1st homework set, Due March 8

1. (4p.) Determine the I-divergences $\mathrm{D}(\mathbb{P}||\mathbb{Q})$ and $\mathrm{D}(\mathbb{Q}||\mathbb{P})$ if

   (a) $\mathbb{Q}$ is an arbitrary distribution over a finite set A and $\mathbb{P}(a) = \frac{\mathbb{Q}(a)}{\mathbb{Q}(B)}$ if $a \in B$ and 0 otherwise, where $B \subset A$ and $\mathbb{Q}(B) = \sum_{a \in B} Q(a) > 0$.

   (b) $\mathbb{P}$ and $\mathbb{Q}$ are defined as follows. A hat contains k_1 slips of paper marked with 1, k_2 slips of paper marked with 2 ... and k_r slips of paper marked with r. Let n be equal to $k_1 + \cdots + k_r$. We draw n times from the hat (i) without replacement (ii) with replacement. For a given sequence $\mathbf{x} = x_1 \ldots x_n \in \{1, \ldots, r\}^n$ let $\mathbb{P}(\mathbf{x})$ and $\mathbb{Q}(\mathbf{x})$ denote the probability of drawing the given sequence in cases (i) and (ii), respectively. Is there any connection with part (a)?

   (c) Calculate also the entropies of $\mathbb{P}$ and $\mathbb{Q}$ defined in part (b)! Which entropy is the larger?

2. (4p.) Prove the Pinsker inequality (also called Csiszár-Kemperman-Kullback-Pinsker inequality)

$$\mathrm{D}(\mathbb{P}||\mathbb{Q}) \geq \frac{1}{2\ln 2}\,\mathrm{d}^2(\mathbb{P},\mathbb{Q}),$$

   where $d(\mathbb{P},\mathbb{Q})$ is the variational distance of distributions $\mathbb{P}$ and $\mathbb{Q}$, i.e.,

$$\mathrm{d}(\mathbb{P},\mathbb{Q}) = \sum_{a \in \mathcal{A}} |\mathbb{P}(a) - \mathbb{Q}(a)|.$$

   Show that this bound is tight in the sense that the ratio of $\mathrm{D}(\mathbb{P}||\mathbb{Q})$ and $d^2(\mathbb{P},\mathbb{Q})$ can be arbitrarily close to $\frac{1}{2\ln 2}$.

   Hint: With $B \triangleq \{a : \mathbb{P}(a) \geq \mathbb{Q}(a)\}$, $\widetilde{\mathbb{P}} \triangleq (\mathbb{P}(B), \mathbb{P}(A-B))$, $\widetilde{\mathbb{Q}} \triangleq (\mathbb{Q}(B), \mathbb{Q}(A-B))$, we have $\mathrm{D}(\mathbb{P}||\mathbb{Q}) \geq \mathrm{D}(\widetilde{\mathbb{P}}||\widetilde{\mathbb{Q}})$, $\mathrm{d}(\mathbb{P},\mathbb{Q}) = d(\widetilde{\mathbb{P}},\widetilde{\mathbb{Q}})$. Hence it suffices to consider the case $A = \{0,1\}$, i.e., to determine the largest c such that

$$p\log\left(\frac{p}{q}\right) + (1-p)\log\left(\frac{1-p}{1-q}\right) - 4c(p-q)^2 \geq 0, \text{ for every } 0 \leq q \leq p \leq 1.$$

   For $q = p$ the equality holds; further, the derivative of the left-hand side with respect to q is negative for $q < p$ if $c \leq \frac{1}{2\ln 2}$ while for $c > \frac{1}{2\ln 2}$ and $p = \frac{1}{2}$ it is positive in the neighborhood of p.

   Remark: When checking the statements of the hint above, pay attention to the fact that the base of the logarithm is 2, hence, $(\log x)' = \frac{1}{x \ln 2}$.

3. (2p.) (A reversed Pinsker inequality)

   Let $\mathbb{P}$ and $\mathbb{Q}$ be probability distributions on the finite set A. Let $A_+ = \{a : \mathbb{Q}(a) > 0\}$ and let

$$\alpha_{\mathbb{Q}} = \min_{a \in A_+} \mathbb{Q}(a).$$

   Prove that if $\mathrm{D}(\mathbb{P}||\mathbb{Q}) < \infty$ then

$$\mathrm{D}(\mathbb{P}||\mathbb{Q}) \leq \frac{\mathrm{d}^2(\mathbb{P},\mathbb{Q})}{\alpha_{\mathbb{Q}} \cdot \ln 2}.$$

   Hint: First prove that

$$\mathrm{D}(\mathbb{P}||\mathbb{Q}) \leq \sum_{a \in A_+} \frac{\mathbb{P}(a)}{\ln 2}\left(\frac{\mathbb{P}(a)}{\mathbb{Q}(a)} - 1\right) = \frac{1}{\ln 2}\sum_{a \in A_+} \frac{|\mathbb{P}(a) - \mathbb{Q}(a)|^2}{\mathbb{Q}(a)}.$$