CONVERGENCE PROBLEMS OF ORTHOGONAL SERIES

by

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PREFACE TO THE ENGLISH EDITION

The text of this book is an improved and extended version of the German original, for, since the issue of the latter many interesting results were published which I have thought necessary to include in the text. At the same time, I have corrected some errors and misprints of the original text. In this task I have derived much assistance from the valuable remarks of ÁKOS CSÁSZÁR and GÉZA FREUD. I am particularly indebted to KÁROLY TANDORI for having revised the complete English text and read its proof-sheets. Finally, I express my gratitude to ISTVÁN FÖLDES for the careful translation.

PREFACE

The questions of convergence and summation of the general orthogonal series forms perhaps the most impressive domain of application of the Lebesgue or of the Stieltjes—Lebesgue concept of integral, respectively. Many methods of inquiry owe their discovery to the investigation of this sphere of problems. In spite of their great generality, some of the results obtained provide wider knowledge of the convergence features than the remaining theorems, shaped specially to the expansion in question, even in case of applications to classical orthogonal expansions. Thus, for instance, the MENCHOFF—RADEMACHER convergence theorem for general orthogonal series ensures the convergence almost everywhere of certain Fourier series with irregularly distributed lacunarities, while the special theorems achieved up to the present time on the convergence of the Fourier series are unable to answer this question in such cases. Furthermore, the convergence problems of the orthogonal series are very tightly bound up with some other branches of Analysis, especially with probability theory. It may even be stated that a set of theorems from the theory of orthogonal series and from the theory of probability are basically only bilingual terms for the same mathematical fact.

The large range and the depth of the convergence theory of the orthogonal series justifies a systematic treatment of this theory. Although a programme of such a kind was excellently carried out in the well-known book of KACZMARZ and STEINHAUS, the zeal of the mathematicians has opened a way to new, beautiful and important discoveries during the 25 years which have elapsed since its publication. In view of this circumstance, it seems to me reasonable to hope that this book will not call forth the sentiment that it is superfluous.

I have attempted to represent the actual state of the theory of convergence and summation of the general orthogonal series with hints as to the connexion of the general theory with the corresponding questions of the classical expansions. On the other hand, however, I have not dealt with other important ranges of ideas, unrelated to questions of convergence, as for instance those connected with the theorem of YOUNG—HAUSDORFF or the theorem of PALEY.

I endeavoured to formulate the text in such a way that any reader, acquainted with the most important facts from the theory of functions of a real variable and from the theory of the Fourier series, will find all the rest completely proved in this book, excepting only the parts printed in smaller type, containing various topics: theorems with complete or with only sketched proof or even without proof, references to unsolved problems, remarks for the better classification of the main text, etc. I have also striven to give a hint of the origin of the several theorems, not only by indicating the place where the theorem in question has been formulated for the first time in its most general form, but frequently also by referring to the older literature in which the fundamental idea of the proof has first appeared.

I am very much indebted, relative to the form as well as to the content, to the monograph of KACZMARZ and STEINHAUS, to the book of SZEGŐ on orthogonal polynomials and to the well-known work of ZYGMUND on trigonometrical series whose new, greatly extended edition, however, could unfortunately not actually be utilized for this book. The appendix of GUTER and ULIANOFF provided for the Russian translation of the book of KACZMARZ and STEINHAUS (Moscow, 1958) has likewise been very useful for our purposes.

I avail myself of the opportunity to express my deepest gratitude to my colleagues B. SZ.-NAGY and K. TANDORI for their generous help and very valuable comments during the writing of this book. Their remarks, advice and assistance have contributed appreciably to the improvement of the text. I have also to thank to the publishing house of the Hungarian Academy of Sciences, as well as to the printing office of Szeged, for their careful production of the book.

Budapest

G. ALEXITS.