Topology (TOP) / Alex Küronya / Spring 2010

Homework 6

Due date: April 6th

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. ** Let X be a non-empty compact Hausdorff space with no isolated points. Show that X must be uncountable.

2. Prove that a connected metric space is either uncountable or has at most one point.

Definition. Let X be a topological space, $x \in X$. We say that X is *locally compact at x*, if x has a compact neighbourhood. The space X is called *locally compact* if it is locally compact at every one of its points.

3. Check that X is locally compact at x if and only if x has a compact neighbourhood basis.

4. * Prove that $\mathbb{R}^n$ is locally compact, but $\mathbb{Q}$ is not.

5. Decide whether the continuous image of a locally compact space is locally compact as well.

6. Let $f : X \to Y$ be a closed surjective map. Prove that X is normal whenever Y is.

7. Let (X, τ) be a regular topological space. Show that every pair of points have open neighbourhoods whose closures are disjoint.

8. Check that a closed subspace of a normal topological space is again normal.

9. * Prove that a locally compact Hausdorff space is regular.

10. (Urysohn's Lemma[1] for metric spaces) Let (X, d) be a metric space, $A, B \subseteq X$ disjoint closed subsets. Prove that there exists a continuous map

$$f : X \longrightarrow [0, 1]$$

for which $f|_A \equiv 0$ and $f|_B \equiv 1$.

11. Let (X, d) be a metric space, $\mathcal{C}$ the set of Cauchy sequences in X.

(1) For $(x_n), (y_n) \in \mathcal{C}$, set $(x_n) \sim (y_n)$ if

$$\lim_{n\to\infty} d(x_n, y_n) = 0 .$$

Show that $\sim$ is an equivalence relation.

(2) Let $\tilde{X}$ be the collection of equivalence classes of $\sim$. Verify that the function

$$\begin{aligned} \tilde{d} : \tilde{X} \times \tilde{X} &\longrightarrow \mathbb{R}_{\geq 0} \\ ([(x_n)], [(y_n)]) &\mapsto \lim_{n\to\infty} d(x_n, y_n) \end{aligned}$$

is a distance function on $\tilde{X}$, thus making $(\tilde{X}, \tilde{d})$ into a metric space.

(3) Prove that the function $i : (X, d) \to (\tilde{X}, \tilde{d})$ mapping $x \in X$ to the class of the constant sequence (x) is injective, preserves the distance, and has a dense image.

(4) Show that every Cauchy sequence in $\tilde{X}$ converges (that is, $(\tilde{X}, \tilde{d})$ is *complete*).

The metric space constructed above is the *completion of* (X, d).

[1] Urysohn's lemma is a very important non-trivial result in topology. It holds for all normal topological spaces.