mathrm{I}_{f(c(\tau(l_a)))}|}}.$$

Theorem 7. *Consider a curve lying on the hypersurface determining a vector field in V. Using the arc-length as parameter, the absolute values of the first derivative (tangent) vectors are equal to 1, moreover the second derivative vector fields are orthogonal to the first one. (With respect to the Minkowski product of V.)*

Proof. By definition the tangent vectors are non-light-like. Thus the required differential is

$$D\left((f \circ c) \circ (\tau(l_a))\right) = D(f \circ c) \circ (\tau(l_a)) \cdot (\tau(l_a))' = D(f \circ c) \circ (\tau(l_a)) \frac{1}{\sqrt{|\mathrm{I}_{f(c(\tau(l_a)))}|}},$$

implying that

$$\left[D\left((f \circ c) \circ (\tau(l_a))\right), D\left((f \circ c) \circ (\tau(l_a))\right)\right]^+ = \frac{\mathrm{I}_{f(c(\tau(l_a)))}}{|\mathrm{I}_{f(c(\tau(l_a)))}|} = \mathrm{sign}(\mathrm{I}_{f(c(\tau(l_a)))}).$$

If we would like to consider the derivative of the tangent vector field, we have to compute

$$D\left(D\left((f \circ c) \circ (\tau(l_a))\right)\right).$$

By Lemma 4 we get

$$D\left(\sqrt{|\mathrm{I}_{f(c(\tau(l_a)))}|}\right) = \frac{\mathrm{sign}(\mathrm{I}_{f(c(\tau(l_a)))})}{2\sqrt{|\mathrm{I}_{f(c(\tau(l_a)))}|}} \left\{ \frac{[D^2(f \circ c) \circ (\tau(l_a)), D(f \circ c) \circ (\tau(l_a))]}{\sqrt{|\mathrm{I}_{f(c(\tau(l_a)))}|}} + \right.$$

$$\left. + [D(f \circ c) \circ (\tau(l_a)), \cdot]'_{D((Df \circ c) \circ \tau(l_a))}(D(f \circ c) \circ (\tau(l_a))) \right\},$$

and since the s.i.p is continuously differentiable we have by Theorem 5 that

$$[D(f \circ c) \circ (\tau(l_a)), \cdot]'_{D((Df \circ c) \circ \tau(l_a))}(D(f \circ c) \circ (\tau(l_a))) =$$

$$= \frac{[D^2(f \circ c) \circ (\tau(l_a)), D(f \circ c) \circ (\tau(l_a))]}{\sqrt{|\mathrm{I}_{f(c(\tau(l_a)))}|}}.$$

Thus the complete differential is

$$D\left(D(f \circ c) \circ (\tau(l_a)) \frac{1}{\sqrt{|\mathrm{I}_{f(c(\tau(l_a)))}|}}\right) = \frac{D^2(f \circ c) \circ (\tau(l_a))}{|\mathrm{I}_{f(c(\tau(l_a)))}|} -$$

$$-\mathrm{sign}(\mathrm{I}_{f(c(\tau(l_a)))}) \frac{[D^2(f \circ c) \circ (\tau(l_a)), D(f \circ c) \circ (\tau(l_a))]}{(\mathrm{I}_{f(c(\tau(l_a)))})^2} D(f \circ c) \circ (\tau(l_a)).$$

Now we can see that

$$\left[D\left(D(f \circ c) \circ (\tau(l_a)) \frac{1}{\sqrt{\mathrm{I}_{f(c(\tau(l_a)))}}}\right), D(f \circ c) \circ (\tau(l_a))\right] = 0$$

as we stated. *QED*

Inner metric determines the geodesics of the hypersurface in a standard way. Since we use the concept of arc-length parametrization for a general (not necessary space-like) curve, we can introduce its **velocity and acceleration vectors fields** as the first and second derivative vector fields of the natural parametrization, respectively. We can introduce the concept of geodesics as the solutions of the Euler-Lagrange equation with respect to the hypersurface. More precisely:

Definition 16. We say that the C^2-curve $f \circ c$ (with almost all non-light-like tangent vectors) is a *geodesic* of the hypersurface F, if its acceleration vector field is orthogonal to the tangent hyperplane of F at each point of the curve. So there exists a function $\alpha(\tau(l_a)) : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$D\left(D\left((f \circ c) \circ (\tau(l_a))\right)\right) = \alpha(\tau(l_a))(n^0 \circ c)(\tau(l_a)).$$

The curvature of a curve can be defined as the square root of the absolute value of the derivative of its tangent vectors with respect to this parametrization.

Definition 17. The *curvature of the curve* $f \circ c$ is the non-negative function

$$\gamma_{f \circ c}(\tau(l_a)) := \sqrt{|[D\left(D\left((f \circ c) \circ (\tau(l_a))\right)\right), D\left(D\left((f \circ c) \circ (\tau(l_a))\right)\right)]^+|} =$$

$$= |\alpha(\tau(l_a))|.$$

If the curvature is non-zero then we can define the vector $(m \circ c)(\tau(l_a))$ by the equality:

$$(m \circ c)(\tau(l_a)) = \frac{D\left(D\left((f \circ c) \circ (\tau(l_a))\right)\right)}{\gamma_{f \circ c}(\tau(l_a))}.$$

From this equality immediately follows that

$$[(m \circ c)(\tau(l_a)), (m \circ c)(\tau(l_a))]^+ =$$

$$= \frac{[D\left(D\left((f \circ c) \circ (\tau(l_a))\right)\right), D\left(D\left((f \circ c) \circ (\tau(l_a))\right)\right)]^+}{\gamma^2_{f \circ c}(\tau(l_a))}.$$

Using the equality

$$D\left(D\left((f \circ c) \circ (\tau(l_a))\right)\right) = D\left(D(f \circ c) \circ (\tau(l_a)) \frac{1}{\sqrt{|\mathrm{I}_{f(c(\tau(l_a)))}|}}\right) =$$

$$= \frac{D^2(f \circ c) \circ (\tau(l_a))}{|\mathrm{I}_{f(c(\tau(l_a)))}|} -$$

$$-\mathrm{sign}(\mathrm{I}_{f(c(\tau(l_a)))}) \frac{[D^2(f \circ c) \circ (\tau(l_a)), D(f \circ c) \circ (\tau(l_a))]}{(\mathrm{I}_{f(c(\tau(l_a)))})^2} D(f \circ c) \circ (\tau(l_a)),$$

computed in Theorem 7, and the orthogonality property of the vectors $D(f \circ c)$ and $n^0 \circ c$, we get a connection analogous to the Meusnier's theorem:

$$\gamma_{f \circ c}(\tau(l_a))[(m \circ c)(\tau(l_a)), (n^0 \circ c)(\tau(l_a)]^+ = \left[\frac{D^2(f \circ c) \circ (\tau(l_a))}{|\mathrm{I}_{f(c(\tau(l_a)))}|}, (n^0 \circ c)(\tau(l_a)\right]^+$$

meaning that

$$\gamma_{f \circ c}(\tau(l_a))[(m \circ c)(\tau(l_a)), (n^0 \circ c)(\tau(l_a)]^+ = \frac{\mathrm{II}_{f(c(\tau(l_a)))}}{|\mathrm{I}_{f(c(\tau(l_a)))}|}.$$

The product form of this equality is

$$\gamma_{f \circ c}(\tau(l_a))[(m \circ c)(\tau(l_a)), (n^0 \circ c)(\tau(l_a)]^+ |\mathrm{I}_{f(c(\tau(l_a)))}| = \mathrm{II}_{f(c(\tau(l_a)))}.$$

This form for light-like vectors is also valid, if we define their acceleration vectors as vectors of zero length. By definition, for a geodesic curve

$$[(m \circ c)(\tau(l_a)), (m \circ c)(\tau(l_a))]^+ = \frac{(\alpha(\tau(l_a)))^2}{(\gamma_{f \circ c}(\tau(l_a))^2} \left[(n^0 \circ c)(\tau(l_a)), (n^0 \circ c)(\tau(l_a))\right]^+,$$

showing that $m \circ c$ and $n^0 \circ c$ have the same casual characters and thus

$$m \circ c = \mathrm{sign}(\alpha(\tau(l_a)))(n^0 \circ c).$$

Thus the product form of the Meusnier's theorem simplified into the equality

$$\alpha(\tau(l_a))[(n^0 \circ c)(\tau(l_a)), (n^0 \circ c)(\tau(l_a)]^+ |\mathrm{I}_{f(c(\tau(l_a)))}| = \mathrm{II}_{f(c(\tau(l_a)))}.$$

Equivalently we get

$$\alpha(\tau(l_a)) = [(n^0 \circ c)(\tau(l_a)), (n^0 \circ c)(\tau(l_a)]^+ \frac{\mathrm{II}_{f(c(\tau(l_a)))}}{|\mathrm{I}_{f(c(\tau(l_a)))}|} =$$

$$= [(n^0 \circ c)(\tau(l_a)), (n^0 \circ c)(\tau(l_a)]^+ \mathrm{sign}(\mathrm{I}_{f(c(\tau(l_a)))})\rho(D(f \circ c)).$$

If all tangent vectors are space-like vectors and the normal ones are time-like vectors, respectively, then the extremal values of the function

$$\alpha(\tau(l_a)) = -\rho(D(f \circ c))$$

on a two plane are the negatives of the principal curvatures. By the homogeneity properties of the fundamental forms, the investigated functions can be restricted to such a special subset, on which all of the possible values attain, to the unit circle of this plane. This set is compact and thus there are two extremal values and at least two corresponding unit vectors, respectively. The convexity of such a hypersurface implies that the signs of the extremal values of $\alpha(\tau(l_a))$ are equals, so the two principal curvatures has the same signs and thus the sectional curvature is negative.

On the other hand, the characters of such a tangent plane would be only two types; either it is a space-like plane containing only space-like vectors or it has two non-paralel light-like vectors partitioning the plane two double cones one of them contains the space-like vectors and the other one the time-like vectors, respectively.

In the second case, we can restrict our function onto the union of the imaginary unit circle, the de Sitter circle, and the two lines containing the light-like vectors, respectively. We omit the two direction of the light-like vectors and we can determine the extremal values of the second fundamental form on the de Sitter sphere and on the imaginary unit sphere, respectively. For example if the signs of the functions $\alpha(\tau(l_a))$ and $\mathrm{I}_{f(c(\tau(l_a)))}$ are equals, and the normal vectors are space-time vectors, then the principal curvatures have the same signs, implying that their product is positive. In this case, the sectional curvature is positive.

3 Four interesting premanifolds

In this section we give the most important hypersurfaces of a generalized space-time model and determine their geometries, respectively.

3.1 Imaginary unit sphere

Then by Theorem 6 (H^+, ds^2) is a Minkowski-Finsler space, where for the vectors u_1 and u_2 of T_v we have

$$ds^2_v(u_1, u_2) = [u_1, u_2]^+_v$$

with the Minkowski product $[\cdot,\cdot]^+_v$ of the tangent space T_v. This gives a possibility to examine the geometric property of H^+ on the base of the standard differential geometry of a space-time hypersurface. First we prove the following theorem:

Theorem 8. *H^+ is always convex. It is strictly convex if and only if the s.i.p. space S is a strictly convex space.*

Proof. Let $w = s' + t'$ be a point of H^+ and consider the product

$$[w - v, v]^+ = [s' - s, s] + [t' - t, t] = [s', s] - [s, s] - (\lambda' - \lambda)\lambda = [s', s] - \lambda'\lambda + 1,$$

where $t' = \lambda' e_n$, $t = \lambda e_n$ and $s', s \in S$ with positive λ' and λ, respectively. Since

$$\sqrt{1 + [s', s']} = \lambda' \text{ and } \sqrt{1 + [s, s]} = \lambda$$

thus

$$[w - v, v]^+ = [s', s] - \sqrt{1 + [s', s']}\sqrt{1 + [s, s]} + 1 \leq$$
$$\leq \sqrt{[s', s'][s, s]} - \sqrt{1 + [s', s']}\sqrt{1 + [s, s]} + 1 \leq 0,$$

because of the relation

$$[s', s'][s, s] + 2\sqrt{[s', s'][s, s]} + 1 \leq [s', s'][s, s] + ([s', s'] + [s, s]) + 1.$$

(We used here the inequality between the arithmetic and geometric means of two positive numbers.) Remark that equality holds if and only if the norms of s' and s are equal to each other and thus $\lambda' = \lambda$, too. So we have

$$[s', s] - [s, s] = 0,$$

or equivalently

$$[s', s] = \sqrt{[s', s'][s, s]}.$$

From the characterization of the strict convexity of an s.i.p. space we get H^+ contains only the point v of the tangent space T_v if and only if the s.i.p. space S is strictly convex. QED

To determine the first fundamental form consider the map $h = s + \mathfrak{h}(s)e_n$ giving the points of H^+. (Here $\mathfrak{h}(s) = \sqrt{1+[s,s]}$ is a real valued function.) Then we get that

$$\mathrm{I} = [\dot{c}(t) + (\mathfrak{h}\circ c)'(t)e_n, \dot{c}(t) + (\mathfrak{h}\circ c)'(t)e_n]^+ =$$

$$= [\dot{c}(t), \dot{c}(t)] - [(\mathfrak{h}\circ c)'(t)]^2,$$

where $\dot{c}(t)$ means the tangent vector of the curve c of S at its point $c(t)$. Using Lemma 3 and Theorem 5 we have

$$\mathrm{I} = [\dot{c},\dot{c}] - \frac{\left([\dot{c}(t),c(t)] + [c(t),\cdot]'_{\dot{c}(t)}(c(t))\right)^2}{4(1+[c(t),c(t)])} = [\dot{c},\dot{c}] - \frac{[\dot{c}(t),c(t)]^2}{1+[c(t),c(t)]}.$$

From this formula, by the Cauchy-Schwartz inequality, we can get a new proof for the fact that this form is positive.

The second fundamental form of H^+ is

$$\mathrm{II} := [\ddot{c}(t)+(\mathfrak{h}\circ c)''(t)e_n, c(t)+(\mathfrak{h}\circ c)(t)e_n]^+_{(\mathfrak{h}\circ c)(t)} = [\ddot{c}(t),c(t)] - (\mathfrak{h}\circ c)''(t)\mathfrak{h}(c(t)),$$

since

$$n\circ c = h\circ c = c(t) + (\mathfrak{h}\circ c)(t)e_n.$$

First we compute the derivative of

$$(\mathfrak{h}\circ c)'(t) : \mathbb{R} \longrightarrow \mathbb{R}$$

at its point t. We use again the formulas of Lemma 3 and Lemma 4 getting

$$(\mathfrak{h}\circ c)''(t) = \left((\mathfrak{h}\circ c)'\right)'(t) = \left(\frac{[\dot{c}(t),c(t)]}{\sqrt{1+[c(t),c(t)]}}\right)' =$$

$$= \frac{[\dot{c}(t),c(t)]'}{\sqrt{1+[c(t),c(t)]}} - \frac{\frac{[\dot{c}(t),c(t)]}{\sqrt{1+[c(t),c(t)]}}[\dot{c}(t),c(t)]}{(1+[c(t),c(t)])}$$

and so

$$(\mathfrak{h}\circ c)''(t)\mathfrak{h}(c(t)) = [\dot{c}(t),c(t)]' - \frac{[\dot{c}(t),c(t)]^2}{1+[c(t),c(t)]} =$$

$$\left([\ddot{c}(t),c(t)] + [\dot{c}(t),\cdot]'_{\dot{c}(t)}(c(t))\right) - \frac{[\dot{c}(t),c(t)]^2}{1+[c(t),c(t)]}.$$

Thus the second fundamental form is

$$\mathrm{II} = -[\dot{c}(t),\cdot]'_{\dot{c}(t)}(c(t)) + \frac{[\dot{c}(t),c(t)]^2}{1+[c(t),c(t)]},$$

or using the formula

$$\|y\|\|\cdot\|''_{x,z}(y) = [x,\cdot]'_z(y) - \frac{\Re[x,y]\Re[z,y]}{\|y\|^2},$$

we get the following equivalent form:

$$\mathrm{II} = -\|c(t)\|\|\cdot\|''_{\dot{c}(t),\dot{c}(t)}c(t) - \frac{[\dot{c}(t),c(t)]^2}{\|c(t)\|^2(1+\|c(t)\|^2)}.$$

If we also assume that the norm is a C^2 function of its argument then we can use Theorem 5 and we get

$$\mathrm{II} = -[\dot{c}(t),\dot{c}(t)] + \frac{[\dot{c}(t),c(t)]^2}{1+[c(t),c(t)]} = -\mathrm{I}.$$

By the positivity of the first fundamental form on H^+, we get that the second fundamental form is negative definite and

$$\rho(u,v)_{\max} = \rho(u,v)_{\min} = -1.$$

This implies that the sectional curvatures are equal to -1, the Ricci and scalar curvatures in any direction at any point is $-(n-2)$ and $-\binom{n-1}{2}$, respectively. Thus we proved:

Theorem 9. *If S is a continuously differentiable s.i.p. space then the imaginary unit sphere H^+ has constant negative curvature.*

Observe that our definitions in the case when the Minkowski product is an i.i.p. go to the usual concepts of hypersurfaces of a semi-Riemann manifolds (see [4], [16] or [18]) so we can regard H^+ a natural generalization of the usual hyperbolic space. Thus we can say that H is premanifold with constant negative curvature and H^+ is a **prehyperbolic** space.

3.2 de Sitter sphere

In this subsection we shall investigate the set G of those points of a generalized space-time model which scalar square is equal to one. In a pseudo-euclidean space this set was called by the **de Sitter sphere**. The tangent hyperplanes of the de Sitter space are pseudo-euclidean spaces. G is not a hypersurface of V but we can restrict our investigation to the positive part of G defined by

$$G^+ = \{s+t \in G : t = \lambda e_n \text{ where } \lambda > 0\}.$$

We remark that the local geometry of G^+ and G is agree by the symmetry of G in the subspace S. G^+ is already a hypersurface defined by the function

$$g(s) = s + \mathfrak{g}(s)e_n,$$

where

$$\mathfrak{g}(s) = \sqrt{-1+[s,s]} \text{ for } [s,s] > 1.$$

First we calculate the directional derivatives of the function

$$\mathfrak{g} : s \longmapsto \sqrt{-1+[s,s]}$$

giving the corresponding tangent vectors of form

$$u = \alpha(e + \mathfrak{g}'_e(s)e_n).$$

Since between $\mathfrak{g}$ and $\mathfrak{f} : s \longmapsto \sqrt{1+[s,s]}$, there is a connection in form

$$\mathfrak{f}^2(s) + \mathfrak{g}^2(s) = 2[s,s],$$

the derivative of $\mathfrak{g}$ in the direction of the unit vector $e \in S$ (by Lemma 1 and Lemma 3) can be calculated from the equality

$$2\mathfrak{f}(s)\mathfrak{f}'_e(s) + 2\mathfrak{g}(s)\mathfrak{g}'_e(s) = 4\|s\| \| \cdot \|'_e(s) = 4[e,s].$$

Thus

$$\mathfrak{g}'_e(s) = \frac{[e,s]}{\mathfrak{g}(s)} = \frac{[e,s]}{\sqrt{-1+[s,s]}}$$

meaning that

$$[u,u]^+ = \alpha^2 \left(1 - \frac{[e,s]^2}{(-1+[s,s])}\right) = \alpha^2 \frac{-1+[s,s]-[e,s]^2}{-1+[s,s]}.$$

From this we can see immediately that

$$\begin{array}{ll} [u,u]^+ > 0 & \text{if } -1+[s,s] > [e,s]^2 \\ [u,u]^+ = 0 & \text{if } -1+[s,s] = [e,s]^2 \\ [u,u]^+ < 0 & \text{if } -1+[s,s] < [e,s]^2. \end{array}$$

So a vector s' of the $n-2$-subspace of S orthogonal to s determines a space-time tangent vector in the tangent space and a tangent vector corresponding to αs is a time-like one. To determine the light-like tangent vectors consider a unit vector $e \in S$ of the form

$$e = \frac{\pm\sqrt{-1+[s,s]}}{[s,s]} s + s', \text{ where } s' \in s^\perp.$$

Such a unit vector lying in the intersection of the unit sphere of S by the union of $(n-2)$-dimensional affine subspaces

$$s^\perp + \frac{\pm\sqrt{-1+[s,s]}}{[s,s]} s.$$

Since $s^{\perp}$ is the orthogonal complement of s in S and

$$\left(\frac{\pm\sqrt{-1+[s,s]}}{[s,s]}\right)^2 [s,s] = \frac{-1+[s,s]}{[s,s]} < 1,$$

this intersection is the union of two spheres of dimension $n-3$.

Thus the directions of the light-like vectors form a cone of the tangent hyperplane; the cone of the points:

$$u = \alpha\left(\left(\pm\sqrt{-1+[s,s]}s + [s,s]s'\right) \pm [s,s]e_n\right).$$

Recall that we considered the tangent hyperplane as a subspace of the original vector space and observe that thus we can admit it an inner Minkowskian structure, with respect to the positive and negative subspaces

$$S' := s^{\perp} \cap S = s^{\perp} \text{ and } T' = \alpha\left(\sqrt{-1+[s,s]}s + [s,s]e_n\right).$$

First we note the following:

Theorem 10. *G^+ and its tangent hyperplanes are intersecting, consequently there is no point at which G would be convex.*

Proof. At an arbitrary point of G^+ there are two sets lying on G^+ and having in distinct halfspaces with respect to the corresponding tangent hyperplane. The first set is the intersection of the 2-plane spanned by e_n and $s+t \in M$; and the other one is an arbitrary curve of the $(n-2)$-hypersurface defined by the intersection of G and the hyperplane $S+(s+t)$. In fact, a normal vector of the tangent hyperplane at $s+t$ is itself $s+t$, because we have

$$\left[e + \frac{[e,s]}{\sqrt{-1+[s,s]}}e_n, s + \sqrt{-1+[s,s]}e_n\right]^+ = 0.$$

Thus with $\alpha > \frac{1}{\sqrt{[s,s]}}$ we have

$$\left[\left(\alpha s + \sqrt{-1+[\alpha s,\alpha s]}e_n\right) - \left(s + \sqrt{-1+[s,s]}e_n\right), s + \sqrt{-1+[s,s]}e_n\right]^+ =$$

$$= (\alpha-1)[s,s] + (\sqrt{-1+[s,s]} - \sqrt{-1+[\alpha s,\alpha s]})\sqrt{-1+[s,s]} =$$

$$= -1 + \alpha[s,s] - \sqrt{(-1+[\alpha s,\alpha s])(-1+[s,s])} =$$

$$= \alpha[s,s] - 1 - \sqrt{1-(1+\alpha^2)[s,s]+\alpha^2[s,s]^2} \geq 2(\alpha[s,s]-1) > 2(\|s\|-1) \geq 0.$$

On the other hand if $s'+t\in M$ arbitrary, then $\|s'\|=\|s\|$ thus

$$[s'-s+(t-t),s+t]^+=[s',s]-[s,s]\leq\sqrt{[s',s']}\sqrt{[s,s]}-[s,s]=0,$$

with equality if and only if $s'=\pm s$. *QED*

Continue our investigation with the computation of the fundamental forms. Using the function g, the first fundamental form has the form

$$\mathrm{I}=[\dot{c}(t)+(\mathfrak{g}\circ c)'(t)e_n,\dot{c}(t)+(\mathfrak{g}\circ c)'(t)e_n]^+=[\dot{c}(t),\dot{c}(t)]-[(\mathfrak{g}\circ c)'(t)]^2.$$

Using Lemma 3 and Theorem 5 we get

$$\mathrm{I}=[\dot{c},\dot{c}]-\frac{\left([\dot{c}(t),c(t)]+[c(t),\cdot]'_{\dot{c}(t)}(c(t))\right)^2}{4(-1+[c(t),c(t)])}=[\dot{c},\dot{c}]-\frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]}.$$

Furthermore we also have that $n\circ c=g\circ c=c(t)+(\mathfrak{g}\circ c)(t)e_n$ thus we get:

$$\mathrm{II}:=[\ddot{c}(t)+(\mathfrak{g}\circ c)''(t)e_n,c(t)+(\mathfrak{g}\circ c)(t)e_n]^+_{(\mathfrak{g}\circ c)(t)}=[\ddot{c}(t),c(t)]-(\mathfrak{g}\circ c)''(t)\mathfrak{g}(c(t)).$$

The derivative of the real function

$$(\mathfrak{g}\circ c)'(t)=D(\mathfrak{g}\circ c)(t):\mathbb{R}\longrightarrow\mathbb{R}$$

at its point t is:

$$(\mathfrak{g}\circ c)''(t)=\frac{[\dot{c}(t),c(t)]'}{\sqrt{-1+[c(t),c(t)]}}-\frac{\frac{[\dot{c}(t),c(t)]}{\sqrt{-1+[c(t),c(t)]}}[\dot{c}(t),c(t)]}{(-1+[c(t),c(t)])}$$

so by Lemma 4

$$(\mathfrak{g}\circ c)''(t)\mathfrak{g}(c(t))=[\dot{c}(t),c(t)]'-\frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]}=$$

$$=\left([\ddot{c}(t),c(t)]+[\dot{c}(t),\cdot]'_{\dot{c}(t)}(c(t))\right)-\frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]}.$$

Thus we have

$$\mathrm{II}=-[\dot{c}(t),\cdot]'_{\dot{c}(t)}(c(t))+\frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]}.$$

If we assume again that the norm is a C^2 function of its argument then we can use again Theorem 5 and we get

$$\mathrm{II}=-[\dot{c}(t),\dot{c}(t)]+\frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]}=-\mathrm{I},$$

as in the case of H^+. The principal curvatures are equal to -1. But the scalar squares of the normal vectors is positive at all points of G^+ implying that the sectional curvatures are equal to 1. The Ricci curvatures in any directions and at any points are equal to $(n-2)$, moreover the scalar curvatures at any points are equal to $\binom{n-1}{2}$ showing that:

Theorem 11. *The de Sitter sphere G has constant positive curvature if S is a continuously differentiable s.i.p space.*

On the basis of this theorem we can say about G as a premanifold of constant positive curvature and we may say that it is a **pre-sphere**.

3.3 The light cone

The inner geometry of the light cone L can be determined, too. Let L^+ be the positive part of this double cone determined by the function:

$$l(s) = s + \sqrt{[s,s]}e_n.$$

If S is a uniformly continuous s.i.p. space, then the tangent vectors at s are of the form:

$$u = \alpha\left(e + \|\cdot\|'_e(s)e_n\right) = \alpha\left(e + \frac{[e,s]}{\sqrt{[s,s]}}e_n\right).$$

Thus all tangents orthogonal to $l(s)$ which is also a tangent vector. (Choose $e = s^0$ and $\alpha = \|s\|$!) But the orthogonal companion of a neutral (isotropic or light-like) vector in a s.i.i.p space is an $(n-1)$-dimensional degenerated subspace containing it ([6] (Theorem 7)) Tangent hyperplanes are exist at every points of L^+ and it is an $(n-1)$-dimensional degenerated subspace of V. This also a support hyperplane of L. In fact, by $v = s + t$ and $w = s' + t'$ we get

$$[w - v, v]^+ = [s', s] + [t', t] = [s', s] - \lambda'\lambda$$

where $t' = \lambda' e_n$, $t = \lambda e_n$ and $s', s \in S$ with positive λ' and λ, respectively. Since

$$\sqrt{[s',s']} = \lambda' \text{ and } \sqrt{[s,s]} = \lambda$$

thus

$$[w - v, v]^+ = [s', s] - \sqrt{[s',s']}\sqrt{[s,s]} \leq 0$$

holds by the Cauchy-Schwartz inequality. We remark that equality holds if and only if $s' = \alpha s$ meaning that there is only one line of L^+ in the tangent space T_v. Thus the light cone is convex and thus the second fundamental form is semi-definite quadratic form. It also follows that any other vectors of the tangent hyperplane are space-like ones and there are two types of tangent 2-planes; one of them space-like plane and the other one contains space-like vectors and a doubled line of light-like vectors. In the first case, the corresponding principal and sectional curvatures is well defined and have negative values, respectively. To determine it we compute the fundamental forms.

In the case when S is continuously differentiable, the first fundamental form is

$$\mathrm{I} = [\dot{c}, \dot{c}] - \frac{\left([\dot{c}(t), c(t)] + [c(t), \cdot]'_{\dot{c}(t)}(c(t))\right)^2}{4[c(t), c(t)]} = [\dot{c}, \dot{c}] - \frac{[\dot{c}(t), c(t)]^2}{[c(t), c(t)]},$$

and the second one is

$$\mathrm{II} = -[\dot{c}(t), \cdot]'_{\dot{c}(t)}(c(t)) + \frac{[\dot{c}(t), c(t)]^2}{[c(t), c(t)]} = -[\dot{c}(t), \dot{c}(t)] + \frac{[\dot{c}(t), c(t)]^2}{[c(t), c(t)]} = -\mathrm{I}.$$

Thus the principal curvatures are -1 as in the cases of the unit spheres. However our definition gives at such a point zero sectional curvature for it, because of the zero lengthes of the normal vectors. The above computation can be used in the second case, too. Agreed that we calculate the fundamental forms only non-light-like directions, so on the plane of the second type the principal curvatures are also -1 and the sectional curvatures are zero, too. This implies that the Ricci and scalar curvatures are also zero, respectively. We have got

Theorem 12. *The light cone L^+ has zero curvatures if S is a continuously differentiable s.i.p space.*

Hence L is a premanifold with zero sectional, Ricci and scalar curvatures, respectively. We may also say that it is a **pre-Euclidean** space.

3.4 The unit sphere of the s.i.p. space $(V, [\cdot,\cdot]^-)$

In this subsection we shall investigate the set K of those points of the generalized space-time model which collects the unit sphere of the embedding s.i.p. space. In a pseudo-euclidean space it is the unit sphere of the embedding euclidean space. Its tangent hyperplanes are pseudo-euclidean one. K is not a hypersurface but we can also restrict our investigation to the positive part of K defined by

$$K^+ = \{s + t \in K \,:\, t = \lambda e_n \text{ where } \lambda > 0\}.$$

K^+ is a hypersurface defined by the function

$$k(s) = s + \mathfrak{k}(s) e_n,$$

where

$$\mathfrak{k}(s) = \sqrt{1 - [s, s]} \text{ for } [s, s] < 1.$$

The directional derivatives of the function

$$\mathfrak{k} : s \longmapsto \sqrt{1-[s,s]} \text{ for } [s,s] < 1$$

gives the corresponding tangent vectors of form

$$u = \alpha(e + \mathfrak{k}'_e(s)e_n).$$

Since by the function

$$\mathfrak{f} : s \longmapsto \sqrt{1+[s,s]},$$

we have the equality

$$\mathfrak{f}^2(s) + \mathfrak{k}^2(s) = 2$$

the derivative in the direction of the unit vector $e \in S$ is

$$\mathfrak{k}'_e(s) = -\frac{[e,s]}{\sqrt{1-[s,s]}}$$

meaning that

$$[u,u]^+ = \alpha^2\left(1 - \frac{[e,s]^2}{(1-[s,s])}\right) = \alpha^2\frac{1-[s,s]-[e,s]^2}{1-[s,s]}.$$

From this we can see immediately that

$$\begin{array}{ll} [u,u]^+ > 0 & \text{if } 1-[s,s] > [e,s]^2 \\ [u,u]^+ = 0 & \text{if } 1-[s,s] = [e,s]^2 \\ [u,u]^+ < 0 & \text{if } 1-[s,s] < [e,s]^2. \end{array}$$

It follows that the vector s' of the $n-2$-subspace of S orthogonal to s gives a space-like tangent vector and the vector corresponding to αs is a time-like one.

As in the case of the imaginary unit sphere we note the following:

Theorem 13. *K^+ is convex. If S is a strictly convex space, then K^+ is also strictly convex.*

Proof. Let $w = s' + t'$ be a point of K^+ and consider the product

$$[w-v, n_v]^+ = [s'-s, s''] + [t'-t, t''] = [s', s''] - [s, s''] - (\lambda' - \lambda)\lambda'',$$

where $t'' = \lambda'' e_n$, $t' = \lambda' e_n$, $t = \lambda e_n$ and $s'', s', s \in S$ with positive λ'', λ' and λ, respectively. Since

$$\sqrt{1-[s',s']} = \lambda' \text{ and } \sqrt{1-[s,s]} = \lambda$$

and

$$n_v = s - \sqrt{1-[s,s]}e_n$$

thus

$$[w-v,n_v]^+ = [s',s] + \sqrt{1-[s',s']}\sqrt{1-[s,s]} - 1 \leq$$
$$\leq \sqrt{[s',s'][s,s]} + \sqrt{1-[s',s']}\sqrt{1-[s,s]} - 1 \leq 0,$$

because

$$2\sqrt{[s',s'][s,s]} \leq [s',s'] + [s,s]).$$

We remark that equality holds in the inequalities if and only if the norms of s' and s are equal to each other. So we have the equality

$$[s',s] - [s,s] = 0,$$

or equivalently

$$[s',s] = \sqrt{[s',s'][s,s]}.$$

We also get that v is the only point of K^+ lying on the tangent space T_v if and only if the s.i.p. space S is strictly convex. QED

Using the function k the first fundamental form has the form

$$\mathrm{I} = [\dot{c}(t),\dot{c}(t)] - [(\mathfrak{k}\circ c)'(t)]^2.$$

Using Lemma 3 and Theorem 5 we have

$$\mathrm{I} = [\dot{c},\dot{c}] - \frac{\left([\dot{c}(t),c(t)] + [c(t),\cdot]'_{\dot{c}(t)}(c(t))\right)^2}{4(1-[c(t),c(t)])} = [\dot{c},\dot{c}] - \frac{[\dot{c}(t),c(t)]^2}{1-[c(t),c(t)]},$$

and assuming that $2[c(t),c(t)] \neq 1$ we get

$$\mathrm{II} = \left[\ddot{c}(t) + (\mathfrak{k}\circ c)''(t)e_n, \frac{c(t)-(\mathfrak{k}\circ c)(t)e_n}{\sqrt{|-1+2[c(t),c(t)]|}}\right]^+_{(\mathfrak{k}\circ c)(t)} =$$

$$= \frac{1}{\sqrt{|-1+2[c(t),c(t)]|}}\left([\ddot{c}(t),c(t)] + (\mathfrak{k}\circ c)''(t)\mathfrak{k}(c(t))\right).$$

Lemma 4 implies that

$$(\mathfrak{k}\circ c)''(t)\mathfrak{k}(c(t)) = -[\dot{c}(t),c(t)]' + \frac{[\dot{c}(t),c(t)]^2}{1-[c(t),c(t)]} =$$

$$= -\left([\ddot{c}(t),c(t)] + [\dot{c}(t),\cdot]'_{\dot{c}(t)}(c(t))\right) + \frac{[\dot{c}(t),c(t)]^2}{1-[c(t),c(t)]}.$$

thus we have

$$\mathrm{II} = \frac{1}{\sqrt{|-1+2[c(t),c(t)]|}}\left(-[\dot{c}(t),\cdot]'_{\dot{c}(t)}(c(t)) + \frac{[\dot{c}(t),c(t)]^2}{1-[c(t),c(t)]}\right).$$

Assuming that S is continuously differentiable and using Theorem 5 we get

$$\mathrm{II} = \frac{1}{\sqrt{|-1+2[c(t),c(t)]|}}\left(-[\dot{c}(t),\dot{c}(t)] + \frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]}\right) =$$

$$= -\frac{1}{\sqrt{|-1+2[c(t),c(t)]|}}\mathrm{I}.$$

The principal curvatures at a point $k(c(t))$ are

$$\rho_{\max}(u,v) = \rho_{\min}(u,v) = -\frac{1}{\sqrt{|-1+2[c(t),c(t)]|}}$$

giving the sectional curvatures

$$\kappa(u,v) := [n^0(c(t)), n^0(c(t))]^+ \rho(u,v)_{\max}\rho(u,v)_{\min} = \frac{1}{-1+2[c(t),c(t)]}.$$

The Ricci curvatures in any directions at the point $k(c(t))$ are equal to

$$\mathrm{Ric}(v)_{k(c(t))} := (n-2)\cdot E(\kappa_{k(c(t))}(u,v)) = \frac{n-2}{-1+2[c(t),c(t)]}$$

and the scalar curvature of the hypersurface K^+ at its point $k(c(t))$ is

$$\Gamma_{k(c(t))} := \binom{n-1}{2}\cdot E(\kappa_{f(c(t))}(u,v)) = \frac{\binom{n-1}{2}}{-1+2[c(t),c(t)]}.$$

Finally we remark that at the points of K^+ having the equality $2[c(t),c(t)] = 1$ all of the curvatures can be defined as in the case of the light cone and can be regard to zero.

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References

[1] J. Alonso, C. Benitez: *Orthogonality in normed linear spaces: a survey. Part I. Main properties.* Extracta Math. **3** (1988), 1–15.

[2] J. Alonso, C. Benitez: *Orthogonality in normed linear spaces: a survey. Part II. Relations between main orthogonalities.* Extracta Math. **4** (1989), 121–131.

[3] B. Csikós: *Personal communication.* (2010).

[4] M. W. Dawis, G. Moussong: Notes on Nonpositively Curved Polyhedra, Bolyai Society Mathematical Studies, 8, Budapest, 1999.

[5] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov: Modern Geometry- Methods and Applications, Part I. The geometry of Surfaces, Transformation Groups, and Fields. Second Edition, Springer-Verlag, 1992.

[6] Á. G. Horváth: *Semi-indefinite inner product and generalized Minkowski spaces.* Journal of Geometry and Physics **60** (2010) 1190–1208.

[7] J. R. Giles: *Classes of semi-inner-product spaces.* Trans. Amer. Math.Soc. **129/3** (1967), 436–446.

[8] I. Gohberg, P. Lancester, L. Rodman: Indefinite Linear Algebra and Applications. Birkhäuser, Basel-Boston-Berlin 2005.

[9] P. M. Gruber, C. C. Lekkerkerker: Geometry of Numbers. North-Holland Amsterdam-New York-Oxford-Tokyo 1987.

[10] G. Lumer: *Semi-inner product spaces.* Trans. Amer. Math. Soc. **100** (1961), 29-43.

[11] G. Lumer: *On the isometries of reflexive Orlicz spaces.* Ann. Inst. Fourier, Grenoble **13** (1963) 99–109.

[12] H. Martini: *Shadow boundaries of convex bodies.* Discrete Math. **155** (1996), 161-172.

[13] H. Martini, K. Swanepoel, G. Weiss: *The geometry of Minkowski spaces - a survey. Part I.* Expositiones Mathematicae **19** (2001), 97-142.

[14] H. Martini, K. Swanepoel: *The geometry of Minkowski spaces - a survey. Part II.* Expositiones Mathematicae **22(2)** (2004), 93-144.

[15] E. J. McShane: *Linear functionals on certain Banach spaces.* Proc. Amer. Math. Soc., Vol. **1** (1950), 402–408.

[16] G. O'Neill: Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New-York, 1983.

[17] L. Tamássy: *Finsler spaces corresponding to distance spaces.* Proc. of the Conf., Contemporary Geometry and Related Topics, Belgrade, Serbia and Montenegro, June 26–July 2, (2005), 485–495.

[18] S. Verpoort: The Geometry of the Second Fundamental Form: Curvature Properties and Variational Aspects, Katholike Universiteit Leuven, Dissertation 2008. electronic version: http://bib.kuleuven.be.