Topology (TOP) / Alex Küronya / Spring 2010

# Homework 9

**Due date: May 4th**

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Which of the following topological properties are preserved under homotopy equivalence of topological spaces (i.e. if $X \simeq Y$ and $X$ has the property in question then does $Y$ have it automatically?): compactness, connectedness, path-connectedness, first countability, second countability, $T_0$, $T_1$, $T_2$, $T_3$, $T_4$?

2. Which of the following properties are preserved under homotopy of maps: injective, surjective, open, closed, perfect, proper?

3. Prove or disprove the following statements.

(1) If $X$ is a trivial topological space, $x_0 \in X$, then any two loops based at $x_0$ are homotopic to each other.
(2) Let $X$ be a discrete topological space with $n$ points. Then for any $x \in X$, there are exactly $n$ distinct homotopy classes of loops based at $x$.
(3) If $X$ is a contractible space, $X \simeq Y$, then $Y$ is contractible as well.
(4) If $X$ is contractible, then every map $f : Y \to X$ is homotopic to a constant map.
(5) If $X$ is a connected topological space, $Y \simeq X$, then $Y$ is connected as well.

4.* Let $G$ be a topological group, $f, g : (I, \partial I) :\to (G, 1_G)$ loops. Let $f \cdot g$ denote the pointwise product

$$t \mapsto f(t)g(t) \quad t \in [0,1]$$

of the loops $f$ and $g$. Show that $f \cdot g$ is a loop as well, and $f \cdot g \simeq f * g$.

5. Let $f, g : X \to \mathbb{S}^n$ be two maps with the property that $f(x) \neq -g(x)$ for every $x \in X$. Prove that $f \simeq g$.

6.* Let $G$ be a topological group, $H \leqslant G$ a subgoup. Set

$$G/H \stackrel{\text{def}}{=} \{xH \mid x \in G\}$$

the set of left cosets with respect to $H$. Consider the surjective function

$$\begin{aligned} q : G &\longrightarrow G/H \\ x &\mapsto xH\ . \end{aligned}$$

We give $G/H$ the quotient topology with respect to $q$. Show that $q$ is open and that $G/H$ is a homogeneous space.

7. ** Let $H \lhd G$ be a normal subgroup in the topological group $G$. Prove that the set $G/H$ equipped with the quotient topology and the quotient group structure is a topological group.

8. Let $G$ be a topological group, $\mathcal{N} \subseteq \tau_G$ an open neighbourhood basis of $1_G \in G$. Verify the following claims.

(1) For every $g \in G$, $\mathcal{N}(g) \stackrel{\text{def}}{=} \{xU \mid U \in \mathcal{N}\}$ is a neighbourhood basis of $g$.
(2) For every $U, V \in \mathcal{N}$ there exists $W \in \mathcal{N}$ for which $W \subseteq U \cap V$.
(3) For every $U \in \mathcal{N}$ and $g \in U$ there is an element $V \in \mathcal{N}$ such that $aV \subseteq U$.
(4) For every $U \in \mathcal{N}$ there is $V \in \mathcal{N}$ for which $V^{-1}V \subseteq U$.
(5) For every $U \in \mathcal{N}$, $g \in G$, there exists $V \in \mathcal{N}$ with the property that $g^{-1}Vg \subseteq U$.

9. (Properties of Hausdorff topological groups) Let $G$ be a Hausdorff topological group. Prove the following[1].

(1) $\Delta : X \to X \times X$ (taking $g$ to $(g, g)$) has a closed image.
(2) If $\phi : H \to G$ is a morphism of topological groups, then $\ker \phi \subseteq H$ is closed.
(3) $\{1_G\} \subseteq G$ is closed.
(4) $G$ is $T_1$.
(5) The intersection of all open neighbourhoods of $1_G$ equals $\{1_G\}$.

10. ** Prove that if the intersection of all open neighbourhoods of $1_G$ in the topological group $G$ is equal to $\{1_G\}$, then $G$ is Hausdorff.

[1]As it can be shown with a little more work, all of these are in fact equivalent to $G$ being Hausdorff.