5th homework set, Due !!!June 9!!!

(Of course you can submit your homework earlier, in this case I will correct it earlier)

1. (2p.) Find out whether there exists a binary prefix code with code lengths

   (a) 2,3,3,3,4,4,4,5,5,5,5,5,6,7,7

   (b) 2,3,3,4,4,4,4,4,4,5,5,5,5,6

   If yes, then define such a coding!

2. (2p.+ 2p. + 3p.) (Hypothesis testing with both errors exponentially decreasing)

   (a) We observe independent drawings from an unknown distribution $\mathbb{Q}$ on the finite set A. Let γ be a positive number and let $\mathbb{P}_0$ and $\mathbb{P}_0$ be strictly positive distributions on A with $D(\mathbb{P}_1||\mathbb{P}_0) > \gamma$. To test the (simple) null hypothesis $\mathbb{Q} = \mathbb{P}_0$ against the simple alternative hypothesis $\mathbb{Q} = \mathbb{P}_1$, let the acceptance region $A_n \subset \mathcal{X}^n$ be the union of all type classes $|\mathcal{T}^n_{\mathbb{P}}|$ with $D(\mathbb{P}||\mathbb{P}_0) \le \gamma$. Show that then the probability of type 1 error decreases with exponent γ, i.e.,

   $$\mathbb{P}_0^n(\mathcal{X}^n - A_n) = 2^{-n\gamma+o(n)} \quad \left(\text{or, i.e., } \lim_{n\to\infty} \frac{1}{n}\log \mathbb{P}_0^n(\mathcal{X}^n - A_n) = -\gamma\right), \tag{1}$$

   whereas the type 2 error probability ($\mathbb{P}_1^n(A_n)$) decreases with exponent $\delta = D(\mathbb{P}^*||\mathbb{P}_1)$ where $\mathbb{P}^*$ is the I-projection of P_1 onto the "divergence ball"

   $$B(\mathbb{P}_0, \gamma) = \{\mathbb{P} : D(\mathbb{P}||\mathbb{P}_0) \le \gamma\}. \tag{2}$$

   Hint: Apply Sanov's theorem! Note that $B(\mathbb{P}_0, \gamma)$ is closed and its interior is $\{\mathbb{P} : D(\mathbb{P}||\mathbb{P}_0) < \gamma\}$.

   (b) Show that the above is the best possible, i.e., for any $\tilde{A}_n \subset \mathcal{X}^n$ satisfying (1), always

   $$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1^n(\tilde{A}_n) \ge -\delta. \tag{3}$$

   Hint: Fix an $\varepsilon > 0$. (1) implies that $\exists N$ such that $\mathbb{P}_0^n(\mathcal{X}^n - \tilde{A}_n) \le 2^{-n(\gamma-\varepsilon)}$ if $n > N$. Let Q be an arbitrary n-type in $B(\mathbb{P}_0, \gamma - 2\varepsilon)$. Show that $\tilde{A}_n$ contains at least half of $\mathcal{T}^n_Q$ if n is large enough!

   (c) With the notation used above, show that the I-Projection $\mathbb{P}^*$ of $\mathbb{P}_1$ onto $B(\mathbb{P}_0, \gamma)$ equals the I-projection of both $\mathbb{P}_0$ and $\mathbb{P}_1$ onto the linear family

   $$\mathcal{L} = \{\mathbb{P} : \sum_{a\in A} \mathbb{P}(a) \log \frac{\mathbb{P}_0(a)}{\mathbb{P}_1(a)} = \delta - \gamma\} = \{\mathbb{P} : D(\mathbb{P}||\mathbb{P}_1) - D(\mathbb{P}||\mathbb{P}_0) = \delta - \gamma\}, \tag{4}$$

   and also equals the I-projection of $\mathbb{P}_0$ onto $B(\mathbb{P}_1, \delta)$. Give a geometric interpretation. Finally conclude that $\mathbb{P}^*$ is of the form $\mathbb{P}^*(a) = c \cdot \mathbb{P}_0^{\theta}(a) \cdot \mathbb{P}_1^{1-\theta}(a)$ for some $0 < \theta < 1$.
   Hint: We learned that $D(\mathbb{Q}||\mathbb{P})$ is strictly convex in $\mathbb{Q}$ when $\mathbb{P}$ is fixed and strictly positive. Using this fact first prove that $\mathbb{P}^*$ is on the border of $B(\mathbb{P}_0, \gamma)$, i.e., $D(\mathbb{P}^*||\mathbb{P}_0) = \gamma$! After that, prove that $B(\mathbb{P}_0, \gamma) \cap \mathcal{L} = \{\mathbb{P}^*\}$! Then prove that the I-projection of $\mathbb{P}_1$ onto $\mathcal{L}$ equals $\mathbb{P}^*$. Finally, prove the remaining statements!

3. (2p.) (A reversed Pinsker inequality)

   Let $\mathbb{P}$ and $\mathbb{Q}$ be probability distributions on the finite set A. Let $A_+ = \{a : \mathbb{Q}(a) > 0\}$ and let

   $$\alpha_{\mathbb{Q}} = \min_{a\in A_+} \mathbb{Q}(a).$$

   Prove that if $\mathrm{D}(\mathbb{P}||\mathbb{Q}) < \infty$ then

   $$\mathrm{D}(\mathbb{P}||\mathbb{Q}) \le \frac{\mathrm{d}^2(\mathbb{P}, \mathbb{Q})}{\alpha_{\mathbb{Q}} \cdot \ln 2}.$$

   Hint: First prove that

   $$\mathrm{D}(\mathbb{P}||\mathbb{Q}) \le \sum_{a\in A_+} \frac{\mathbb{P}(a)}{\ln 2} \left(\frac{\mathbb{P}(a)}{\mathbb{Q}(a)} - 1\right) = \frac{1}{\ln 2} \sum_{a\in A_+} \frac{|\mathbb{P}(a) - \mathbb{Q}(a)|^2}{\mathbb{Q}(a)}.$$