On Existence of Polygons with Equal Angles

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Abstract

Consider a pencil of rays in the Euclidean or Hyperbolic plane. The question may arise whether a polygon with equal angles can be constructed in such a way that the vertices are located on the given set of rays. We will discuss the solutions for triangles and quadrilaterals where the conditions are exactly given.

1 Introduction

In this work we focus on the investigation of polygons in the Euclidean and hyperbolic plane with the property that all the vertices are located on a given set of rays (directed half-lines) and their angles are equal. The earlier results [4], [5] and [2] described the properties of these kind of polygons in $\mathbf{H^2}$, however the question of existence has not been dealt with. The present publication aims to give the exact conditions which allow the construction of the above characterised polygons for the case of three and four rays.

2 The case of three rays

We start with the absolute observation that a triangle having equal angles is also a regular one as its sides are equal, too.

The problem is now whether the existence of a regular triangle can be guaranteed if a set of rays with common initial point is given.

In the hyperbolic plane the answer is the following

Theorem 1 *If r_1, r_2 and r_3 are three rays originated from a common point O then one can always find a regular triangle $A_1A_2A_3\triangle$ with $A_i \in r_i$, $i = 1, 2, 3$.*

Proof

Let the angles formed by the rays be denoted by $\beta_1, \beta_2, \beta_3$ in a non-decreasing sequence $0 \leq \beta_1 \leq \beta_2 \leq \beta_3 \leq 2\pi$ and let r_i be the common leg of β_j and β_k; $i, j, k = 1, 2, 3$ and $i \neq j \neq k \neq i$. We solve the problem using continuity principle by distinguishing two subcases.

1. $\beta_3 < \pi$. It follows easily that $\frac{2\pi}{3} \leq \beta_3$ and $\frac{\pi}{2} < \beta_2$ also hold. Now choosing an arbitrary point A_1 on r_1 we rotate the broken line r_1, O, r_3 around A_1 in such a direction that the rotated image r_3' of r_3 intersects the ray r_2. This point of intersection will be denoted by A_2 and its rerotated image on r_3 by A_3. Since $|A_1A_2| = |A_1A_3|$, the angles $A_1A_2A_3\angle$ and $A_2A_3A_1\angle$ are equal, too (Fig. 1).

   Continuously rotating r_3 the intersection point A_2 slides from O to the infinity, while the angle $A_2A_1O\angle$ (and so $A_2A_1A_3\angle$) increases from zero to a well defined value α^*. Parallelly the angle $A_1A_2A_3\angle$ decreases from $\frac{\pi}{2}$ to 0. The continuity of these functions guarantees a position where all the angles are equal. We note that for every point of r_1 there does exist a solution.

2. $\beta_3 \geq \pi$. Consequently $\beta_1 \leq \frac{\pi}{2}$. In this case first we reflect the line r_2 to r_3, this results t. We choose a point A_3 on the ray r_3 such that the angle $TA_3O\angle$ be less than $\frac{\beta_1}{2}$, where T is the orthogonal projection of A_3 onto t. (Fig. 2) Denote by O^* the reflected image of O with respect to T. In the isosceles triangle $OO^*A_3\triangle$ the relations between the angles are $\beta_1 = O^*OA_3\angle = A_3O^*O\angle > \alpha^* = O^*A_3O\angle$. Rotating r_2 around A_3 with an angle α, $\alpha' \geq \alpha \geq \alpha^*$ we obtain an intersection point with the ray r_1, denoted by A_1. Its rerotated image on r_2 is A_2. If α increases from α^* to α' (α' is the angle of rotation when the rotated ray r_2' is parallel to the ray r_1) the angles $A_3A_2A_1\angle = A_3A_1A_2\angle$ decrease from β_1 to zero. (With respect to the rotation by α^*, $A_1 = 0$ and A_2 is the rerotated image of $O = A_1$.) Since $\alpha^* < \beta_1$ and the values change continuously there exists a triangle with equal angles that is necessarily a regular one. We emphasise that now A_3 can not be chosen arbitrarily. $\square$

For the Euclidean case we can see at once that the above arguments fails to hold: travelling along r_3 the angle $TA_3O\angle$ remains the same!

Therefore we follow another approach which allows us not merely prove the existence but let the triangle be constructed directly.

First we change our point of view. We take a regular triangle and try to adjust a pencil to it that is identical to the required one. It is straightforward that if a single

construction exists then there are infinitely many solutions in homotetic positions. The number of solutions therefore refers to the number of basicly different solutions.

Theorem 2 *Given a pencil of three rays* (r_1, r_2, r_3) *in* $\mathbf{E^2}$. *A regular triangle* $A_1A_2A_3\triangle$ *with* $A_i \in r_i$, $i = 1, 2, 3$ *can be constructed, if and only if one of the following requirements holds:*

- $\beta_2 < \frac{\pi}{3}$ *and* $\beta_1 < \frac{\pi}{3} - \beta_2$
- $\beta_2 < \frac{\pi}{3}$ *and* $\beta_1 \geq \frac{\pi}{3} - \beta_2$
- $\beta_1 = \frac{\pi}{3}$
- $\beta_2 = \frac{\pi}{3}$
- $\frac{\pi}{3} < \beta_1, \beta_2 \leq \frac{2\pi}{3}$.

Moreover, the number of corresponding different solutions are $2, 1, 1, 1, 1$, *respectively.*

Proof We recall the well-known fact that for any segment AB the locus of all points P such that the angle $APB\angle$ is constant consists of two circular arcs symmetric to AB. Let us suppose that the medium size angle β_2 corresponds to the side AB, whilest β_1 to AC. (Since in Euclidean cases the role of vertices differs from the hyperbolic cases (see e. g. Lemma 2 and 3, later on) we use another notations, however now we have set $A := A_3$, $B := A_1$, $C := A_2$.) We denote the arc-pairs by K_{β_1} and K_{β_2}, respectively. Obviously the existence of the pencil is equivalent to the fact that the arc-pairs intersect each other. It is easy to see that O can not be located outside the two smaller plane quarters bounded by the lines of AC and AB (otherwise β_1 would be a subset of β_2, excluded). Obviously the problem has solution if $\beta_2 < \frac{\pi}{3}$: travelling by X along the arc VB the angle $CXA\angle$ increases from V to B and changes between 0 and $\frac{\pi}{3}$ (Fig. 3a). On the other hand there can be a solution in the other region (bottom left), too. Moving the point Y from U to A on the arc, the leg YA of $CYA\angle$ tends to the tangent line t of the circle, therefore $0 < CYA\angle < \frac{\pi}{3} - \beta_2$ as it can be seen in Fig. 3a.

If β_1 or $\beta_2 = \frac{\pi}{3}$ then we have the trivial solution.

If $\frac{\pi}{3} < \beta_2$ then K_{β_2} has no points in the bottom left region, so the number of solutions is at most 1. We have to distinguish two subcases. If $\frac{\pi}{3} < \beta_2 < \frac{2\pi}{3}$ then K_{β_2} intersects the sides AC and CB (see Fig. 3b). While X slides from D to B the angle $AXC\angle$ decreases from π to $\frac{\pi}{3}$. Taking into consideration that $\beta_1 \leq \beta_2$ we get the condition $\frac{\pi}{3} < \beta_1, \beta_2 < \frac{2\pi}{3}$. If, on the contrary $\beta_2 \geq \frac{2\pi}{3}$ then the moving point X gives angles between $\beta_2 - \frac{2\pi}{3}$ and $\frac{\pi}{3}$ (Fig. 3c). But $\beta_2 \leq \pi$ ought to hold, too, that would lead to $\frac{\pi}{3} \leq \beta_1 \leq \frac{\pi}{3}$ a contradiction. $\square$

To illustrate the theorem we represent the solutions in a cartesian coordinate system. The conditions $\beta_3 \geq \frac{2\pi}{3} \iff \beta_2 \leq \frac{4\pi}{3} - \beta_1$, $\beta_3 \geq \beta_2 \iff \pi - \frac{\beta_1}{2} \geq \beta_2$ and $\beta_2 \geq \beta_1$ describe the set PQR of possible solutions (Fig. 4). The darker triangle at the corner shows two solutions, the lighter area refers to one solution. White areas serve no solutions. Dotted borders illustrate one solution, other number of solutions are indicated just below the lines in question.

3 The case of four rays

In this section we deal with the existence of *rectangles* in $\mathbf{H^2}$ and $\mathbf{E^2}$, respectively, if a pencil of four rays is given. We recall some important properties of rectangles: they are quadrilaterals with equal angles having two axes of symmetry perpendicular to each other, the opposite sides and the diagonals have the same length. A *parallelogram* is a quadrilateral with center of symmetry. The rays are labelled by r_1, r_2, r_3, r_4 in a cyclic order, the angle of r_i and r_{i+1} is β_i, $(i = 1, 2, 3, 4, r_5 \equiv r_1)$.

We start with the hyperbolic solution.

Theorem 3 *For any pencil of four rays in* $\mathbf{H^2}$ *there exists a rectangle whose vertices are located on the rays.*

Proof The main idea of the proof is the following: we fix the vertex A_2 on r_2 in a suitable position and vary the positions of the other vertices on the rays in time such that the vertices form parallelograms and moreover by continuous deformation they are transformed onto each other. Then we show that in one position $|A_1A_3| < |A_2A_4|$ whilst in other position the opposite relation holds.

We examine three subcases:

- case A: $\beta_i + \beta_{i+1} > \frac{\pi}{2}$ for $(i = 1, 2)$,
- case B: $\beta_1 + \beta_2 \leq \frac{\pi}{2}$ and $\beta_2 + \beta_3 > \frac{\pi}{2}$,
- case C: $\beta_i + \beta_{i+1} \leq \frac{\pi}{2}$ for $(i = 1, 2)$,

Case A: The position of A_2 on r_2 is arbitrary but fixed. (Fig. 5) We form the image of the ray r_4 by a contraction with center A_2 and ratio $\frac{1}{2}$. Choose the cyclic ordering of the rays in such a way that this image (denoted by k) intersects the ray r_3. The curve k (a hyperbole of the hyperbolic plane) will be the orbit of the centers of parallelograms. For $t = 0$ the center $K(0)$ is $r_2 \cap k$, for $t = 1$ the center $K(1)$ is $r_3 \cap k$. This implies that the vertex $A_4(0)$ is the common initial point O of the rays, while $A_4(1)$ is a well defined point on r_4 $(A_2 = A_2(0) = A_2(1))$. For any $t \in [0, 1]$ $A_1(t)$ and $A_3(t)$ can be constructed by the rays r_1' and r_3' which are the

reflected images of the rays r_1 and r_3 to the point $K(t)$ setting $A_1(t) = r_1 \cap r_3'$ and $A_3(t) = r_1' \cap r_3$. On one hand since $\beta_1 + \beta_2 > \frac{\pi}{2}$, $|A_1(0)A_3(0)| > |A_2(0)A_4(0)|$ holds, on the other hand $\beta_2 + \beta_3 > \frac{\pi}{2}$ implies $|A_1(1)A_3(1)| < |A_2(1)A_4(1)|$. By continuity principle there exists a value $t \in]0, 1[$ where the equality is gained. We note that it may happen that the intersection points in question are not real ones. In this case we will say that the length of the diagonals is infinite and the statement still holds.

Case B: In this case the position of A_2 is also fixed, but is not arbitrarily chosen (Fig. 6). First we construct the common parallel line $p_{1,3}$ to r_1 and r_3. Let $K' := p_{1,3} \cap r_2$ and A_2^* will denote the reflected image of O to K'. If K' is the center of a parallelogram then the rays r_1' and r_3, r_1 and r_3' are parallel, respectively. If we choose A_2 on the segment OA_2^* then the intersection points A_1 and A_3 can be constructed, otherwise not. If A_2 lies close enough to A_2^* then $|A_1A_3| > |A_2A_4|$, this will be the starting position $t = 0$. Let $t = 1$ be the same as above: $K(1) := r_3 \cap k$ and the parallelogram is constructed as usual. In this position obviously $|A_1(1)A_3(1)| < |A_2(1)A_4(1)|$ showing the veracity of the statement.

Case C: We choose the suitable point A_2 on the segment OA_2^*, again. We have to distinguish three subcases here.

- $\beta_1 < \beta_2 \leq \beta_3$ (Fig. 7) Define the point $A_2(0) = A_2(t)$ on the segment OA_2^* as above. If we reflect any point of r_2 to r_3 then the image belongs to the region β_3. Therefore the midpoint of the segment from $A_2(0)$ orthogonal to r_3 is in β_3, too. Let the intersection point be K''. Obviously $r_3 \cap k = K(t)$ belongs to the segment OK''. Since $K''A_2O\angle < \beta_1 < \beta_2$, we have $|OK(t)| < |OK''| < |K''A_2(t)| < |K(t)A_2(t)|$ that implies $|A_1(t)A_3(t)| = 2|OK(t)| < 2|K(t)A_2(t)| = |A_2(t)A_4(t)|$. On the other hand if A_2 lies close enough to A_2^* then for $t = 0$ the opposite relation holds.

- $\beta_2 > \beta_1, \beta_2 > \beta_3$ Let A_2^{**} be the point on r_2 with parallel angle β_1 with respect to r_4. Obviously this point lies on the segment OA^*. Denote by $K(1)$ the intersection point $r_3 \cap k$ and choose $A_2(0) = A_2(1)$ from the segment $A_2^*A_2^{**}$. Now $K(1)A_2A_1(1)\angle < \beta_1$, and moreover $|A_1(1)A_3(1)| = 2|K(1)A_1(1)| < 2|K(1)A_2(1)| = |A_2(1)A_4(1)|$. But if A_2 is close to A_2^* then for $t = 0$ the other diagonal would be longer.

- $\beta_2 < \beta_1, \beta_2 < \beta_3$ (Fig. 8) Here we follow a more sophisticated way giving first the positions belonging to $t = 0$ and $t = 1$, then we give the continuous deformation by the orbit of the centre $K(t)$ and the method how to construct the vertices. Denote now by A_2^* the point of r_2 for which the ray parallel to r_3 (r') makes an angle β_1 with r_2 and similarly, let A_3^* be the point of r_3 for which the ray parallel to r_2 (r'') makes an angle β_3 with r_3. Obviously if $A_2^* = A_2(0)$ then in the corresponding parallelogram the diagonal $A_1(0)A_3(0)$ should be longer, whereas for $A_3^* = A_3(1)$ the diagonal $A_2(1)A_4(1)$ has the same

property. Denote now the intersection point of r' and r'' by M. It is easy to see that the quadrilateral $OA_2^*MA_3^*$ is convex implying that its diagonal $A_2^*A_3^*$ lies in its interior. The orbit of the centers $K(t)$ consists of the midpoints of the segments $OO(t)$, where $O(t) \in A_2^*A_3^*$. The "intermediate" parallelograms are obtained by the intersections $A_3(t) := r_1'(t) \cap r_3$ and $A_2(t) := r_4'(t) \cap r_2$ and by their reflected images to $K(t)$. The following lemma guaranties the existence of the intersections above.

Lemma 1 *Take an inner point P in an asymptotic triangle with angles $\beta_2 < \beta_3 < \frac{\pi}{2}$. Let the midpoint of the segment OP be denoted by F and reflect OO' to F: PP'. If for the point Q; $P'PQ\angle = \beta_3$ holds, then the ray PQ does not intersect the side r'', opposite to O (Fig. 9).*

Proof Denote the common perpendicular of OO' and PP' across F by n. Because of the angle conditions the point P belongs to the region bounded by n and r'. The ray QP intersects the line OO' in R. If R is not on the segment OO' then the segment PR intersects r'' and thus PQ will not intersect. Assume now that R is on the segment OO'. In the shaded quadrilateral the condition for the sum of angles is $(\pi - \beta_3) + \frac{\pi}{2} + \frac{\pi}{2} + PRO\angle < 2\pi$, implies $PRO\angle < \beta_3$. If the ray RQ intersects r'' then in that triangle the inner angle at O' would be greater than the outer angle at R, a contradiction. □

In this way we have completed the proof of the theorem. □

Now we turn to the Euclidean case. First we prove a similar statement for rectangular isosceles triangles as for regular ones above.

Lemma 2 *The triangle diagram of Fig. 10 represents the conditions for which a rectangular isosceles triangle can be constructed for three given rays with angles $\beta_1, \beta_2, 2\pi - \beta_1 - \beta_2$. The numbers in the diagram refer to the number of different solutions.*

(Remark: Any point in the triangular domain uniquely describes a pencil of three rays and vice versa. For a given point the β_i values can be read off by drawing parallel lines to the sides. The dotted lines in Fig. 10 represent the trivial solution.)

Proof For simplification let us denote the hypotenuse by AB and the corresponding angle by β_2. Similarly let β_1 be the angle of AC. We emphasize that β_1 can be greater than β_2, there is no ordering for the angles.

Similar arguments as in the case of regular triangles implies the following:

There is a solution if one of the following conditions holds:

- $\beta_2 < \frac{\pi}{4}$ and $\beta_1 < \frac{\pi}{4} - \beta_2$
- $\beta_2 < \frac{\pi}{2}$ and $\beta_1 < \frac{\pi}{4}$
- $\beta_1 = \frac{\pi}{4}$
- $\beta_2 = \frac{\pi}{2}$
- $\frac{\pi}{2} < \beta_2 < \frac{3\pi}{4}$ and $\frac{\pi}{4} < \beta_1 < \pi$
- $\frac{3\pi}{4} < \beta_2 < \pi$ and $\frac{\pi}{4} < \beta_1 < \frac{7\pi}{4} - \beta_2$.

If we release the correspondence between sides and angles we obtain the diagram. The number of solutions for given triplet $(\beta_1, \beta_2, \beta_3 := 2\pi - \beta_1 - \beta_2)$ immediately follows from the enumeration how many times they fulfill the conditions above. In some cases we have to halve because we construct isosceles triangles. □

We can generalize this observation.

Lemma 3 *One can construct an obtuse-angled isosceles triangle with greatest angle σ if the angles $\beta_1, \beta_2, 2\pi - \beta_1 - \beta_2$ describe a suitable point in Fig. 11. The number of solutions can be read off, as well.*

Proof Let β_2 be the angle to the longest side and β_1 the other one. Similarly as before we have solutions if either

- $\beta_2 < \sigma$ and $\beta_1 < \frac{\pi}{2} - \sigma$ or
- $\beta_2 < \frac{\pi-\sigma}{2}$ and $\beta_1 < \frac{\pi-\sigma}{2} - \beta_2$ or
- $\beta_1 = \frac{\pi-\sigma}{2}$ or
- $\beta_2 = \sigma$ or
- $\sigma < \beta_2 < \frac{\pi+\sigma}{2}$ and $\frac{\pi-\beta_2}{2} < \beta_1 < \pi$ or
- $\frac{\pi+\sigma}{2} < \beta_2 < \pi$ and $\frac{\pi-\sigma}{2} < \beta_1 < \frac{3\pi+\sigma}{2} - \beta_2$.

The arguments is the same as before. □

Now we are ready to formulate our results first for square and later for other regular $n-$gons in E^2.

Theorem 4 *If a right-angled isosceles triangle is constructable for $\beta_1, \beta_2, 2\pi-\beta_1-\beta_2$ in E^2 then for each solution there exists a unique partition of one of the angles into two parts such that there exists a square with vertices on the rays and there is no construction for other angles.*

Proof Let us take an isosceles rectangular triangle and construct the corresponding pencil. Complete now the triangle to square. This uniquely cuts an angle into two parts. If we blow up or shrinken the square the fourth vertex remains on the same ray. $\square$

For the general answer we need a new concept: the *vertex-triangle* of a regular $n-$gon in E^2 ($n > 4$) is nothing but the isosceles triangle of three consequtive vertices.

Theorem 5 *Let β_1, β_2 be given. If one can construct a vertex-triangle of a regular $n-$gon for the angles $\beta_1, \beta_2, 2\pi - \beta_1 - \beta_2$ in E^2 then there is a unique partition of the angle(s) into altogether n parts such that the rays of the corresponding pencil contain the vertices of the regular $n-$gon and there is no solution for other angles.*

Proof It is an easy consequence of the fact that a vertex-triangle uniquely defines the other $n-3$ vertices of a regular $n-$gon. If the common initial point of the rays falls into the opposite halfplane comparing to the polygon with respect to both of the side lines of the vertex-triangle then two angles should be cut into parts otherwise it is enough to partitionate just one of them. $\square$

Theorem 6 *Let $\beta_1, \beta_2, \ldots, \beta_k$ be given in a cyclic ordering ($\sum \beta_i = 2\pi, \beta_i > 0$). Form three groups from these angles in the following way: take β_j alone and group the remaining angles into two parts: $\beta_j^+ := \cup_{l=1}^{t} \beta_{j+l}$; $\beta_j^- := \cup_{l=t+1}^{k-1} \beta_{j+l}$. If there is no solution for none of these partitions $\beta_j^-, \beta_j, \beta_j^+$ then the construction is not possible for the initial angles (Fig. 12).*

This statement is just an easy consequence of the above observations. $\square$

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