Topology (TOP) / Spring 2010 / Alex Küronya

Homework 2

Due date: March 2nd

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Let (X,τ) be a topological space, $A \subseteq B \subseteq X$ subsets. Show that $\tau|_A = (\tau|_B)|_A$. Prove in addition that if A is closed in B and B is closed in X, then A is also closed in X.

2. * Let X,Y be topological space, $f : X \to Y$ a function. Show that f is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset $A \subseteq X$.

3. Prove the following statements.
   (1) If $f : X \to Y$ is a map, $A \subseteq X$ a subspace, then
   $$f|_A : A \longrightarrow Y$$
   given by $f|_A(x) = f(x)$ whenever $x \in A$, is a continuous function.
   (2) Let X,Y be topological space, $Y_1 \subseteq Y_2 \subseteq Y$ subspaces, $f : X \to Y_2$ a continuous function. Then f as a function $X \to Y$ is also continuous. If $f(X) \subseteq Y_1$, then f as a function from X to Y_1 is continuous as well.

4. Let X be a topological space, Y a metric space, $f_n : X \to Y$ a sequence of functions. We say that f_n *converges uniformly to a function* $f : X \to Y$, if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that
$$d(f_n(x), f(x)) < \epsilon$$
for every $x \in X$ whenever $n \geq N_\epsilon$.
   Prove that the limit function of a uniformly convergent sequence of continuous functions is also continuous.

5. Let $f : X \to Y$ be a function between topological spaces, $\mathcal{S}$ a subbasis for Y. Show that f is continuous if and only if for every $U \in \mathcal{S}$ the set $f^{-1}(U)$ is open in X.

6. Let (X,d) be a metric space, $A \subseteq X$ an arbitrary subset. Show that $\overline{A}$ = the set of limit points of convergent sequences of points in X.

7. Let X be an arbitrary topological space, $A,B \subseteq X$ subsets.
   (1) Show that $\mathrm{int}(A) = \{ x \in X \mid \exists U \subseteq X \text{ open for which } x \in U \subseteq A \}$.
   (2) Prove that $\overline{A} = \{ x \in X \mid \forall U \subseteq X \text{ open with } x \in U : U \cap A \neq \emptyset \}$.
   (3) Show that A is open if and only if $A = \mathrm{int}(A)$, and A is closed if and only in $\overline{A} = A$. Verify furthermore that $X - \mathrm{int}(A) = \overline{X - A}$, and $X - \overline{A} = \mathrm{int}(X - A)$.
   (4) Prove the following identities: $\mathrm{int}(A) \cap \mathrm{int}(B) = \mathrm{int}(A \cap B)$, $\overline{A} \cup \overline{B} = \overline{A \cup B}$.
   (5) Verify that
   $$\bigcap_{\alpha \in I} \mathrm{int}(A_\alpha) \supseteq \mathrm{int}\left(\bigcap_{\alpha \in I} A_\alpha\right) = \mathrm{int}\left(\bigcap_{\alpha \in I} \mathrm{int}(A_\alpha)\right) \quad , \quad \bigcup_{\alpha \in I} \mathrm{int}(A_\alpha) \subseteq \mathrm{int}\left(\bigcup_{\alpha \in I} A_\alpha\right) .$$
   (6) Show that $A \subseteq B$ implies $\mathrm{int}(A) \subseteq \mathrm{int}(B)$, and $\overline{A} \subseteq \overline{B}$.

8. Consider the topology on $\mathbb{R}$ generated by the sets $[x,y)$ and $(x,y]$, for all $x,y \in \mathbb{R}$. Prove that it coincides with the discrete topology.

9. For a topological space X and an arbitrary subset $A \subseteq X$ one has
$$X = \mathrm{int}(A) \amalg \partial A \amalg (X - \overline{A}) .$$

10. * Let $A \subseteq X$ be a nowhere dense set. What can we say about $X - A$? Show that a finite union of nowhere dense sets is again nowhere dense.

11. Let (X,d) be a metric space. Show that X is second countable if and only if it has a countable dense subset.

12. ** How many pairwise non-homeomorphic topologies are there on a three-element set?