Random matrices and the geometry of their random eigenstates

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Outline

- Introduction, examples of random matrices in physics

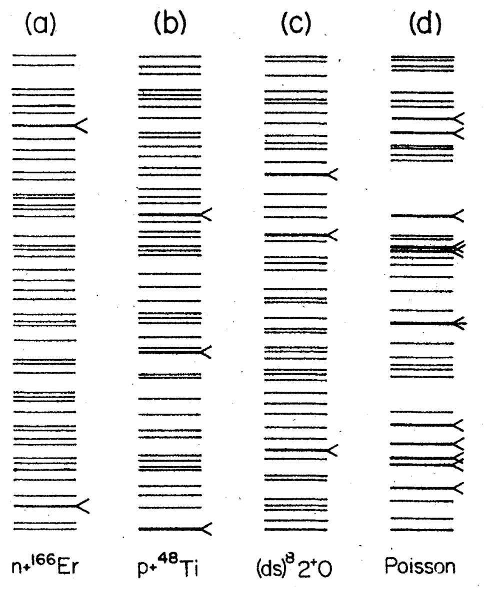
  Wigner's idea and eigenvalue statistics
  Scattering properties and circular ensembles

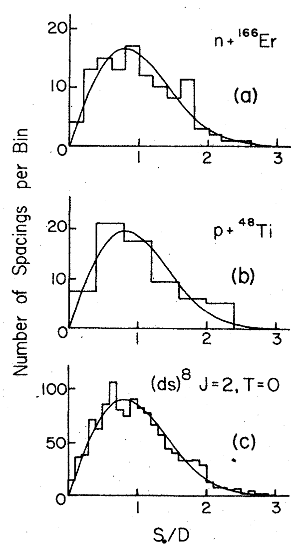
- Random matrix models for quantum work
- Quantum Geometric Tensor and its statistics
- Summary

Introduction: Eigenvalue statistics

Wigner's observation

- Spectrum of nuclei is corrlated, and universal...





Level repulsion !

Atomic spectra

Spectrum of Hydrogen in magnetic field

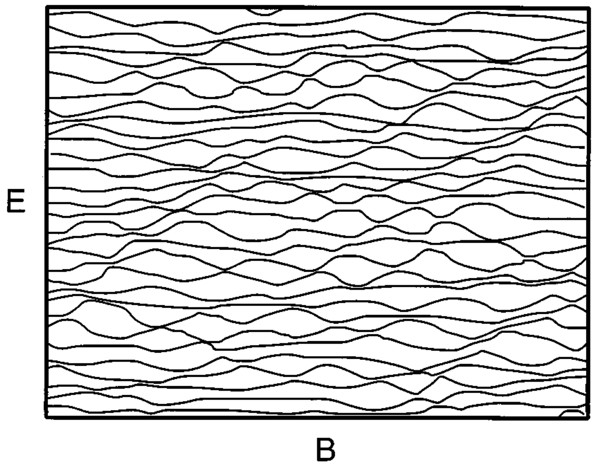
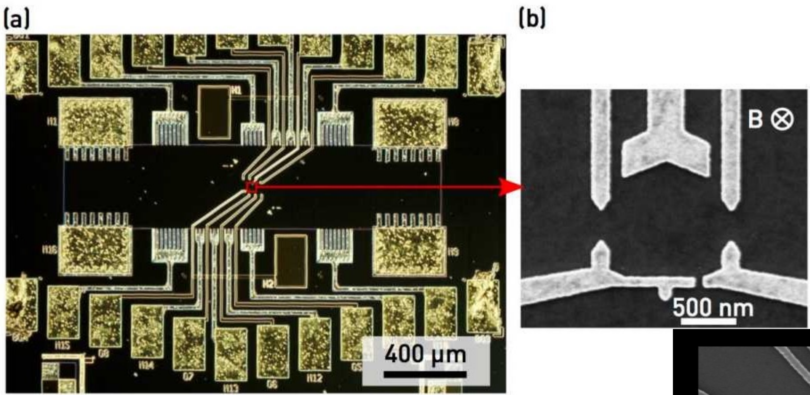
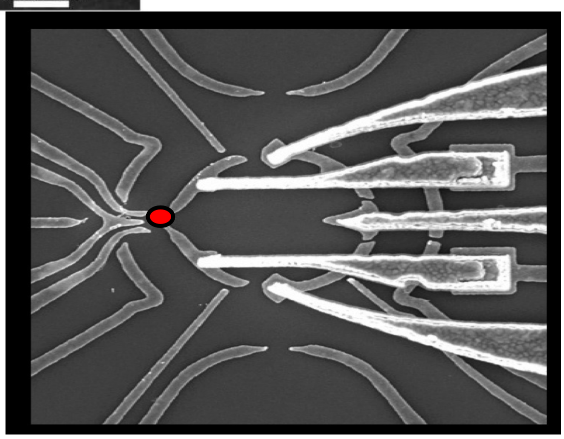


FIG. 2. Illustration of the magnetic-field dependence of energy levels in a chaotic system (magnetic field B and energy E in arbitrary units). This plot is based on a calculation of the spectrum of the hydrogen atom in a strong magnetic field by Goldberg *et al.* (1991).

„Artificial atoms"



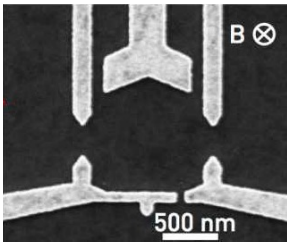
Ennslin group (ETH Zurich)



A. J. Keller, L. Peeters, C. P. Moca, I. Weymann, D. Mahalu, V. Umansky, G.Z., and D. Goldhaber-Gordon, Nature 526, 237 (2015)

Sensitivity of eigenvalues

- Accidental degeneracies are rare...



Hamilton operator

$$H = -\frac{\hbar^2 \Delta_{2D}}{2\,m} + U(x,y)$$

Spectrum $\quad H\,\psi_i = E_i\,\psi_i(x,y)$

$\Rightarrow \quad E_0,\ E_1,\ E_2,\ \dots$ (levels)

Levels are **sensitive** to external voltages and potentials

$$E_2 = E_2(V_1, V_2, \dots, B)$$

Rare degeneracies

degeneracies are rare

Assume $\quad E_k = E_{k+1} = E$

In appropriate basis

$$H \rightarrow \{H_{ij}\} = \begin{array}{cc} & \begin{array}{cc} k & k+1 \end{array} \\ & \begin{bmatrix} \ddots & & & \\ & E & & \\ & & E & \\ & & & \ddots \end{bmatrix} \end{array}$$

change, e.g. $\quad V_1 \rightarrow V_1'$

$$\Rightarrow \begin{bmatrix} \ddots & & & \\ & E+a & c & \\ & c^* & E+b & \\ & & & \ddots \end{bmatrix}$$

$a = a(V_1, V_2, \dots)$
$c = c(V_1, V_2, \dots)$
$b = b(V_1, V_2, \dots)$

To see degeneracy, fine-tuning of parameters needed !

Wigner's idea

- Hamilton matrix is a random Hermitian matrix
- Universality ⟺ symmetries matter only
- Basis is irrelevant for spectrum !

➡ **Distribution :** $d\mu(H) \cdot P(H)$

measure in
space of Hamiltonian matrices

Basic symmetry classes

Time reversal symmetry $(B = \emptyset)$

- $\exists$ basis where H_{ij} is **real**
- Spectrum is basis invariant:

$$d\mu(H) = d\mu(O H O^T)$$ with O any **orthogonal** matrix

➡ $$d\mu(H) = \prod_{i \leq j} dH_{ij}$$

- $P(H)$ is not essential (only sets energy scale)

➡ $$P(H) \rightarrow e^{-\frac{N}{4} \operatorname{tr}(H^2)}$$

Gaussian **O**rthogonal **E**nsemble = **GOE**

Basic symmetry classes

- No time reversal symmetry $(B \neq 0)$

H_{ij} Hermitian

$$d\mu(H) = d\mu(U^{+} H U)$$

with any **unitary** matrix

Gaussian **U**nitary **E**nsemble = **GUE**

- Time reversal symmetry 2

$$d\mu(H) = d\mu(S^{+} H S)$$

with any **symplectic** matrix

Gaussian **S**ymplectic **E**nsemble = **GSE**

Joint eigenvalue probability density

$$p(\{E_i\}) \sim \prod_{i<j} |E_i - E_j|^{\beta} \, e^{-\frac{N\beta}{4} \sum_{i=1}^{N} E_i^2}$$

- GOE : $\beta = 1$
- GUE : $\beta = 2$
- GSE : $\beta = 4$

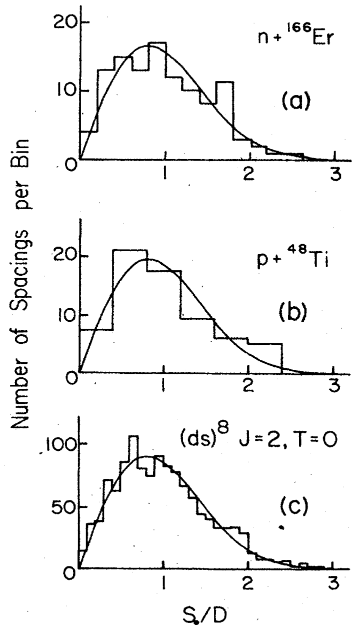
Ramark: level repulsion in prefactor...

Wigner surmise

Two level's separation:

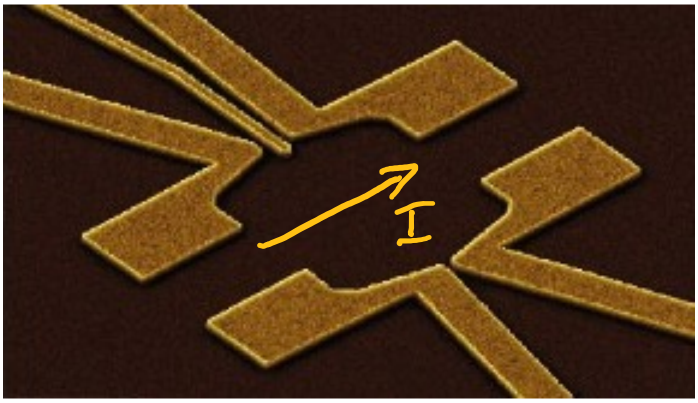
Wigner surmise:

$$P(s) \sim s^{\beta} \cdot e^{-cst \cdot s^2}$$



Scattering: Circular ensembles

Scattering problem

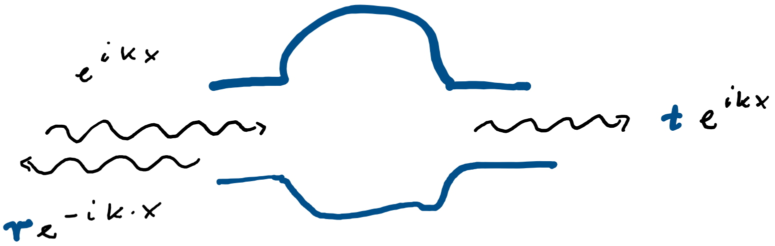


GaAs/AlGaAs

Nanophysics Group, ETH-Zurich

$$G = \frac{I}{V} = ?$$

Scattering problem



transmitted wave

reflected

Scattering matrix

$$S = \begin{bmatrix} r & t' \\ t & r' \end{bmatrix}$$

S random unitary : $S S^{+} = 1$

Scattering problem

Conductance (Landauer-Büttiker)

$$G = \frac{e^2}{h} \mathrm{tr}\{ t\, t^+ \}$$

Dimensionless conductance: $T = \mathrm{tr}\, \{t\, t^+\}$

Symmetry classes

$$\text{Circular} \left\{ \begin{array}{l} \text{Orthogonal} \\ \text{Unitary} \\ \text{Symplectic} \end{array} \right\} \text{Ensemble}$$

Some interesting results

Average conductance *N* channels in, *N* channels out

CUE	$\langle T \rangle = \frac{N}{2}$	**random**
COE	$\langle T \rangle = \frac{N}{2} - \frac{N}{4N+2}$	**„localization"**
CSE	$\langle T \rangle = \frac{N}{2} + \frac{N}{2N-2}$	**„anti-localization"**

Conductance distribution of a device

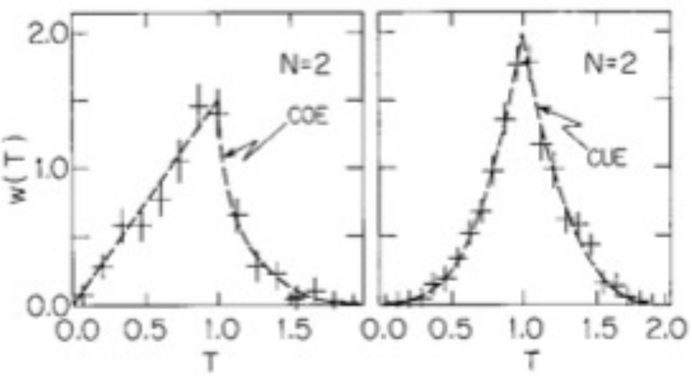
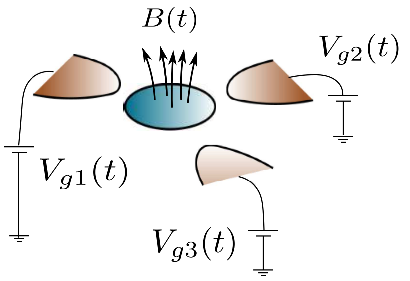


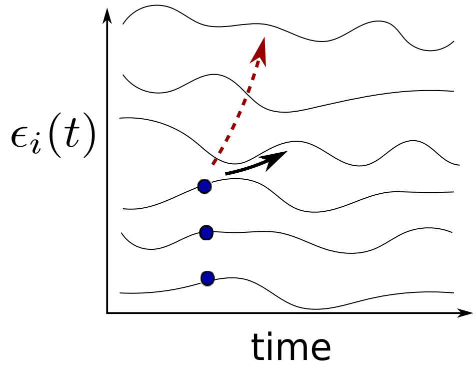
Fig. 7. Distribution of T for $N = 2$: the same numerical simulation of the previous figures is contrasted with the theoretical prediction arising from COE and CUE.

Random matrix models for quantum work

Theory of quantum work

Some nanocircuit...





Questions

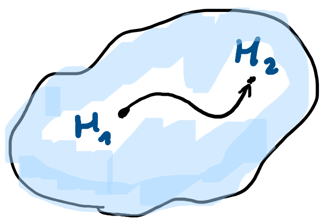
- Role of avoided level crossings ?

  ➡ Landau-Zener transitions

- Full energy distribution after „quench" ?
- Probability of adiabatic evolution ?

Random Matrix Setup

Motion in random matrix ensemble



$$H(t) = H_1 \cos(\lambda(t)) + H_2 \sin(\lambda(t))$$

from GUE / GOE / GSE

Time dependent dynamics

$$i\,\partial_t \varphi(t) = H(t)\,\varphi(t)$$

➡ determinant formula for $P(W) = P(\,E_{\text{final}} - E_{\text{initial}}\,)$

I. Lovas, A.Grabarits, M. Kormos, and G.Z., Phys. Rev. Research, **2** 2643 (2020);
, A.Grabarits, I. Lovas, M. Kormos, and G.Z., submitted (2021).

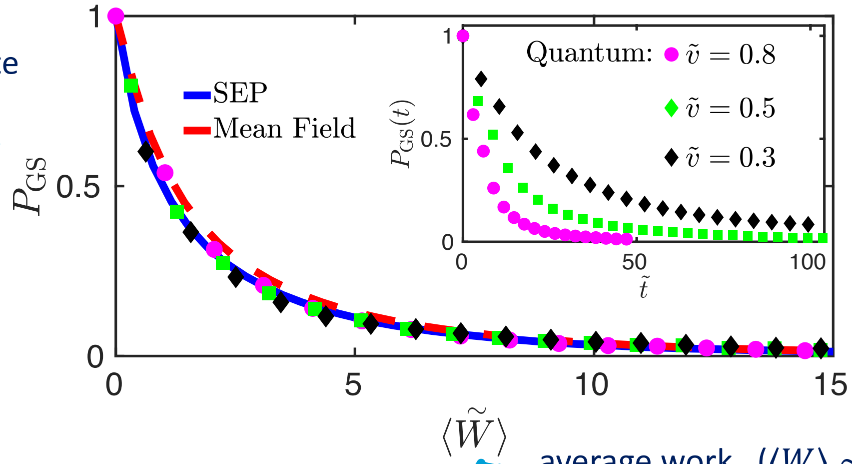
Probability of adiabaticity

$$P_{GS}(t) \approx \frac{1}{(8\pi\, \widetilde{D}\, \tilde{t})^{1/4}}\, e^{-c\sqrt{\widetilde{D}\,\tilde{t}}}$$

$$\langle\, \widetilde{W}\, \rangle \;=\; \widetilde{D}\, \tilde{t}$$

diffusion constant in energy space

Probability to stay in ground state



average work $(\langle W \rangle \sim t\,)$

Quantum geometric tensor of random eigenstates

Fubini-Study metric

Distance of states in Hilbert space?

Consider smooth family of (normalized) states in Hilbert space
(e.g., ground state as a function of external parameters)

$$|\varphi(\underline{\lambda})\rangle \qquad \underline{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_P)$$

Distance (Fubini-Study):

$$ds^2 = 1 - \left|\langle\varphi(\underline{\lambda} + d\underline{\lambda}\,|\varphi(\underline{\lambda})\rangle\right|^2 = d\lambda_\alpha d\lambda_\beta \, Re \, g_{\alpha\beta}(\underline{\lambda})$$

Quantum Geometric Tensor (gauge invariant)

$$g_{\alpha\beta}(\underline{\lambda}) = \langle\partial_\alpha \, \varphi|\partial_\beta \varphi\rangle - \langle\partial_\alpha \, \varphi|\varphi\rangle\langle\varphi|\partial_\beta \varphi\rangle$$

Physical content of Quantum Geometric Tensor

Quantum Geometric Tensor (gauge invariant)

- Antisymmetric part: **Berry curvature**

  Berry connection $\quad A_\alpha(\underline{\lambda}) = i\langle\varphi|\partial_\alpha \, \varphi\rangle$

  $$\Rightarrow \quad B_{\alpha\beta}(\underline{\lambda}) = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

- Diagonal part:

  Fidelity susceptibility

  measures sensitivity of a quantum state

  Conductance

Random Matrix theory of QGT

Consider family of random Hamiltonians (tuned by external nobs)

$H(\underline{\lambda})$ ➡ eigenstates $|\varphi_n(\underline{\lambda})\rangle$ $E_n(\underline{\lambda})$

Each level is characterized by some random QGT : $g^{(n)}_{\alpha\beta}(\underline{\lambda})$

what is $P(\underline{g}) = ?$

RMT approach (Berry and Shukla, (2020))

$$H(x, y) = H_0 + x\,H_x + y\,H_y$$

- $H_{0,x,y}$ random matrices
- (x, y) two-parameter manyfold

Supersymmetric calculation for GUE

Exact distribution (A. Penner, F. von Oppen, G.Z., M. Zirnbauer)

In large *N* limit, using supersymmetry approaches

$$P(g) = \int \frac{d\alpha_0}{2\pi}\frac{d\boldsymbol{\alpha}}{(2\pi)^3} e^{i[\alpha_0 g_0 + \boldsymbol{\alpha}\cdot\mathbf{g}]}\tilde{P}(\alpha_0, \boldsymbol{\alpha})$$

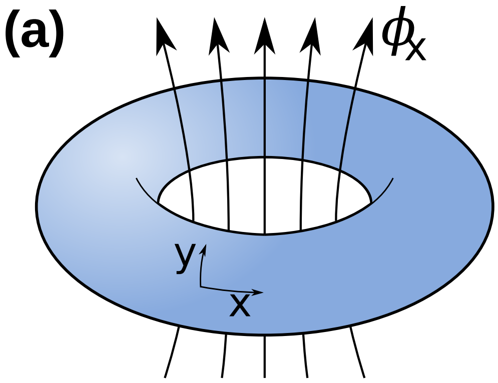
$$\tilde{P}(\alpha_0, \boldsymbol{\alpha}) = r(\tfrac{1}{2}\sqrt{\gamma\alpha_+}, \tfrac{1}{2}\sqrt{\gamma\alpha_-})e^{-\frac{1-i}{2}(\sqrt{\gamma\alpha_+}+\sqrt{\gamma\alpha_-})}$$

$$r(a,b) = 1 + (1-i)(a+b) - \frac{2i}{3}(a^2+3ab+b^2) - \frac{1+i}{12}\frac{a^4+9a^3b+17a^2b^2+9ab^3+b^4}{a+b}$$

$$-\frac{1}{30}ab(5a^2+16ab+5b^2) + \frac{1-i}{180}\frac{a^2b^2(13a2+29ab+13b^2)}{a+b} + \frac{i}{30}a^3b^3 + \frac{1+i}{240}\frac{a^4b^4}{a+b} + \frac{1}{2160}\frac{a^5b^5}{(a+b)^2}$$

Comparison with real systems…

disordered random flux model



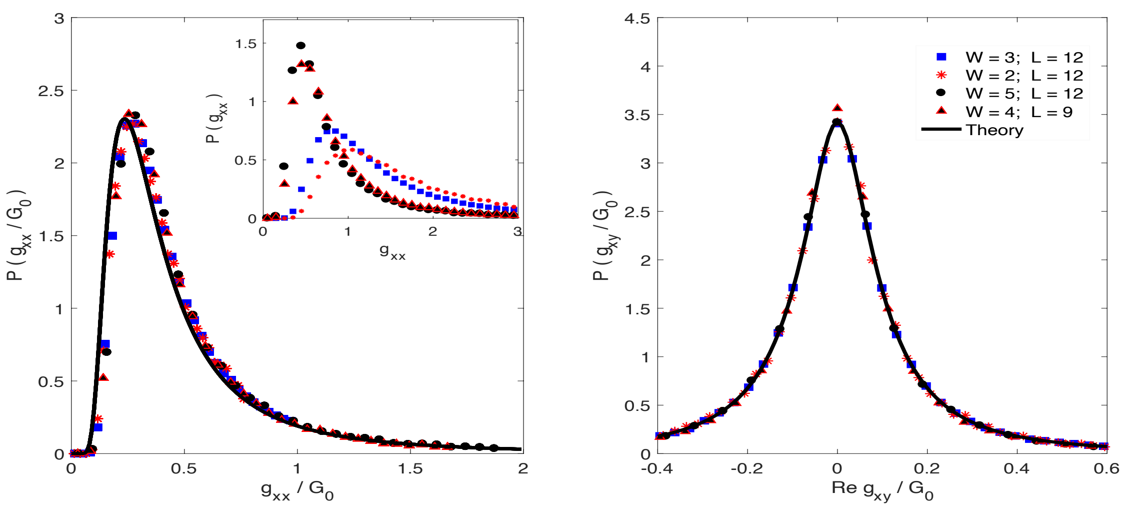
Random hopping of electrons on surface

$$\underline{\lambda} = (\Phi_x, \Phi_y)$$

$$H = \sum_i \epsilon_i \; c_i^+ c_i + \sum_{i<j} t_{ij} \; c_i^+ c_j + h.c.$$

Comparison with real systems…

3-dimensional disordered random flux model (GUE)



L = 12 (12 x 12 x 12 system)

Some references

Random matrices provide a fascinatig framework to study generic properties of random, chaotic and/or interacting **complex systems**

Some interesting applications

- Spectral properties, scattering problems in nanostructures
- Motion under **deformation**

➡ Quantum work statistics

➡ Quantum geometrical tensor

➡ Topological structure of degeneracies

Thank you for your attention !