Topology (TOP) / Alex Küronya / Spring 2010

Homework 10

Due date: May 21st

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. * (Functorial properties of f_*) Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ be continuous maps. Show that $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is a homomorphism of groups, and

$$(g \circ f)_* = g_* \circ f_* \ .$$

Moreover, verify that the identity map it induces the identity homomorphism on the fundamental groups.

2. Prove that if $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

is an isomorphism.

3. ** Show that if X is a path-connected topological space, $h : X \to Y$ a continuous map, then h_* is independent of the base points chosen (up to appropriate isomorphisms). (Note: Your first task is to write it down precisely what it means for f_* to be independent of base points.)

4. * Prove that for any topological group G, the fundamental group $\pi_1(G, 1)$ is abelian.

5. Decide if the following statements are true:
   (1) If X is a discrete topological space with n elements (n a positive integer), $x_0 \in X$, then $\pi_1(X, x_0)$ is isomorphic to the cyclic group with n elements.
   (2) The fundamental group of a trivial topological space at an arbitrary base-point is trivial.

6. Let X be a path-connected topological space, $x_0, x_0' \in X$, h and h' two paths from x_0 to x_0'. Show that the $\tau_h = \tau_{h'} : \pi_1(X, x_0') \to \pi_1(X, x_0)$ whenever h is path-homotopic to h'.

7. ** Let $f_0, f_1 : (X, x_0) \to (Y, y_0)$ continuous maps homotopic via $F : X \times I \to Y$, denote

$$\begin{array}{rcl} \gamma : I & \longrightarrow & Y \\ t & \mapsto & F(x_0, t) \ . \end{array}$$

Prove that

$$(f_1)_* = \tau_\gamma \circ (f_0)_* \ .$$

As a consequence show that $(f_1)_*$ is injective/surjective/trivial whenever $(f_0)_*$ is.

8. Let $f : X \to Y$ be a map, which is homotopic to a constant map. Show that f_* is the trivial homomorphism.

9. (Homotopy invariance of the fundamental group) Let $f : (X, x_0) \to (Y, y_0)$ be a homotopy equivalence. Verify that $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.