Volume of convex hull of two bodies and related problems

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Geometry and Symmetry, Veszprém, Hungary, 2015

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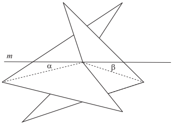
Regular triangles in action

Given two regular triangles with common centre in the 3-dimensional space.



Determine the volume function of the convex hull of the six vertices!

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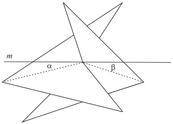
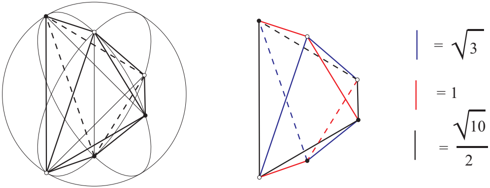


Determine the volume function of the convex hull of the six vertices!

$$v = \frac{r^3}{2} \sin\gamma(\cos\alpha + \cos\beta).$$

Optimal arrangement

Combinatorial (non-regular) octahedron



It is clear that the regular octahedron gives the maximal volume polytope inscribed in the unit sphere with six vertices.

Problems

1. Determine the maximal volume polytopes inscribed in the unit sphere with given number of vertices!

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1. Determine the maximal volume polytopes inscribed in the unit sphere with given number of vertices! (For $n = 6$ the solution is the regular octahedron).
2. Given n points in the unit sphere with prescribed conditions. Determine those arrangements whose convex hull have maximal volume! (For $n = 6$ if the point system is the union of the vertex sets of two regular triangles we exclude the regular octahedron among the possible solutions.)

P. Brass, W. Moser and J. Pach, *Research Problems in Discrete Geometry*, Springer, New York, 2005.

Croft, H. T., Falconer K.J., Guy, R.K., *Unsolved Problems in Geometry*, Vol. 2, Springer, New York, 1991.

László Fejes-Tóth

Fejes-Tóth, L., *Regular Figures*, The Macmillan Company, New York, 1964.

Inequalities of László Fejes-Tóth

Two important results of the genetics of the Platonic solids are contained in the following

Theorem

If V denotes the volume, r the inradius and R the circumradius of a convex polyhedron having f faces, v vertices and e edges, then

$$\frac{e}{3}\sin\frac{\pi f}{e}\left(\tan^2\frac{\pi f}{2e}\tan^2\frac{\pi v}{2e}\right)r^3 \leq V$$

and

$$V \leq \frac{2e}{3}\cos^2\frac{\pi f}{2e}\cot\frac{\pi v}{2e}\left(1-\cot^2\frac{\pi f}{2e}\cot^2\frac{\pi v}{2e}\right)R^3.$$

Equality holds in both inequalities only for regular polyhedra.

a polyhedron with a given number of faces f is always a limiting figure of a trihedral polyhedron with f faces. Similarly, a polyhedron with a given number v of vertices is always the limiting figure of a trigonal polyhedron with v vertices.

Solution of the first cases

$$\omega_n = \frac{n}{n-2}\frac{\pi}{6}$$

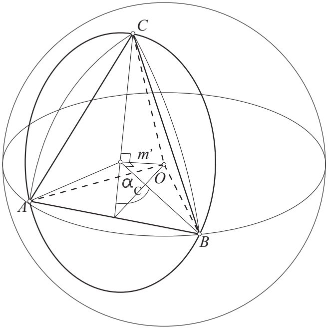
we get

$$(f-2)\sin 2\omega_f \left(3\tan^2\omega_f - 1\right) r^3 \leq V \leq \frac{2\sqrt{3}}{9}(f-2)\cos^2\omega_f \left(3-\cot^2\omega_f\right) R^3$$

$$\frac{\sqrt{3}}{2}(v-2)\left(3\tan^2\omega_v - 1\right) r^3 \leq V \leq \frac{1}{6}(v-2)\cot\omega_v \left(3-\cot^2\omega_v\right) R^3.$$

Equality holds in the first two inequalities only for regular tetrahedron, hexahedron and dodecahedron (f=4, 6, 12) and in the last two inequalities only for the regular tetrahedron, octahedron and icosahedron (v=4, 6, 12).

Spherical and rectilineal triangles, central angles



A generalization of the icosahedron inequality

Let α_A, α_B and α_C denote the resp. angles of the rectilineal triangle ABC. These are the *central angles* of the spherical edges BC, AC and AB, respectively.

Lemma

Let ABC be a triangle inscribed in the unit sphere. Then there is an isosceles triangle $A'B'C'$ inscribed in the unit sphere with the following properties:

- *the greatest central angles and also the spherical areas of the two triangles are equal to each other, respectively;*
- *the volume of the facial tetrahedron with base $A'B'C'$ is greater than or equal to the volume of the facial tetrahedron with base ABC.*

Upper bounds on the volume

Proposition

Let the spherical area of the spherical triangle ABC be τ. Let α_C be the greatest central angle of ABC corresponding to AB. Then the volume V of the Euclidean pyramid with base ABC and apex O holds the inequality

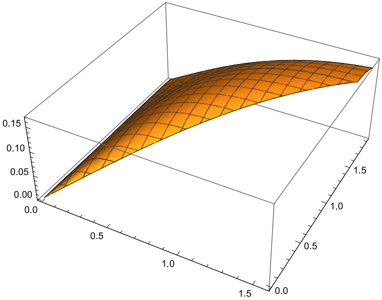
$$V \leq \frac{1}{3}\tan\frac{\tau}{2}\left(2-\frac{|AB|^2}{4}\left(1+\frac{1}{(1+\cos\alpha_C)}\right)\right). \qquad (1)$$

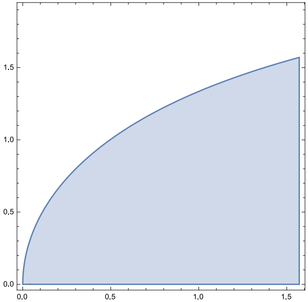
In terms of τ and $c := AB$ we have

$$V \leq v(\tau,c) := \frac{1}{6}\sin c\,\frac{\cos\frac{\tau-c}{2}-\cos\frac{\tau}{2}\cos\frac{c}{2}}{1-\cos\frac{c}{2}\cos\frac{\tau}{2}}. \qquad (2)$$

Equality holds if and only if $|AC| = |CB|$.

Domain of concavity and the function $f(\tau)$





$$\mathcal{D} := \{0 < \tau < \pi/2, \tau \le c < \min\{f(\tau), 2\sin^{-1}\sqrt{2/3}\}\}$$

Theorem

Assume that $0 < \tau_i < \pi/2$ *holds for all* i*. For* $i = 1, \ldots, f'$ *we require the inequalities* $0 < \tau_i \leq c_i \leq \min\{f(\tau_i), 2\sin^{-1}\sqrt{2/3}\}$ *and for all* j *with* $j \geq f'$ *the inequalities* $0 < f(\tau_j) \leq c_j \leq 2\sin^{-1}\sqrt{2/3}$*, respectively. Let denote* $c' := \frac{1}{f'}\sum_{i=1}^{f'} c_i$, $c^\star := \frac{1}{f-f'}\sum_{i=f'+1}^{f} f(\tau_i)$ *and* $\tau' := \sum_{i=f'+1}^{f} \tau_i$*, respectively. Then we have*

$$v(P) \leq \frac{f}{6}\sin\left(\frac{f'c' + (f-f')c^\star}{f}\right) \times$$

$$\times \frac{\cos\left(\frac{4\pi - f'c' - (f-f')c^\star}{2f}\right) - \cos\frac{2\pi}{f}\cos\left(\frac{f'c' + (f-f')c^\star}{2f}\right)}{1 - \cos\frac{4\pi}{2f}\cos\left(\frac{f'c' + (f-f')c^\star}{2f}\right)}.$$

When $f' = f$ we have the following formula:

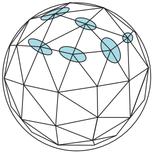
$$v(P) \leq \frac{f}{6} \sin c' \frac{\cos\left(\frac{2\pi}{f} - \frac{c'}{2}\right) - \cos\frac{2\pi}{f}\cos\frac{c'}{2}}{1 - \cos\frac{c'}{2}\cos\frac{2\pi}{f}}, \tag{3}$$

where $c' = \frac{1}{f}\sum_{i=1}^{f} c_i$. In this case the upper bound is sharp if all face-triangles are isosceles ones with the same area and maximal edge lengths. Consider the corresponding triangulation of the sphere. Observe that a polyhedron related to such a tiling, in general, could not be convex.

Problem

Give such values τ and c that the isosceles spherical triangle with area τ and unique maximal edge length c can generate a tiling of the unit sphere.

Local extremity of a point system



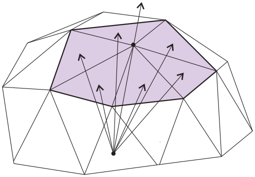
Definition

Let $P \in \mathcal{P}_d(n)$ be a d-polytope with $V(P) = \{p_1, p_2, \ldots, p_n\}$. If for each i, there is an open set $U_i \subset \mathbb{S}^{d-1}$ such that $p_i \in U_i$, and for any $q \in U_i$, we have

$$\mathrm{vol}_d \left(\mathrm{conv}\left(\left(V(P) \setminus \{p_i\}\right) \cup \{q\}\right)\right) \leq \mathrm{vol}_d \left(P\right),$$

then we say that P satisfies *Property Z*.

Main lemma



Lemma

Let P with vertices $p_1, \dots, p_n$ have property Z. Let $C(P)$ be any oriented complex associated with P such that $\mathrm{vol}(C(P)) \geq 0$. *Suppose $s_{12}, \dots, s_{1r}$ are all the edges of $C(P)$ incident with p_1 and that p_2, p_3, p_1; p_3, p_4, p_1; ... ;p_r, p_2, p_1 are orders for faces consistent with the orientation of $C(P)$.*

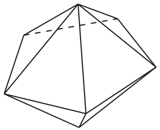
i, *Then $p_1 = m/|m|$ where $m = n_{23} + n_{34} + \cdots + n_{r2}$.*

ii, *Furthermore, each face of P is triangular.*

Optimal configurations ($n \leq 7$)

The maximal volume polyhedron for $n = 4$ is the regular simplex. For $n = 5, 6, 7$ they are so-called double n-pyramids, respectively.

By a *double n-pyramid* (for $n \geq 5$), is meant a complex of n vertices with two vertices of valence $n - 2$ each of which is connected by an edge to each of the remaining $n - 2$ vertices, all of which have valence 4. The $2(n - 2)$ faces of a double n-pyramid are all triangular. A polyhedron P is a *double n-pyramid* provided each of its faces is triangular and some $C(P)$ is a double n-pyramid.



Lemma

If P is a double n-pyramid with property Z then P is unique up to congruence and its volume is $[(n - 2)/3] \sin 2\pi/(n - 2)$.

Optimal configurations (n=8)

For $n = 8$ there exists only two non-isomorphic complexes which have no vertices of valence 3. One of them the double 8-pyramid and the other one has four valence 4 vertices and four valence 5 vertices, and therefore it is the medial complex. It has been shown that if this latter has Property Z then P is uniquely determined up to congruence and its volume is $\sqrt{\left[\frac{475+29\sqrt{145}}{250}\right]}$ giving the maximal volume polyhedron with eight vertices.

Problems

Problem

For which types of polyhedra does Property Z determine a unique polyhedron. More generally, for each isomorphism class of polyhedra is there one and only one polyhedron (up to congruence) which gives a relative maximum for the volume?

Problem

For $n = 4, \ldots, 7$ the duals of the polyhedra of maximum volume are just those polyhedra with n faces circumscribed about the unit sphere of minimum volume. For $n = 8$ the dual of the maximal volume polyhedron (described above) is the best known solution to the isoperimetric problem for polyhedra with 8 faces. Is this true in general?

Berman, J. D., Hanes, K., Volumes of polyhedra inscribed in the unit sphere in E^3, *Math. Ann.* **188** (1970), 78–84.

The results of a computer based search

N : the cardinality of vertices
V : the value of the volumes
F : the number of the faces
degree : the number of that vertices which have a given valence in the polyhedra
E_{min} : the minimal edge lengths of the polyhedron
E_{max} : the maximal edge lengths of the polyhedron

N	V	F	degree	E_{min}	E_{max}	E_{min}/E_{max}
4	0.51320010	4	3×4	1.63261848	1.63335658	0.99954810
5	0.86602375	6	3×2 4×3	1.41273620	1.73244016	0.81546032
6	1.33333036	8	4×6	1.41301062	1.41573098	0.99807848
7	1.58508910	10	4×5 5×2	1.17439900	1.41629677	0.82920403
8	1.81571182	12	4×4 5×4	1.13754324	1.45682579	0.78083684
9	2.04374046	14	4×3 5×6	1.12352943	1.36344511	0.82403716
10	2.21872888	16	4×2 5×8	1.04153932	1.26202346	0.82529315
11	2.35462915	18	4×2 5×8 6×1	0.96536493	1.26366642	0.76393969
12	2.53614471	20	5×12	1.04956370	1.05406113	0.99573324
13	2.61282570	22	4×1 5×10 6×2	0.80234323	1.14003700	0.70378701
14	2.72096433	24	5×12 6×2	0.89290608	1.05849227	0.84356410
15	2.80436840	26	5×12 6×3	0.81809612	1.04523381	0.78269198
16	2.88644378	28	5×12 6×4	0.81890957	0.97608070	0.83897732
17	2.94750699	30	5×12 6×5	0.74657798	1.02119982	0.73107923
18	3.00958510	32	5×12 6×6	0.75499655	0.96805125	0.77991382
19	3.06319073	34	5×12 6×7	0.72816306	0.99849367	0.72926157
20	3.11851200	36	5×12 6×8	0.74113726	0.95901998	0.77280690
21	3.16440426	38	5×12 6×9	0.69438111	0.94733206	0.73298597
22	3.20820707	40	5×12 6×10	0.69345933	0.89581626	0.77410889
23	3.24694072	42	5×12 6×11	0.66928634	0.87244988	0.76713442
24	3.28399413	44	5×12 6×12	0.69163182	0.87601499	0.78952053
25	3.31626151	46	5×12 6×13	0.66118725	0.86554529	0.76389677
26	3.34935826	48	5×12 6×14	0.65046670	0.85140448	0.76399258
27	3.38027449	50	5×12 6×15	0.65839644	0.82328392	0.79971978
28	3.40577470	52	5×12 6×16	0.59817265	0.81912078	0.73026185
29	3.42990751	54	5×12 6×17	0.58296887	0.80547257	0.72376005
30	3.45322727	56	5×12 6×18	0.59082147	0.79788645	0.74048315

Note I

N. Mutoh, The polyhedra of maximal volume inscribed in the unit sphere and of minimal volume circumscribed about the unit sphere, *JCDCG, Lecture Notes in Computer Science* **2866** (2002), 204–214.

Remark

It seems to be that the conjecture of Grace on medial polyhedron is falls because the optimal ones in the cases $n = 11$ *and* $n = 13$ *are not medial ones, respectively.*

Note II

Remark

"Goldberg conjectured that the polyhedron of maximal volume inscribed to the unit sphere and the polyhedron of minimal volume circumscribed about the unit sphere are dual. A comparison of Table 1 and 3 shows that the number of vertices and the number of faces of the two class of polyhedra correspond with each other. The degrees of vertices of the polyhedra of maximal volume inscribed in the unit sphere correspond to the numbers of vertices of faces of the polyhedra of minimal volume circumscribed about the unit sphere. Indeed, the volume of polyhedra whose vertices are the contact points of the unit sphere and the polyhedra circumscribed about the unit sphere differs only by 0.07299% from the volume of the polyhedra inscribed in the unit sphere."

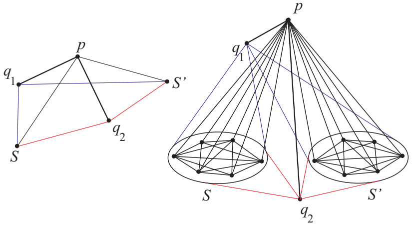
Lemmas

Lemma

Consider a polytope $P \in \mathcal{P}_d(n)$ satisfying Property Z. For any $p \in V(P)$, let $\mathcal{F}_p$ denote the family of the facets of $\mathcal{C}(P)$ containing p. For any $F \in \mathcal{F}_p$, set $A(F,p) = \mathrm{vol}_{d-1}\left(\mathrm{conv}\left((V(F) \cup \{o\}) \setminus \{p\}\right)\right)$, and let $m(F,p)$ be the unit normal vector of the hyperplane, spanned by $(V(F) \cup \{o\}) \setminus \{p\}$, pointing in the direction of the half space containing p.

- *Then we have $p = m/|m|$, where $m = \sum_{F \in \mathcal{F}_p} A(F,p) m(F,p)$.*
- *Furthermore P is simplicial.*

Lemmas



Lemma

Let $P \in \mathcal{P}_d(n)$ satisfy Property Z, and let $p \in V(P)$. Let $q_1, q_2 \in V(P)$ be adjacent to p. Assume that any facet of P containing p contains at least one of q_1 and q_2, and for any $S \subset V(P)$ of cardinality $d-2$, $\operatorname{conv}(S \cup \{p, q_1\})$ is a facet of P not containing q_2 if, and only if $\operatorname{conv}(S \cup \{p, q_2\})$ is a facet of P not containing q_1. Then $|q_1 - p| = |q_2 - p|$.

Results on simplices

Corollary

If $P \in \mathcal{P}_d(d+1)$ and $\mathrm{vol}_d(P) = v_d(d+1)$, then P is a regular simplex inscribed in $\mathbb{S}^{d-1}$.

Böröczky, K., On an extremum property of the regular simplex in $\mathcal{S}^d$. *Colloq. Math. Soc. János Bolyai* **48** Intuitive Geometry, Siófok, 1985, 117–121.

Theorem

The above result is true in spherical geometry, too.

Results on simplices

Haagerup, U., Munkholm, H. J., Simplices of maximal volume in hyperbolic n-space. *Acta. Math.* **147** (1981), 1- 12.

Theorem

In hyperbolic n-space, for $n \geq 2$, a simplex is of maximal volume if and only if it is ideal and regular.

Recent observations:

Proposition

For $d = 2$ a triangle is of maximal area

- *inscribed in the unit circle if and only if it is regular,*
- *inscribed in a hypercycle, if and only if its two vertices are ideal ones.*

There is no triangle inscribed in a paracycle of maximal area.

We note that an cyclic n-gon is of maximal area if and only if it is regular.

n=d+2

Theorem

Let $P \in \mathcal{P}_d(d+2)$ have maximal volume over $\mathcal{P}_d(d+2)$. Then $P = \mathrm{conv}(P_1 \cup P_2)$, where P_1 and P_2 are regular simplices of dimensions $\lfloor \frac{d}{2} \rfloor$ and $\lceil \frac{d}{2} \rceil$, respectively, inscribed in $\mathbb{S}^{d-1}$, and contained in orthogonal linear subspaces of $\mathbb{R}^d$. Furthermore,

$$v_d(d+2) = \frac{1}{d!} \cdot \frac{\left(\lfloor d/2 \rfloor + 1\right)^{\frac{\lfloor d/2 \rfloor + 1}{2}} \cdot \left(\lceil d/2 \rceil + 1\right)^{\frac{\lceil d/2 \rceil + 1}{2}}}{\lfloor d/2 \rfloor^{\frac{\lfloor d/2 \rfloor}{2}} \cdot \lceil d/2 \rceil^{\frac{\lceil d/2 \rceil}{2}}}$$

n=d+3

Theorem

Let $P \in \mathcal{P}_d(d+3)$ satisfy Property Z. If P is even, assume that P is not cyclic. Then $P = \operatorname{conv}\{P_1 \cup P_2 \cup P_3\}$, where P_1, P_2 and P_3 are regular simplices inscribed in $\mathbb{S}^{d-1}$ and contained in three mutually orthogonal linear subspaces of $\mathbb{R}^d$. Furthermore:

- *If d is odd and P has maximal volume over $\mathcal{P}_d(d+3)$, then the dimensions of P_1, P_2 and P_3 are $\lfloor d/3 \rfloor$ or $\lceil d/3 \rceil$. In particular, in this case we have*

$$(v_d(d+3) =) \operatorname{vol}_d(P) = \frac{1}{d!} \cdot \prod_{i=1}^{3} \frac{(k_i+1)^{\frac{k_i+1}{2}}}{k_i^{\frac{k_i}{2}}},$$

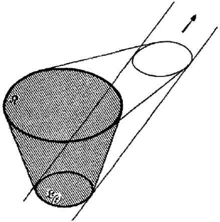
  where $k_1 + k_2 + k_3 = d$ and for every i, we have $k_i \in \left\{ \lfloor \frac{d}{3} \rfloor, \lceil \frac{d}{3} \rceil \right\}$.
- *The same holds if d is even and P has maximal volume over the family of not cyclic elements of $\mathcal{P}_d(d+3)$.*

Problem

Is it true that if $P \in \mathcal{P}_d(d+3)$*, where* d *is even, has volume* $v_d(d+3)$*, then* P *is not cyclic?*

Main lemma on the volume function

I. Fáry & L. Rédey (1950)
C.A. Rogers & G.C. Shephard (1958)
H. Ahn, P.Brass & C. Shin (2008)



Lemma (Main lemma)

The real valued function g of the real variable x defined by the fixed vector t and the formula

$$g(x) := \mathrm{Vol}(\mathrm{conv}(K \cup (K' + t(x))), \text{ where } t(x) := xt,$$

is convex.

Fáry, I., Rédei, L. Der zentralsymmetrische Kern und die zentralsymmetrische Hülle von konvexen Körpern. *Math. Annalen.* **122** (1950), 205-220.

Fáry and Rédey introduced the concepts of inner symmetricity (or outer symmetricity) of a convex body with the ratio (or inverse ratio) of the maximal (or minimal) volumes of the centrally symmetric bodies inscribed in (or circumscribed about) the given body.

Inner symmetricity of a simplex S is

$$c_{\star}(S) = \frac{1}{(n+1)^n} \sum_{0 \leq \nu \leq \frac{n+1}{2}} (-1)^{\nu} \binom{n+1}{\nu} (n+1-2\nu)^n. \tag{4}$$

Outer symmetricity of S is

$$c^{\star}(S) = \frac{1}{\binom{n}{n_0}}, \tag{5}$$

where $n_0 = n/2$ if n is even and $n_0 = (n-1)/2$ if n is an odd number. The above values attain when we consider the volume of the intersection (or the convex hull of the union) of S with its centrally reflected copy S_O.

Definition

For two convex bodies K and L in $\mathbb{R}^n$, let

$$c(K,L) = \max\left\{\mathrm{vol}(\mathrm{conv}(K' \cup L')) : K' \cong K, L' \cong L \text{ and } K' \cap L' \neq \emptyset\right\},$$

where vol denotes n-dimensional Lebesgue measure. Furthermore, if $\mathcal{S}$ is a set of isometries of $\mathbb{R}^n$, we set

$$c(K|\mathcal{S}) = \frac{1}{\mathrm{vol}(K)} \max\left\{\mathrm{vol}(\mathrm{conv}(K \cup K'))\right\}$$

is taken the maximum for those bodies K' for which $K \cap K' \neq \emptyset$ and $K' = \sigma(K)$ for some $\sigma \in \mathcal{S}$.

Rogers, C.A., Shephard G.C., Some extremal problems for convex bodies. *Mathematika* **5/2** (1958), 93–102.

A quantity similar to $c(K, L)$ was defined by Rogers and Shephard, in which congruent copies were replaced by translates. It has been shown that the minimum of $c(K|\mathcal{S})$, taken over the family of convex bodies in $\mathbb{R}^n$, is its value for an n-dimensional Euclidean ball, if $\mathcal{S}$ is the set of translations or that of reflections about a point.

Generalization of the Main Lemma

Definition

Let I be an arbitrary index set, with each member i of which is associated a point a_i in n-dimensional space, and a real number λ_i, where the sets $\{a_i\}_{i\in I}$ and $\{\lambda_i\}_{i\in I}$ are each bounded. If e is a fixed point and t is any real number, $A(t)$ denotes the set of points

$$\{a_i + t\lambda_i e\}_{\imath\in I},$$

and $C(t)$ is the least convex cover of this set of points, then the system of convex sets $C(t)$ is called a *linear parameter system*.

It was proved that the volume $V(t)$ of the set $C(t)$ of a linear parameter system is a convex function of t.

Results of Rogers and Shepard

Theorem

$$1 + \frac{2J_{n-1}}{J_n} \leq \frac{\mathrm{vol}(R^\star K)}{\mathrm{vol}(K)} \leq 2^n,$$

where J_n is the volume of the unit sphere in n-dimensional space, $R^\star K$ is the number to maximize with respect to a point a of K the volumes of the least centrally symmetric convex body with centre a and containing K. Equality holds on the left, if K is an ellipsoid; and on the right, if, and only if, K is a simplex. If K is centrally symmetric, then the upper bound is $1+n$. Equality holds on the left if K is an ellipsoid, and on the right if K is any centrally symmetric double-pyramid on a convex base.

Results of Rogers and Shepard

Theorem

If K is a convex body in n-dimensional space, then

$$1+\frac{2J_{n-1}}{J_n}\leq \frac{\mathrm{vol}(T^{\star}K)}{\mathrm{vol}(K)}\leq 1+n,$$

where $T^{\star}K$ denotes the so-called translation body of K. This is the body for which the volume of $K\cap(K+x)\neq\emptyset$ and the volume of $C(K,K+x)$ is maximal one. Equality holds on the left if K is an ellipsoid, and on the right if K is a simplex.

Conjecture

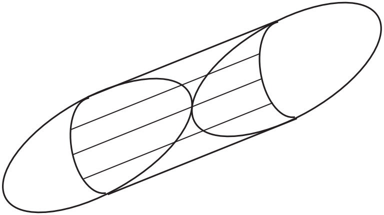
Equality holds on the left if and only if K is an ellipsoid

The volume of the circumscribed cylinder

Then K and $d_K(u)u + K$ touch each other and

$$\frac{\mathrm{vol}(\mathrm{conv}(K \cup (d_K(u)u + K)))}{\mathrm{vol}(K)} = 1 + \frac{d_K(u)\,\mathrm{vol}_{n-1}(K|u^{\perp})}{\mathrm{vol}(K)}. \qquad (6)$$

Clearly, $c^{tr}(K) := \frac{\mathrm{vol}(T^{\star}K)}{\mathrm{vol}(K)}$ is the maximum of this quantity over $u \in \mathbb{S}^{n-1}$.



H. Martini and Z. Mustafaev

Martini, H. and Mustafaev, Z., Some applications of cross-section measures in Minkowski spaces, *Period. Math. Hungar.* **53** (2006), 185-197.

Theorem

For any convex body $K \in \mathcal{K}_n$, there is a direction $u \in \mathbb{S}^{n-1}$ such that, $\frac{d_K(u)\,\mathrm{vol}_{n-1}(K|u^\perp)}{\mathrm{vol}(K)} \geq \frac{2v_{n-1}}{v_n}$, and if for any direction u the two sides are equal, then K is an ellipsoid.

Problem

Characterize those bodies of the d-space for which the quantity on the left is independent from u! (translative constant volume property)

Zsolt Lángi and Á.G.H

Theorem

For any $K \in \mathcal{K}_n$ with $n \geq 2$, we have $c^{tr}(K) \geq 1 + \frac{2v_{n-1}}{v_n}$ with equality if, and only if, K is an ellipsoid.

Theorem

For any plane convex body $K \in \mathcal{K}_2$ the following are equivalent.

(1) *K satisfies the translative constant volume property.*

(2) *The boundary of $\frac{1}{2}(K - K)$ is a Radon curve.*

(3) *K is a body of constant width in a Radon norm.*

Symmetries with respect to r-flats

Theorem

For any $K \in \mathcal{K}_n$ with $n \geq 2$, $c_1(K) \geq 1 + \frac{2v_{n-1}}{v_n}$, with equality if, and only if, K is an ellipsoid.

Theorem

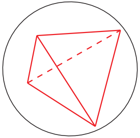
For any $K \in \mathcal{K}_n$ with $n \geq 2$, $c_{n-1}(K) \geq 1 + \frac{2v_{n-1}}{v_n}$, with equality if, and only if, K is a Euclidean ball.

Algorithmic results on the base of the Main Lemma

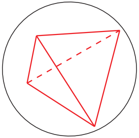
Theorem

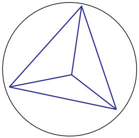
Given two convex polyhedra P and Q in three-dimensional space, we can compute the translation vector t of Q that minimizes $\mathrm{vol}(\mathrm{conv}(P \cup (Q + t)))$ *in expected time $O(n^3 \log^4 n)$. The d-dimensional problem can be solved in expected time $O(n^{d+1-3/d}(logn)^{d+1})$.*

Regular tetrahedra in action

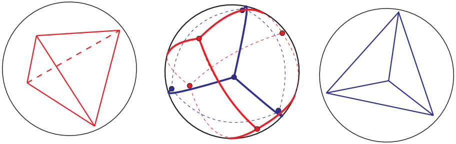


Regular tetrahedra in action

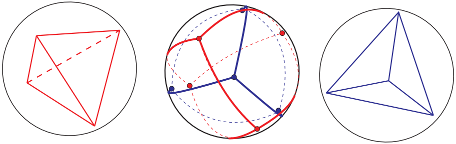




Regular tetrahedra in action



Regular tetrahedra in action



Theorem

The value $v = \frac{8}{3\sqrt{3}} r^3$ *is an upper bound for the volume of the convex hull of two regular tetrahedra are in dual position. It is attained if and only if the eight vertices of the two tetrahedra are the vertices of a cube inscribed in the common circumscribed sphere.*

We can omit the extra condition...

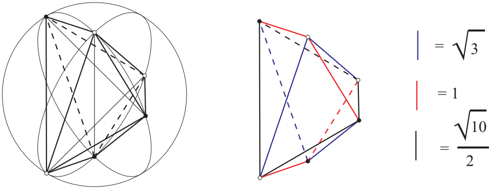
Proposition

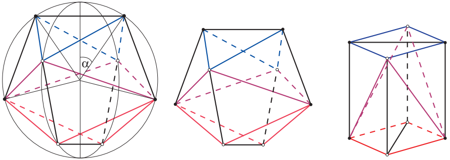
Assume that the closed regular spherical simplices $S(1,2,3)$ *and* $S(4,2,3)$ *contains the vertices* $2', 4'$ *and* $1', 3'$*, respectively. Then the two tetrahedra are the same.*

Theorem

Consider two regular tetrahedra inscribed in the unit sphere. The maximal volume of the convex hull P *of the eight vertices is the volume of the cube* C *inscribed in the unit sphere, so*

$$v(P) \leq v(C) = \frac{8}{3\sqrt{3}}.$$


$= \sqrt{3}$
$= 1$
$= \frac{\sqrt{10}}{2}$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Thank you for your attention!