Derivations and differential operators on rings and fields

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Abstract

Let R be an integral domain of characteristic zero. We prove that a function $D\colon R \to R$ is a derivation of order n if and only if D belongs to the closure of the set of differential operators of degree n in the product topology of R^R, where the image space is endowed with the discrete topology. In other words, f is a derivation of order n if and only if, for every finite set $F \subset R$, there is a differential operator D of degree n such that $f = D$ on F. We also prove that if $d_1, \dots, d_n$ are nonzero derivations on R, then $d_1 \circ \dots \circ d_n$ is a derivation of exact order n.

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1 Introduction and main results

By a ring we mean a commutative ring with unit. An integral domain is a ring with no zero-divisors other than 0. The ring R has characteristic zero if $n \cdot x \neq 0$ for every $x \in R \setminus \{0\}$ and for every positive integer n.

A *derivation* on a ring R is a map $d : R \to R$ such that

$$d(x+y) = d(x) + d(y) \qquad \text{and} \qquad d(xy) = d(x)y + d(y)x \tag{1}$$

for every $x, y \in R$. Derivations of higher order are defined by induction as follows.

Let R be a ring. The identically 0 function defined on R is called the derivation of order 0. Let $n > 0$, and suppose we have defined the derivations of order at most $n-1$. A function $D : R \to R$ is called a *derivation of order at most* n, if D is additive and satisfies

$$D(xy) - D(x)y - D(y)x = B(x, y) \tag{2}$$

for every $x, y \in R$, where $B(x, y)$ is a derivation of order at most $n-1$ in each of its variables. We denote by $\mathcal{D}^n(R)$ the set of derivations of order at most n defined on R. We may write $\mathcal{D}^n$ instead of $\mathcal{D}^n(R)$ if the ring R is clear from the context. We say that the order of a derivation D is n if $D \in \mathcal{D}^n \setminus \mathcal{D}^{n-1}$. (We have $\mathcal{D}^{-1} = \emptyset$ by definition).

Clearly, a function $d\colon R \to R$ is a derivation if and only if $d \in \mathcal{D}_1$.

Now we define differential operators on a ring R. We say that the map $D : R \to R$ is a *differential operator of degree at most* n if D is the linear combination, with coefficients from R, of finitely many maps of the form $d_1 \circ \ldots \circ d_k$, where $d_1, \ldots, d_k$ are derivations on R and $k \le n$. If $k = 0$ then we interpret $d_1 \circ \ldots \circ d_k$ as the identity function on R. We denote by $\mathcal{O}^n(R)$ the set of differential operators of degree at most n defined on R. We may write $\mathcal{O}^n$ instead of $\mathcal{O}^n(R)$ if the ring R is clear from the context. We say that the degree of a differential operator D is n if $D \in \mathcal{O}^n \setminus \mathcal{O}^{n-1}$ (where $\mathcal{O}^{-1} = \emptyset$ by definition).

The term "differential operator" is justified by the following fact. Let $K = \mathbb{Q}(t_1, \ldots, t_k)$, where $t_1, \ldots, t_k$ are algebraically independent over $\mathbb{Q}$. Then K is the field of all rational functions of $t_1, \ldots, t_k$ with rational coefficients. It is clear that $d_i = \frac{\partial}{\partial t_i}$ is a derivation on K for every $i = 1, \ldots, k$. Therefore, every differential operator

$$D = \sum_{i_1+\ldots+i_k \le n} c_{i_1,\ldots,i_k} \cdot \frac{\partial^{i_1+\cdots+i_k}}{\partial t_1^{i_1} \cdots \partial t_k^{i_k}}, \tag{3}$$

where the coefficients $c_{i_1,\ldots,i_k}$ belong to K, is a differential operator of degree at most n. The converse is also true: if D is a differential operator of degree at most n on the field $K = \mathbb{Q}(t_1, \ldots, t_k)$, then D is of the form (3) (see [3, Proposition 3.2] and the proof of Lemma 2.6 below).

Remark 1.1. If d is a derivation on R, then $c \cdot d$ is also a derivation for every $c \in R$. Thus every differential operator is the sum of terms of the form $d_1 \circ \ldots \circ d_k$, where $k \ge 1$ and $d_1, \ldots, d_k$ are derivations, and of a term $c \cdot j$, where $c \in R$ and j is the identity function. Since $d(1) = 0$ for every derivation d, it follows that a differential operator D satisfies $D(1) = 0$ if and only if the term $c \cdot j$ is missing; that is, if D is the sum of terms of the form $d_1 \circ \ldots \circ d_k$, where $k \ge 1$ and $d_1, \ldots, d_k$ are derivations. We denote by $\mathcal{O}_0^n$ the set of all differential operators D of degree at most n satisfying $D(1) = 0$.

Let G be an Abelian semigroup, and let H be an Abelian group. The *difference operator* Δ_g $(g \in G)$ is defined by $\Delta_g f(x) = f(x+g) - f(x)$ for every $f\colon G \to H$ and $x \in G$. A function $f : G \to H$ is a *generalized polynomial*, if there is a k such that $\Delta_{g_1} \ldots \Delta_{g_{k+1}} f = 0$ for every $g_1, \ldots, g_{k+1} \in G$. The smallest k for which this holds for every $g_1, \ldots, g_{k+1} \in G$ is the *degree* of the generalized polynomial f, denoted by $\deg f$. The degree of the identically zero function is -1 by definition. It is clear that the nonzero constant functions are generalized polynomials of degree 0, and the nonconstant additive functions; that is, the nonzero homomorphism from G to H, are generalized polynomials of degree 1.

If X, Y are nonempty sets, then Y^X denotes the set of all maps $f\colon X \to Y$. We endow the space Y with the discrete topology, and Y^X with the product topology. The closure of a set $\mathcal{A} \subset Y^X$ with respect to the product topology is denoted by $\operatorname{cl} \mathcal{A}$. Clearly, a function $f\colon X \to Y$ belongs to $\operatorname{cl} \mathcal{A}$ if and only if, for every finite set $F \subset X$ there is a function $g \in \mathcal{A}$ such that $f(x) = g(x)$ for every $x \in F$.

It is clear that a function $f\colon G \to H$ is a generalized polynomial of degree at most n if and only if, for every finite set $F \subset G$, there is a generalized polynomial h of degree at most n such that $f = h$ on F. This means that *the set of generalized polynomials of degree at most n is closed in H^G.*

If R is a ring, then we denote by R^* the Abelian semigroup $R \setminus \{0\}$ under multiplication. We denote by j the identity function on R.

In this note our aim is to prove that, for every integral domain of characteristic zero and for every positive integer n, we have $\mathcal{D}^n = \operatorname{cl} \mathcal{O}_0^n$. That is, *a map $D\colon R \to R$ is a derivation of order at most n if and only if D belongs to the closure of the set of all differential operators of degree at most n satisfying $D(1) = 0$.* More precisely, we prove the following result.

Theorem 1.1. *Let R be an integral domain of characteristic zero, K its field of fractions, and let n be a positive integer. Then, for every function $D\colon R \to R$, the following are equivalent.*

(i) $D \in \mathcal{D}^n(R)$.

(ii) $D \in \operatorname{cl} (\mathcal{O}_0^n(R))$.

(iii) *D is additive on R, $D(1) = 0$, and D/j, as a map from the semigroup R^* to K, is a generalized polynomial of degree at most n.*

As an immediate consequence of the theorem above we find the following corollary.

Corollary 1.1. *Let R be an integral domain of characteristic zero, K its field of fractions, and let n be a positive integer. Then, for every function $D\colon R \to R$, the following are equivalent.*

(i) $D \in \mathcal{D}^n(R) \setminus \mathcal{D}^{n-1}(R)$.

(ii) $D \in (\operatorname{cl} \mathcal{O}_0^n(R)) \setminus \operatorname{cl} (\mathcal{O}_0^{n-1}(R))$.

(iii) *D is additive on R, $D(1) = 0$, and D/j, as a map from the semigroup R^* to K, is a generalized polynomial of degree n.*

Indeed, suppose $D \in \mathcal{D}^n \setminus \mathcal{D}^{n-1}$. Then, by Theorem 1.1, we have $D \in \operatorname{cl} \mathcal{O}_0^n$. If $D \notin \operatorname{cl}(\mathcal{O}_0^n) \setminus \operatorname{cl}(\mathcal{O}_0^{n-1})$, then $D \in \operatorname{cl} \mathcal{O}_0^{n-1}$. This implies $D \in \mathcal{D}^{n-1}$, which is impossible. Therefore, (i) of Corollary 1.1 implies (ii) of Corollary 1.1. The other implications can be shown similarly.

Remark 1.2. Theorem 1.1 and Corollary 1.1 do not hold without assuming that R is of characteristic zero. Consider the following example.

Let F_2 denote the field having two elements, and let $R = F_2[x]$ be the ring of polynomials with coefficients from F_2. We put

$$D\left(\sum_{i=0}^{n} a_i \cdot x^i\right) = \sum_{i=2}^{n} \frac{i(i-1)}{2} \cdot a_i \cdot x^{i-2}$$

for every $n \geq 0$ and $a_0, \ldots, a_n \in F_2$. It is easy to check that D is a derivation of order at most two on R. Since $D(x) = 0$ and $D(x^2) = 1$, it follows that D is not a derivation, and thus $D \in \mathcal{D}^2 \setminus \mathcal{D}^1$.

On the other hand, if d_1 and d_2 are arbitrary derivations on R, then $d_1 \circ d_2$ is also a derivation. Indeed,

$$d_1(d_2(x^k)) = d_1(k \cdot x^{k-1} \cdot d_2(x)) = k(k-1) \cdot x^{k-2} \cdot d_1(x) \cdot d_2(x) + k \cdot x^{k-1} \cdot d_1(d_2(x))$$

for every $k \geq 2$. Since $k(k-1)$ is even, we find that

$$(d_1 \circ d_2)(x^k) = k \cdot x^{k-1} \cdot a \tag{4}$$

for every $k \geq 2$, where $a = d_1(d_2(x)) \in R$. It is easy to check that (4) is true for $k = 0$ and $k = 1$ as well. Since derivations are additive, (4) gives $d_1(d_2(p)) = a \cdot \frac{\partial p}{\partial x}$ for every $p \in R$, and thus $d_1 \circ d_2 \in \mathcal{O}_0^1$. This implies that $\mathcal{O}_0^2 = \mathcal{O}_0^1$, and thus $\mathcal{D}^2$ is strictly larger than $\mathcal{O}_0^2$.

Remark 1.3. In the proof of Theorem 1.1 the crucial step is to show that if R is of characteristic zero and the transcendence degree of the field of fractions K of R over $\mathbb{Q}$ is finite, then $\mathcal{D}^n = \mathcal{O}_0^n$ (see Lemma 2.7). Comparing to Theorem 1.1 we find that under these conditions, for every function $f \colon R \to R$ we have

$(f \in \mathcal{D}^n \setminus \mathcal{D}^{n-1}) \iff (f \in \mathcal{O}_0^n \setminus \mathcal{O}_0^{n-1}) \iff D$ is additive on K, $D(1) = 0$, and D/j, defined on the group K^*, is a generalized polynomial of degree n.

We also prove that for every integral domain R of characteristic zero, if there are nonzero derivation on R, then the sets $\mathcal{D}^n \setminus \mathcal{D}^{n-1}$ are nonempty; that is, there are derivations of any given order. More precisely, we prove the following.

Theorem 1.2. *Let R be an integral domain of characteristic zero, and let n be a positive integer. If $d_1, \dots, d_n$ are nonzero derivations on K, then $d_1 \circ \dots \circ d_n \in \mathcal{D}^n \setminus \mathcal{D}^{n-1}$.*

(For integral domains of characteristic zero this generalizes [2, Remark 3], where the case $d_1 = \dots = d_n$ is considered.)

Remark 1.4. The statement of the theorem above does not hold without assuming that R is of characteristic zero. Consider the example described in Remark 1.2. Clearly, $d(p) = \frac{\partial p}{\partial x}$ ($p \in R$) defines a nonzero derivation on R. However, as we saw in Remark 1.2, $d \circ d$ is a derivation of order 1.

The statement of the theorem is not true for rings in general; not even for rings of characteristic zero. Let $R = \mathbb{Q}[x] \times \mathbb{Q}[x]$, and put $d_1(p, q) = (\frac{\partial p}{\partial x}, 0)$ and $d_2(p, q) = (0, \frac{\partial q}{\partial x})$ for every $(p, q) \in R$. Then d_1 and d_2 are nonzero derivations on R, but $d_1 \circ d_2 = 0$.

2 Lemmas

Lemma 2.1. *For every ring R and for every nonnegative integer n, the set $\mathcal{D}^n$ is closed in R^R.*

Proof. We prove by induction on n. If $n = 0$, then $\mathcal{D}^0 = \{0\}$ is closed. Let $n > 0$, and suppose that $\mathcal{D}^{n-1}$ is closed. Let $f \in \operatorname{cl} \mathcal{D}^n$ be arbitrary. We have to prove that $f \in \mathcal{D}^n$; that is, for every fixed $y \in R$, the map $x \mapsto g(x) = f(xy) - yf(x) - xf(y)$ belongs to $\mathcal{D}^{n-1}$. By the induction hypothesis, it is enough to show that $g \in \operatorname{cl} \mathcal{D}^{n-1}$; that is, for every finite set $F \subset R$ there is a function $h \in \mathcal{D}^{n-1}$ such that $g(x) = h(x)$ for every $x \in F$.

If F is finite, then so is $A = F \cup \{xy \colon x \in F\} \cup \{y\}$. Since $f \in \operatorname{cl} \mathcal{D}^n$, there is a function $D \in \mathcal{D}^n$ such that $f(z) = D(z)$ for every $z \in A$. If $x \in F$, then $x, y, xy \in A$, and thus

$$g(x) = f(xy) - yf(x) - xf(y) = D(xy) - yD(x) - xD(y).$$

The function $x \mapsto h(x) = D(xy) - yD(x) - xD(y)$ belongs to $\mathcal{D}^{n-1}$, as $D \in \mathcal{D}^n$. Since $g(x) = h(x)$ for every $x \in F$, the lemma is proved. □

Lemma 2.2. *For every ring R we have $\operatorname{cl} \mathcal{O}_0^n \subset \mathcal{D}^n$.*

Proof. Since $\mathcal{D}^n$ is closed by Lemma 2.1, it is enough to show that $\mathcal{O}_0^n \subset \mathcal{D}^n$. Let D be a differential operator of degree at most n satisfying $D(1) = 0$. According to Remark 1.1, D is the sum of terms of the form $d_1 \circ \dots \circ d_k$, where $1 \le k \le n$ and $d_1, \dots, d_k$ are derivations. Since $\mathcal{D}^n$ is a linear space, it is enough to show that $d_1 \circ \ldots \circ d_k \in \mathcal{D}^k$ whenever $k \geq 1$ and $d_1, \ldots, d_k$ are derivations. This, in turn, is easy to prove by induction on k. $\square$

The statement of the following lemma is probably known. In order to make these notes as self-contained as possible, we provide the proof.

Lemma 2.3. *Let G be an Abelian semigroup, and let K be a field. If $p\colon G \to K$ is a generalized polynomial of degree $n \geq 0$ and $a\colon G \to K$ is a nonzero additive function, then $p \cdot a$ is a generalized polynomial of degree at most $n+1$.*

If K is of characteristic zero, then $\deg(p \cdot a) = n + 1$.

Proof. We prove by induction on n. If $n = 0$, then p is a nonzero constant, and $p \cdot a$ is a nonzero additive function, hence a generalized polynomial of degree 1.

Let $n > 0$, and suppose that the statement is true for $n - 1$. Let p be a generalized polynomial of degree n. We have

$$\Delta_g(p \cdot a)(x) = a(x) \cdot \Delta_g p(x) + a(g) \cdot p(x+g) \tag{5}$$

for every $x, g \in G$. Since $\deg \Delta_g p(x) \leq n - 1$, it follows from the induction hypothesis that $\deg(a(x) \cdot \Delta_g p(x)) \leq n$. Therefore, by (5), we have $\deg \Delta_g(p \cdot a) \leq n$ for every $g \in G$, and thus $\deg(p \cdot a) \leq n + 1$. We have to prove that if K is characteristic zero, then $\deg(p \cdot a) \geq n + 1$.

Since the image space K is a torsion free and divisible Abelian group, it follows from Djoković's theorem [1] that $p = P_n + \ldots + P_1 + P_0$, where P_i is a monomial of degree i for every $i = 1, \ldots, n$, and P_0 is constant. Then there is a symmetric function $A(x_1, \ldots, x_n)$, additive in each of its variables, such that $P_n(x) = A(x, \ldots, x)$ $(x \in G)$. Since $q = p - P_n$ is a generalized polynomial of degree $\leq n - 1$, it follows from the induction hypothesis that $\deg(q \cdot a) \leq n$. Therefore, in order to prove $\deg(p \cdot a) \geq n + 1$, it is enough to show that $\deg(P_n \cdot a) = n + 1$.

First we show that there exists an element $g \in G$ such that $P_n(g) \neq 0$ and $a(g) \neq 0$. By assumption, there is an $x \in G$ such that $a(x) \neq 0$. Since $\deg P_n = n \geq 0$, it follows that P_n is nonzero. Let $y \in G$ be such that $P_n(y) \neq 0$. Now $a(kx + y) = k \cdot a(x) + a(y)$ for every positive integer k. Since $a(x), a(y) \in K$ and $a(x) \neq 0$, we have $a(kx + y) \neq 0$ for every k with at most one exception.

Using the fact that $A(x_1, \ldots, x_n)$ is symmetric and additive in each of its variables, we find

$$P_n(kx + y) = \sum_{i=0}^{n} \binom{n}{i} A_i(kx, y) \tag{6}$$

for every positive integer k, where

$$A_i(kx, y) = A(\underbrace{kx, \ldots, kx}_{i}, \underbrace{y, \ldots, y}_{k-i}) = k^i \cdot A(\underbrace{x, \ldots, x}_{i}, \underbrace{y, \ldots, y}_{k-i}).$$

Therefore, by (6), $Q(kx+y)$ is a polynomial of k with coefficients from K. Since the constant term of this polynomial is $A(y,\ldots,y) \neq 0$, $Q(kx+y)$ is not the identically zero polynomial, and thus $P_n(kx+y) \neq 0$ for all but finitely many k. Therefore, we may choose a k such that $P_n(g) \neq 0$ and $a(g) \neq 0$, where $g = kx + y$.

Let $Q = P_n \cdot a$, and suppose that $\deg Q \leq n$. Then $Q = Q_n + \ldots + Q_1 + Q_0$, where Q_i is a monomial of degree i for every $i = 1,\ldots,n$, and Q_0 is constant. For every $i = 1,\ldots,n$, there is there is a symmetric function $B_i(x_1,\ldots,x_i)$, additive in each of its variables, such that $Q_i(x) = B_i(x,\ldots,x)$ $(x \in G)$. Then

$$Q(k \cdot g) = Q_0 + \sum_{i=1}^{n} B_i(kg,\ldots,kg) = Q_0 + \sum_{i=1}^{n} k^i \cdot B_i(g,\ldots,kg)$$

for every positive integer k. Therefore, the map $k \mapsto Q(k \cdot g)$ is a polynomial of degree $\leq n$ with coefficients from K. However,

$$Q(k \cdot g) = k^n \cdot A(g,\ldots,g) \cdot k \cdot a(g) = k^{n+1} \cdot A(g,\ldots,g) \cdot a(g)$$

is a polynomial of degree $n+1$. This is a contradiction, proving $\deg Q = n+1$. $\square$

Lemma 2.4. *Let R be an integral domain, and let K be its field of fractions. If $d_1,\ldots,d_n$ are nonzero derivations on R and $D = d_1 \circ \ldots \circ d_n$, then D/j, as a map from the semigroup R^* to K, is a generalized polynomial of degree at most n.*

If R is of characteristic zero, then $\deg D/j = n$.

Proof. We prove by induction on n. If $n = 1$, then D is a nonzero derivation. It is clear that in this case D/j is additive, hence a generalized polynomial of degree at most 1 on the semigroup R^*. Suppose $\deg D/j \leq 0$. Then D/j is constant on R^*, and thus $D = c \cdot j$ on R, where $c \in R$ is a constant. Since D is a derivation, we have $c = D(1) = 0$ and $d = 0$, a contradiction. Thus $\deg D/j = 1$.

Suppose that $n > 1$, and the statement is true for $n-1$. Let $d_1,\ldots,d_n$ be nonzero derivations on R. By the induction hypothesis, $(d_2 \circ \ldots \circ d_n)/j = p$ is a generalized polynomial of degree at most $n-1$. Since d_1 is a derivation, we have

$$D(x) = (d_1 \circ \ldots \circ d_n)(x) = d_1(p(x) \cdot x) = d_1(p(x)) \cdot x + p(x) \cdot d_1(x)$$

for every $x \in R^*$. Thus

$$D/j = (d_1 \circ p) + p \cdot (d_1/j) \tag{7}$$

on R^*. Since $p\colon R^* \to K$ is a generalized polynomial of degree $\leq n-1$ and $d_1 : R \to R$ is additive, it follows that $d_1 \circ p$ is a generalized polynomial of degree

$\leq n-1$ on R^*. (This is because, if G is an Abelian semigroup, H is an Abelian group, $p\colon G \to H$ is a generalized polynomial of degree k, and $d\colon H \to H$ is additive, then $d \circ p$ is a generalized polynomial of degree at most k.)

If R is of characteristic zero, then so is K. In this case $p\cdot(d_1/j)$ is a generalized polynomial of degree n by Lemma 2.3, since d_1/j is nonzero and additive on R^*. Therefore, D/j is a generalized polynomial of degree n. □

Lemma 2.5. *Let R be an integral domain, and let K be its field of fractions. If $D \in \mathrm{cl}\,\mathcal{O}_0^n(R)$, then D/j, as a map from the semigroup R^* to K, is a generalized polynomial of degree at most n.*

Proof. Let $D \in \mathrm{cl}\,\mathcal{O}_0^n$ be given. As the set of generalized polynomials of degree $\leq n$ is closed, it is enough to show that for every finite set $F \subset R^*$ there is a generalized polynomial $p\colon R^* \to K$ such that $\deg p \leq n$ and $D/j = p$ on F. Since $D \in \mathrm{cl}\,\mathcal{O}_0^n$, there is an $f \in \mathcal{O}_0^n$ such that $D = f$ on F. It is clear from Remark 1.1 and Lemma 2.4 that f/j is a generalized polynomial of degree at most n. Now we have $D/j = f/j$ on F, completing the proof. □

The statement of the following lemma is proved, in a different context, in Lemma 3.3 of [3]. We give the proof adjusted to our purposes.

Lemma 2.6. *Let R be a subring of $\mathbb{C}$, let $K \subset \mathbb{C}$ be its field of fractions, and suppose that the transcendence degree of K over $\mathbb{Q}$ is finite. Let the map $D : R \to R$ be additive. If D/j, as a map from the semigroup R^* to $\mathbb{C}$ is a generalized polynomial of degree at most n, then $D \in \mathcal{O}^n$.*

Proof. Let k be the transcendence degree of K over $\mathbb{Q}$, and let the elements $u_1, \ldots, u_k \in K$ be algebraically independent over $\mathbb{Q}$. Let $u_i = a_i/b_i$, where $a_i, b_i \in R$ for every $i = 1, \ldots, k$. Then the field $\mathbb{Q}(a_1, b_1, \ldots, a_k, b_k)$ has transcendence degree k over $\mathbb{Q}$, and thus we can chose elements $t_1, \ldots, t_k \in \{a_1, b_1, \ldots, a_k, b_k\} \subset R^*$ such that $t_1, \ldots, t_k$ are algebraically independent over $\mathbb{Q}$.

By assumption, the function $p = D/j$ is a generalized polynomial of degree $\leq n$ on R^*. By Djoković's theorem, we have $p = P_n + \ldots + P_1 + P_0$, where P_j is a monomial of degree j for every $j = 1, \ldots, n$, and P_0 is constant. Using the fact that $P_j(x) = A_j(x, \ldots, x)$, where $A_j(x_1, \ldots, x_j)$ is symmetric and additive in each of its variables, it is easy to see that for every $j = 1, \ldots, n$ there is a homogeneous polynomial $\overline{p}_j \in K[x_1, \ldots, x_k]$ of degree j such that

$$P_j\left(t_1^{i_1} \cdots t_k^{i_k}\right) = \overline{p}_j(i_1, \ldots, i_k)$$

whenever $i_1, \ldots, i_k$ are nonnegative integers. (Note that the semigroup operation in R^* is multiplication.) Putting $\overline{p} = P_0 + \sum_{j=1}^n \overline{p}_j$ we find that $\overline{p} \in K[x_1, \ldots, x_k]$, and

$$\overline{p}\left(t_1^{i_1} \cdots t_k^{i_k}\right) = q(i_1, \ldots, i_k)$$

for every $i_1, \ldots, i_k \geq 0$. We shall use the notation $x^{[0]} = 1$ and $x^{[j]} = x(x-1)\cdots(x-j+1)$ for every $j = 1, 2, \ldots$ and $x \in \mathbb{Z}$. It is easy to see that every polynomial belonging to $K[x_1, \ldots, x_k]$ and of degree $\leq n$ can be written in the form $\sum c_j \cdot x_1^{[j_1]} \cdots x_k^{[j_k]}$, where $j = (j_1, \ldots, j_k)$ runs through the set of k-tuples of nonnegative integers with $j_1 + \ldots + j_k \leq n$, and in each term the coefficient c_j belongs to K. Therefore, the polynomial $\overline{p}$ also has such a representation. Then we have

$$\begin{aligned} D\left(t_1^{i_1} \cdots t_k^{i_k}\right) &= p\left(t_1^{i_1} \cdots t_k^{i_k}\right) \cdot t_1^{i_1} \cdots t_k^{i_k} = \\ &= \sum c_j \cdot i_1^{[j_1]} \cdots i_k^{[j_k]} \cdot t_1^{i_1} \cdots t_k^{i_k} = \\ &= \sum c_j \cdot t_1^{j_1} \cdots t_k^{j_k} \cdot i_1^{[j_1]} \cdots i_k^{[j_k]} \cdot t_1^{i_1-j_1} \cdots t_k^{i_k-j_k} = \\ &= E\left(t_1^{i_1} \cdots t_k^{i_k}\right) \end{aligned} \tag{8}$$

for every $i_1, \ldots, i_k \geq 0$, where E is the differential operator

$$\sum c_j \cdot t_1^{j_1} \cdots t_k^{j_k} \cdot \frac{\partial^{j_1+\cdots+j_k}}{\partial t_1^{j_1} \cdots \partial t_k^{j_k}}.$$

By extending the derivations $\partial/\partial t_i$ to K, we can extend E to K as a differential operator $\overline{E}$ of degree at most n. Then $\overline{E}$ is additive on K, and $\overline{E}/j$ is a generalized polynomial on K^* by Lemma 2.4. Let $q(0) = 0$, and let $q(x) = p(x) - \overline{E}(x)/x$ for every $x \in R^*$. Then $q \cdot j = D - \overline{E}$ is additive on R, and q is a generalized polynomial on R^*. Let G denote the semigroup generated by the elements $t_1, \ldots, t_k$. Then q vanishes on G by (8). From these conditions it follows that $q = 0$ on R. This is proved in [3, Lemma 3.6] under the stronger condition that G is the group (and not the semigroup) generated by $t_1, \ldots, t_k$. One can see that the same argument works in our more general case as well; however, for the sake of completeness we give the proof in the appendix. Thus we have $q = 0$; that is, $D = \overline{E}$ on R, which completes the proof. $\square$

Lemma 2.7. *Let R be a subring of $\mathbb{C}$, let $K \subset \mathbb{C}$ be its field of fractions, and suppose that the transcendence degree of K over $\mathbb{Q}$ is finite. Then $\mathcal{D}^n(R) = \mathcal{O}_0^n(R)$.*

Proof. By Lemma 2.2, we only have to show that $\mathcal{D}^n \subset \mathcal{O}_0^n$. It is easy to prove, by induction on n that if $D \in \mathcal{D}^n$, then $D(1) = 0$. Therefore, it is enough to show that if $D \in \mathcal{D}^n$, then D is a differential operator of degree at most n. We prove by induction on n.

The statement is obvious if $n = 0$. Let $n > 0$, and suppose that the statement is true for $n - 1$. Let D be a derivation of order at most n. By Lemma 2.6, it is enough to show that $p = D/j$, defined on the semigroup R^*, is a generalized polynomial of degree at most n. Let $y \in R^*$ be fixed. Dividing (2) by xy we obtain

$$\frac{D(xy)}{xy} - \frac{D(x)}{x} - \frac{D(y)}{y} = \frac{B(x,y)}{xy},$$

and thus $p(xy) - p(x) - p(y) = B(x,y)/xy$ for every $x \in K^*$. Therefore we have

$$\Delta_y p(x) = p(y) + \frac{1}{y} \cdot \frac{B(x,y)}{x} \tag{9}$$

on R^*. The map $x \mapsto B(x,y)$ is a derivation of order at most $n-1$. We also have $B(1,y) = 0$ by $D(1) = 0$. Therefore, by Lemma 2.4, the map $x \mapsto B(x,y)/x$ is a generalized polynomial of degree at most n. Then so is $\Delta_y p$ by (9). Since this is true for every $y \in K^*$, it follows that p is a generalized polynomial of degree at most n. □

3 Proof of Theorems 1.1 and 1.2.

First we prove Theorem 1.1. The implication (ii)⟹(iii) is proved in Lemma 2.5.

(iii)⟹(ii): Suppose that D is additive, $D(1) = 0$, and D/j is a generalized polynomial of degree at most n. In order to prove $D \in \operatorname{cl} \mathcal{O}_0^n$, we have to show that for every finite set $F \subset K$ there is a function $f \in \mathcal{O}_0^n$ such that $D = f$ on F. Let $F \subset K$ be finite, and let L denote the subfield of K generated by F. Obviously, the transcendence degree of L over $\mathbb{Q}$ is finite. It is well-known that every field of characteristic zero and having finite transcendence degree over $\mathbb{Q}$ is isomorphic to a subfield of $\mathbb{C}$. Therefore, we may assume that $L \subset \mathbb{C}$. Thus, by Lemma 2.6, the restriction $D|_L$ of D to the field L is a derivation of order at most n. Since $D(1) = 0$, we also have $D|_L \in \mathcal{O}_0^n(L)$. It is well-known that every derivation on L can be extended to K as a derivation (see [4, pp. 351-352]). This implies that every differential operator on L of degree at most n can be extended to K as a differential operator of degree at most n. If f is such an extension of $D|_L$, then, obviously, $D(x) = f(x)$ for every $x \in F$. This proves (iii)⟹(ii).

(ii)⟹(i): This is Lemma 2.2.

(i)⟹(ii): Let $D \in \mathcal{D}^n$. In order to prove $f \in \operatorname{cl} \mathcal{O}_0^n$ we have to show that for every finite set $F \subset K$ there is a function $f \in \mathcal{O}_0^n$ such that $D = f$ on F. Let L denote the field generated by F. Obviously, the transcendence degree of L over $\mathbb{Q}$ is finite. Thus, by Lemma 2.7, the restriction $D|_L$ of D to the field L is a derivation of order at most n, vanishing at 1. Let f be an extension of $D|_L$ to K as a function $f \in \mathcal{O}_0^n$. Then, obviously, $D(x) = f(x)$ for every $x \in F$. This proves (i)⟹(ii).
□

The statement of Theorem 1.2 is an immediate consequence of Corollary 1.1 and Lemma 2.4. □

4 Appendix

Lemma 4.1. *Let R be a subring of $\mathbb{C}$, and let $K \subset \mathbb{C}$ be its field of fractions. Suppose that the transcendence degree of K over $\mathbb{Q}$ is $k < \infty$, and let the elements $t_1, \ldots, t_k \in R$ be algebraically independent over $\mathbb{Q}$. Let $f\colon R \to \mathbb{C}$ be additive on R (with respect to addition) and such that $q = f/j$, as a map from the semigroup R^* to $\mathbb{C}$ is a generalized polynomial. If $f = 0$ on the semigroup G generated by $t_1, \ldots, t_k$, then $f = 0$ on R.*

Proof. We prove by induction on $\deg q$. If $\deg q = 0$, then q is constant. Since $f = 0$ on G, we have $q = 0$ on G, and thus $q = 0$ on R.

Suppose $m = \deg q > 0$, and that the statement is true for degrees less than m. Let $g \in G$ be fixed, and put $f_1(x) = g^{-1}f(gx) - f(x)$ $(x \in R)$. Then f_1 is additive on R. Also, f_1/j is a generalized polynomial on R^*, since

$$\frac{f_1(x)}{x} = \frac{g^{-1}f(gx) - f(x)}{x} = \frac{f(gx)}{gx} - \frac{f(x)}{x} = q(gx) - q(x) = \Delta_g q(x)$$

for every $x \in R^*$. Since $\deg(f_1/j) = \deg \Delta_g q \leq m - 1$ and $f_1 = 0$ on G, it follows from the induction hypothesis that $f_1 = 0$ on R. Thus $f(gx) = g \cdot f(x)$ for every $g \in G$ and $x \in R$. By the additivity of f we obtain

$$f(cx) = c \cdot f(x) \qquad (c \in \mathbb{Q}[t_1, \ldots, t_k],\ x \in R). \tag{10}$$

Since the transcendence degree of K over $\mathbb{Q}$ is k and $t_1, \ldots, t_k$ are algebraically independent over $\mathbb{Q}$, it follows that every element of K is algebraic over $\mathbb{Q}(t_1, \ldots, t_k)$. Let $\alpha \in R$ be arbitrary. Then α is algebraic over the field $\mathbb{Q}(t_1, \ldots, t_k)$, and there are elements $c_0, \ldots, c_N \in \mathbb{Q}[t_1, \ldots, t_k]$ such that

$$c_N\alpha^N + \ldots + c_1\alpha + c_0 = 0, \tag{11}$$

where $c_N \neq 0$ and N is minimal. Let $f(\alpha^i) = a_i$ $(i = 0, 1, \ldots)$. Multiplying (11) by α^{n-N} for every $n \geq N$ we obtain

$$c_N\alpha^n + \ldots + c_1\alpha^{n-N+1} + c_0\alpha^{n-N} = 0.$$

By (10) and by the additivity of f, this implies

$$c_N a_n + \ldots + c_1 a_{n-N+1} + c_0 a_{n-N} = 0$$

for every $n \geq N$. Therefore, the sequence (a_n) satisfies a linear recurrence relation. It is well-known that a_n can be uniquely represented in the form $a_n = \sum_{\lambda\in\Lambda} p_\lambda(n) \cdot \lambda^n$, where λ runs through Λ, the set of roots of the characteristic polynomial $\chi(x) = c_N x^N + \ldots + c_0$, and for every root $\lambda \in \Lambda$, $p_\lambda \in \mathbb{C}[x]$ is a polynomial of the degree less than the multiplicity of λ.

Since N is minimal, the polynomial χ is irreducible over $\mathbb{Q}(t_1, \ldots, t_k)$. Therefore, every λ is a simple root of χ, and thus

$$a_n = \sum_{\lambda \in \Lambda} d_\lambda \cdot \lambda^n \tag{12}$$

for every n, where d_λ is a constant for every $\lambda \in \Lambda$.

Since q is a generalized polynomial on R^* it follows that the map $n \mapsto q(\alpha^n)$ is a polynomial on $\{0, 1, \ldots\}$. Now, we have $a_n = f(\alpha^n) = q(\alpha^n) \cdot \alpha^n$ for every n. The uniqueness of the representation (12) implies that $\alpha \in \Lambda$, and the function $n \mapsto q(\alpha^n)$ $(n = 0, 1, \ldots)$ is constant. Since $q(1) = f(1) = 0$ by $1 \in G$, it follows that $q(\alpha^n) = 0$ for every n. In particular, $q(\alpha) = 0$ and $f(\alpha) = 0$. Since this is true for every $\alpha \in R$, we obtain $f = 0$ on R. $\square$

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