Semi-indefinite-inner-product and generalized Minkowski spaces

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$$
\begin{tikzcd}
& \boxed{\text{Euclidean space}} & \\
& \boxed{\text{inner-product}} \arrow[u, Rightarrow] & \\
\boxed{\text{semi-inner-product}} \arrow[ur] \arrow[d, Rightarrow] & & \boxed{\text{indefinite-inner-product}} \arrow[ul] \arrow[d, Rightarrow] \\
\boxed{\text{normed space}} \arrow[dr] & \text{and} & \boxed{\text{pseudo-euclidean space}} \arrow[dl] \\
& \boxed{\text{Minkowski space}} &
\end{tikzcd}
$$

The $\left\{ \begin{array}{r} semi-inner-product \\ indefinite-inner-product \end{array} \right.$ on a complex vector space V is a complex function $[x,y] : V \times V \longrightarrow \mathbb{C}$ with the properties $\left\{ \begin{array}{r} s1, s2, s3, s4 \\ i1, i2, i3, i4 \end{array} \right.$

s1=i1: $[x+y,z] = [x,z] + [y,z]$ (additivity of the first argument)

s2=i2: $[\lambda x, y] = \lambda [x,y]$ for every $\lambda \in \mathbb{C}$ (homogeneity of the first argument)

s3: $[x,x] > 0$ when $x \neq 0$ (positivity)

i3: $[x,y] = \overline{[y,x]}$ for every $x, y \in V$ (antisymmetry)

s4: $|[x,y]|^2 \leq [x,x][y,y]$ (Cauchy-Schwartz inequality)

i4: $[x,y] = 0$ for every $y \in V$ then $x = 0$. (nondegeneracy)

History of these concepts

Semi-inner-product

raised by G.Lumer in 1961, in 1967 J.R. Giles prove that the property

s5: $[x, \lambda y] = \bar{\lambda}[x, y]$ for all complex λ (homogeneity in the second argument)

can be imposed.

Indefinite-inner-product

first used by Minkowski, Lorentz, Einstein at the beginning of the twentieth century in the theoretical physics, the first application in mathematics (to the theory of zones of stability for canonical differential equations with periodic coefficients) were obtained by M.G.Krein in 1964,

I.M.Gelfand, N.Levinson, I.Gohberg,...

Connections from among the axioms:

1. $s1, s2, s3, s5 \Longrightarrow i4$

2. $i1, i2, i3, s3 \Longrightarrow s4$

Semi-indefinite-inner-product

The semi-indefinite-inner-product on a complex vector space V is a complex function $[x, y] : V \times V \longrightarrow \mathbb{C}$ with the following properties

1: $[x + y, z] = [x, z] + [y, z]$ (additivity of the first argument)

2: $[\lambda x, y] = \lambda[x, y]$ for every $\lambda \in \mathbb{C}$ (homogeneity of the first argument)

3: $[x, \lambda y] = \bar{\lambda}[x, y]$ for all complex λ (homogeneity in the second argument)

4: $[x, x] \in \mathbb{R}$ for $x \in V$ (the corresponding quadratic form is real valued)

5: $[x, y] = 0$ for every $y \in V$ or $[y, x] = 0$ for every $y \in V$ then $x = 0$. (nondegeneracy)

6: $|[x, y]|^2 \leq [x, x][y, y]$ holds on nonpositive and nonnegative subspaces of V (Cauchy-Schwartz inequality is valid on positive and negative subspaces, resp.)

Remark

$(V, [\cdot,\cdot])$ is a Semi-inner-product space $\Longleftrightarrow$ $[\cdot,\cdot]$ is positive

$(V, [\cdot,\cdot])$ is a Indefinite-inner-product space $\Longleftrightarrow$ $[\cdot,\cdot]$ is antisymmetric

A s.i.i.p. space in which there is no positive (resp. negative) subspace with dimension $n \geq 2$.

Consider the boundary of a cross-polytope:

$$C = \cup\{ \text{ conv } \{\varepsilon_i e_i | i = 1, \ldots, n\} \text{ for all choices of } \varepsilon_i = \pm 1\}.$$

and define a mapping from V into $V^\star$ on the following manner:

for $v \in C$ let $v^\star$ be the functional for which $v^\star(v) = (-1)^k$ where $k = \dim F_v$

$$(\lambda v)^\star = \lambda v^\star$$

$$[u, v] := v^\star(u)$$

Main example for semi-indefinite-inner-product

C is the unit sphere of a normed linear space, $P(C) = C/\sim$.
By the Hahn-Banach theorem there exists at least one continuous linear functional, and we choose exactly one such that $\|\tilde{v}^\star\| = 1$ and $\tilde{v}^\star(v) = 1$ for $v \in C$.
Consider a sign function $\varepsilon(v)$ with value ± 1 on $P(C)$,
and if $\varepsilon([v]) = 1$ denote by $v^\star = \tilde{v}^\star$
and if $\varepsilon([v]) = -1$ define $v^\star = -\tilde{v}^\star$,
homogeneously extend it to V the mapping $v \mapsto v^\star$ by the equality $(\lambda v)^\star = \overline{\lambda} v^\star$.
For the duality mapping
$v \mapsto v^\star$ the equalities $v^\star(v) := \varepsilon([v_0])\|v\|^2$ and $\|v\| = \|v^\star\|$ are hold.

$\boxed{[u, v] = v^\star(u)}$ satisfies **1**-**5**.

If U is a nonnegative subspace then it is positive and we have for all nonzero $u, v \in U$:

$|[u,v]| = |v^\star(u)| = \frac{|v^\star(u)|}{\|u\|}\|u\| \leq \|v^\star\|\|u\| = \|v\|\|u\|$, proving **6**.

The generalized Minkowski space

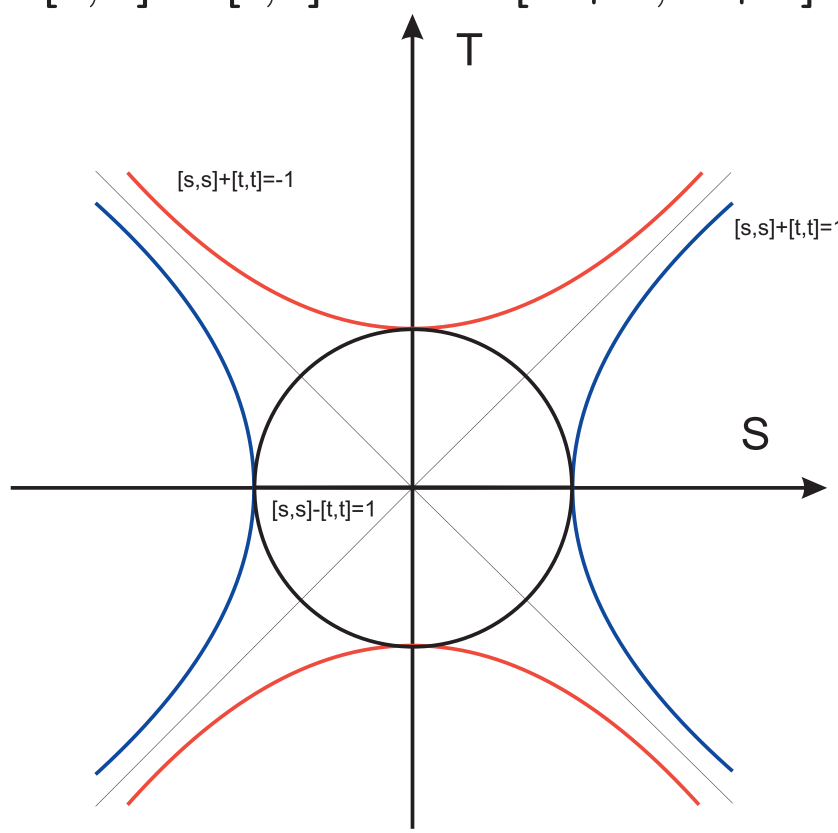
Definition: *Let* $(V, [\cdot,\cdot])$ *be an s.i.i.p. space. Let* $S, T \leq V$ *be positive and negative subspaces, where* T *is a direct complement of* S *with respect to* V. *Define a product on* V *by the equality* $[u,v]^+ = [s_1+t_1, s_2+t_2]^+ = [s_1,s_2] + [t_1,t_2]$, *where* $s_i \in S$ *and* $t_i \in T$, *respectively. Then we say that the pair* $(V, [\cdot,\cdot]^+)$ *is a generalized Minkowski space with Minkowski product* $[\cdot,\cdot]^+$. *We also say that* V *is a real generalized Minkowski space if it is a real vector space and the s.i.i.p. is a real valued function.*

Lemma: *Let* $(S, [\cdot,\cdot]_S)$ *and* $(T, -[\cdot,\cdot]_T)$ *be two s.i.p. spaces. Then the function* $[\cdot,\cdot]^- : (S+T)\times(S+T) \longrightarrow \mathbb{C}$ *defined by*

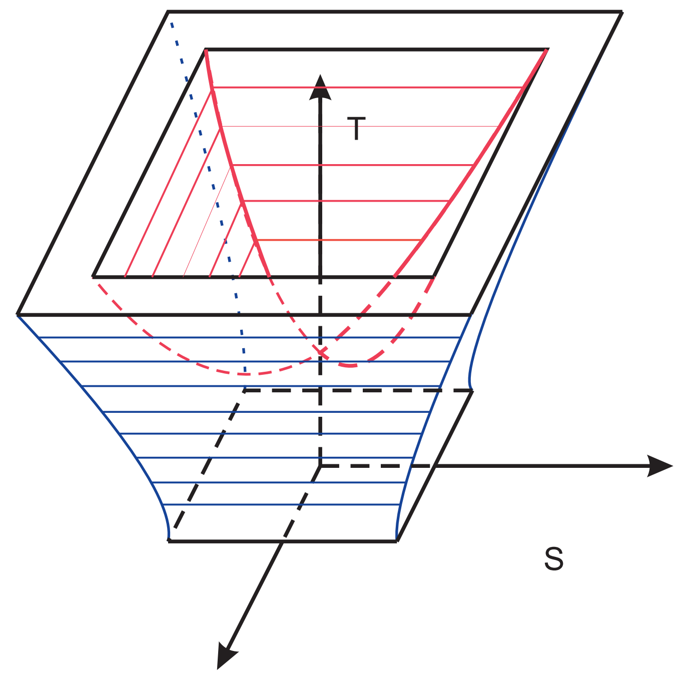
$$[s_1+t_1, s_2+t_2]^- := [s_1,s_2] - [t_1,t_2]$$

is an s.i.p. on the vector space $S+T$.

$$[s+t,s+t]^- := [s,s]-[t,t] \quad \text{and} \quad [s+t,s+t]^+ := [s,s]+[t,t]$$



Unit spheres in dimension two.



The case of the norm L^∞.

Facts on Minkowski spaces

The real generalized Minkowski space is a geometrical Minkowski space if it is finite dimensional and the s.i.i.p. is an s.i.p..The unit ball of this space is $\{v|[v,v]=1\}$.

The finite dimensional real generalized Minkowski space is a pseudo-Euclidean space if the s.i.i.p is an i.i.p, a space-time model if it is pseudo-Euclidean and its negative direct component has dimension 1. Its signature of corresponds to the dimensions of S and T.

By Lemma the s.i.p. $\sqrt{[v,v]^{-}}$ is a norm function on V which can give an embedding space for a generalized Minkowski space.

The Minkowski product satisfies the properties **1**-**5** of the s.i.i.p..

But in general the property **6** does not hold.

A s.i.p. associated to the L^∞ norm is:

$$[x_1e_1+x_2e_2,y_1e_1+y_2e_2]_S := x_1y_1 \lim_{p\to\infty} \frac{1}{\left(1+\left(\frac{y_2}{y_1}\right)^p\right)^{\frac{p-2}{p}}} + x_2y_2 \lim_{p\to\infty} \frac{1}{\left(1+\left(\frac{y_1}{y_2}\right)^p\right)^{\frac{p-2}{p}}}.$$

By Lemma the function

$$[x_1e_1+x_2e_2+x_3e_3,y_1e_1+y_2e_2+y_3e_3]^- := [x_1e_1+x_2e_2,y_1e_1+y_2e_2]_S + x_3y_3$$

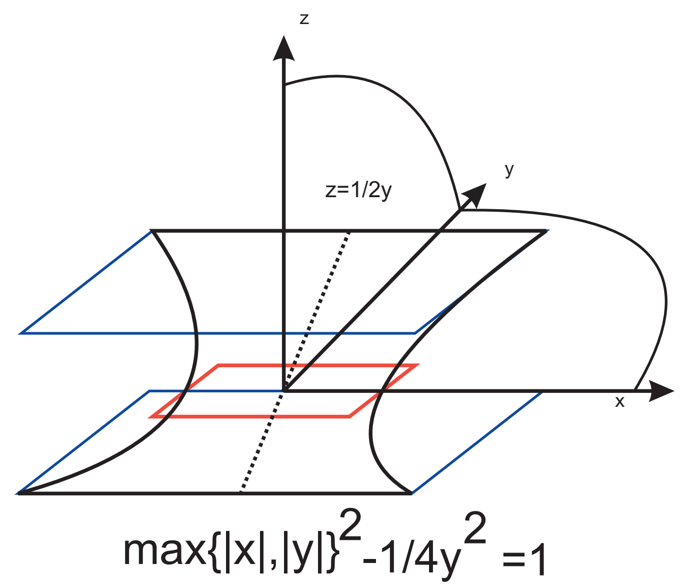
is an s.i.p. on E^3 defining the norm

$$\sqrt{[x_1e_1+x_2e_2+x_3e_3,x_1e_1+x_2e_2+x_3e_3]^-} := \sqrt{\max\{|x_1|,|x_2|\}^2+x_3^2}.$$

Consider such a sign function for which $\varepsilon(v)$ is equal to 1 if v is in $S\cap C$ and is equal to -1 if $v=e_3$ holds. This sign function determine an s.i.i.p. $[\cdot,\cdot]$, a Minkowski product $[\cdot,\cdot]^+$ and a square root function:

$$f(v) := \sqrt{[x_1e_1+x_2e_2+x_3e_3,x_1e_1+x_2e_2+x_3e_3]^+} = \sqrt{\max\{|x_1|,|x_2|\}^2-x_3^2}.$$

The plane $x_3 = \alpha x_2$ for $0 < \alpha < 1$ is positive subspace with respect to the Minkowski product but its unit ball is not convex.



On the other hand $f(v)$ homogeneous thus it is not subadditive. Since Cauchy-Scwartz inequality implies subadditivity, this inequality also false in this positive subspace.

Generalized space-time model

It is a real generalized Minkowski space with $\dim T = 1$.

Let denote by $\mathcal{S}, \mathcal{L}$ and $\mathcal{T}$ the sets of the space-like, light-like and time-like vectors, respectively. In this case $\mathcal{T}$ is a union of its two parts,

$$\mathcal{T} = \mathcal{T}^{+} \cup \mathcal{T}^{-}$$

where

$$\mathcal{T}^{+} = \{s + t \in \mathcal{T} | \text{ where } t = \lambda e_n \text{ for } \lambda \geq 0\} \text{ and}$$

$$\mathcal{T}^{-} = \{s + t \in \mathcal{T} | \text{ where } t = \lambda e_n \text{ for } \lambda \leq 0\}.$$

Theorem: *Let V be a generalized space-time model. Then $\mathcal{T}$ is an open double cone with boundary $\mathcal{L}$ and the positive part $\mathcal{T}^{+}$ (resp. negative part $\mathcal{T}^{-}$) of $\mathcal{T}$ is convex.*

The imaginary unit sphere in a generalized space-time model is a generalized two sheets hyperboloid corresponding the two piece of $\mathcal{T}$.

$$H := \{v \in V | [v,v]^{+} = -1\}$$

Usually we deal only with one sheet of the hyperboloid or identify the two sheets projectively. The positive part of it denoted by H^{+}. The space-time component $s \in S$ of v determines uniquely the time-like one $t \in T$ and thus the function

$$\boxed{s \mapsto v = s + \sqrt{1 + [s,s]} e_n}$$

gives H^{+}. The graph of such a function called by hypersurface.

The directional derivatives of a function $f : S \longmapsto \mathbb{R}$ with respect to a unit vector e of S can be defined, by the existence of the limits for real λ:

$$f'_e(s) = \lim_{\lambda \mapsto 0} \frac{f(s+\lambda e) - f(s)}{\lambda}.$$

Lemma :*Let V be a generalized Minkowski space and assume that the s.i.p. $[\cdot,\cdot]|_S$ is continuous. (So the property* **s6** *holds.) Then the directional derivatives of the real valued function*

$$f : s \longmapsto \sqrt{1+[s,s]},$$

are

$$f'_e(s) = \frac{\Re[e,s]}{\sqrt{1+[s,s]}}.$$

Tangent vectors and tangent hyperplane

u is a $\boxed{\textbf{tangent vector}}$ of the hypersurface F in its point $v = (s + f(s)e_n)$, if it is of the form

$$u = \alpha(e + f'_e(s)e_n) \text{ for real } \alpha \text{ and unit vector } e \in S.$$

The linear hull of the tangent vectors translated into the point s is the tangent space of F in s. If the tangent space has dimension $(n-1)$ we call it $\boxed{\textbf{tangent hyperplane}}$. We also consider it as an affine hyperplane through the examined point. So it also means the set

$$\boxed{\left\{(s + f(s)e_n) + \alpha\left(e + \frac{[e,s]}{\sqrt{1+[s,s]}}e_n\right) \;:\; \alpha \in \mathbb{R}, e \in S\right\}}$$

Definition: *If F is a hypersurface of a generalized space-time model for which the following properties hold:*

i, *in every point v of F, there is an (unique) tangent hyperplane T_v for which the restriction of the Minkowski product $[\cdot,\cdot]_v^+$ is positive,*

ii, *the function $ds_v^2 := [\cdot,\cdot]_v^+ : F \times T_v \times T_v \longrightarrow \mathbb{R}^+$*

$$ds_v^2 : (v, u_1, u_2) \longmapsto [u_1, u_2]_v^+$$

varying differentiable with the vectors $v \in F$ and $u_1, u_2 \in T_v$,

then we say that the pair (F, ds^2) is a Minkowski-Finsler space with semi-metric ds^2 embedding into the generalized space-time model V.

The main result on the imaginary unit sphere

Theorem: *Let V be a generalized space-time model. Let S be a continuously differentiable s.i.p. space then*

(H^+, ds^2) *is a* ***Minkowski-Finsler space.***

To prove this statement we need the concept of orthogonality in a s.i.i.p. space. We also proved:

Lemma: *Let H be the imaginary unit sphere of a generalized space-time model. Then the tangent vectors of the hypersurface H in its point $v = s + \sqrt{1+[s,s]}e_n$ form the* ***orthogonal complement*** *$v^{\perp}$ of v.*

and the theorem:

Theorem: *The* ***orthogonal complement*** *corresponding to the point $v = s + t \in H$ is a positive (n-1)-dimensional subspace of the generalized Minkowski space $(V, [\cdot,\cdot]^+)$.*

The geometry of H^+.

Definition: *A* $\boxed{\textit{linear isometry}}$ $f : H^+ \longrightarrow H^+$ *of* H^+ *is the restriction to* H^+ *of a linear map* $F : V \longrightarrow V$ *which preserves the Minkowski product and which sends* H^+ *onto itself.*

We need the concept of $\boxed{\text{generalized adjoint}}$.

Koehler proved that if the generalized Riesz-Fischer representation theorem is valid in a normed space then for every bounded linear operator A has a $\boxed{\text{generalized adjoint}}$ A^T defined by the equality:

$$[A(x), y] = [x, A^T(y)] \text{ for all } x, y \in V.$$

This mapping is the usual Hilbert space adjoint if the space is an i.p. one. In this more general setting this map is not usually linear but it still has some interesting properties.

Theorem:*Let V be a generalized space-time model. Assume that*

the subspace S is a strictly convex, smooth normed space

with respect to the norm arisen from the s.i.i.p.. Then the s.i.p. space $\{V, [\cdot,\cdot]^-\}$ is also smooth and strictly convex. Let F^T be the generalized adjoint *of the linear mapping F with respect to the s.i.p. space $\{V, [\cdot,\cdot]^-\}$, and define the involutive linear mapping $J : V \longrightarrow V$ by the equalities $J|_S = id|_S$, $J|_T = -id|_T$. The map $F|_H = f : H \longrightarrow H$ is a linear isometry of the upper sheet H^+ of H if and only if it is invertible, satisfies the equality:*

$$F^{-1} = JF^T J,$$

moreover takes e_n into a point of H^+.

Topological isometry

Definition: *Denote by p,q a pair of points in H^+ and consider the set $\Gamma_{p,q}$ of equally oriented piecewise differentiable curves $c(t)$ $a \leq t \leq b$ of H^+ emanating from p and terminating at q. Then the Minkowskian-Finsler distance of these points is:*

$$\rho(p,q) = \inf\left\{\int\limits_a^b \sqrt{[\dot{c}(x),\dot{c}(x)]^+_{c(x)}}dx \text{ for } c \in \Gamma_{p,q}\right\},$$

where $\dot{c}(x)$ means the tangent vector of the curve c in its point $c(x)$. A topological isometry $f : H \longrightarrow H$ of H is a homeomorphism of H which preserves the Minkowski-Finsler distance between each pair of points of H.

$\boxed{\text{Finsler isometry}}$

Definition: *The Minkowski-Finsler semi-metric on H^+ is the function ds^2 which assigns at each point $v \in H^+$ the Minkowski product which is the restriction of the Minkowski product to the tangent space T_v.* $\boxed{\textit{Finsler isometry}}$ *is a diffeomorphism of H onto H which preserves the Minkowski-Finsler semi-metric function.*

Myers-Steenrod:

In a Riemann space $\boxed{\text{topological isometry}} \Leftrightarrow \boxed{\text{Finsler isometry}}$.

Deng-Hou:

In a Finsler space $\boxed{\text{topological isometry}} \Leftrightarrow \boxed{\text{Finsler isometry}}$.

For the hyperboloid model of hyperbolic space embedded in a pseudo-euclidean space

$\boxed{\text{linear isometry}} \Leftrightarrow \boxed{\text{Finsler isometry}} \Leftrightarrow \boxed{\text{topological isometry}}$

We proved on H^+:

linear isometry $\Rightarrow$ Finsler isometry $\Rightarrow$ topological isometry.

moreover if

the subspace S is a strictly convex, smooth normed space,
the group of linear isometries of H^+ acts transitively on H^+

denoting by $d(\cdot, \cdot)$ the Minkowski-Finsler distance of H^+ we have:

$$[a, b]^+ = -ch(d(a, b)) \text{ for } a, b \in H^+.$$