1. MATEMATIKA A2 FELADATSOR - MEGOLDÁSOK

1. Határozza meg az alábbi végtelen sorok értékét:

a. $1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots = \frac{1}{1-\frac{1}{10}} = \frac{10}{9}$

b. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots = 2(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots) = 2\frac{1}{1-\frac{1}{3}} = 3$

c. $\sum_{n=0}^{\infty} \frac{1}{5^n} = 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}$

d. $\sum_{n=0}^{\infty} \frac{2}{3^{n+1}} = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots = \frac{2}{3}(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots) = \frac{2}{3}\frac{1}{1-\frac{1}{3}} = 1$

e. $\sum_{n=1}^{\infty} \frac{5^n}{6^{n-1}} = 5 + \frac{25}{6} + \frac{125}{36} + \cdots = 5(1 + \frac{5}{6} + \frac{25}{36} + \dots) = 5\frac{1}{1-\frac{5}{6}} = 30$

f. $\sum_{n=1}^{\infty} \frac{9^n + 3^n}{10^n} = \frac{9+3}{10} + \frac{81+9}{100} + \frac{729+27}{1000} + \cdots = \frac{9}{10} + \frac{81}{100} + \frac{729}{1000} + \cdots + \frac{3}{10} + \frac{9}{100} + \frac{27}{1000} + \cdots =$
$\frac{9}{10}(1 + \frac{9}{10} + \frac{81}{100} + \frac{729}{1000} + \dots) + \frac{3}{10}(1 + \frac{3}{10} + \frac{9}{100} + \frac{27}{1000} + \dots) = \frac{9}{10}\frac{1}{1-\frac{9}{10}} + \frac{3}{10}\frac{1}{1-\frac{3}{10}} = \frac{66}{7}$

g. $\sum_{n=0}^{\infty} \frac{2^{3n+2} + 3^{2n+3}}{10^n} = \frac{4+27}{1} + \frac{32+243}{10} + \frac{256+2187}{100} + \cdots = 4 + \frac{32}{10} + \frac{256}{100} + \cdots + 27 + \frac{243}{10} + \frac{2187}{100} + \cdots =$
$4(1 + \frac{8}{10} + \frac{64}{100} + \dots) + 27(1 + \frac{9}{10} + \frac{81}{100} + \dots) = 4\frac{1}{1-\frac{8}{10}} + 27\frac{1}{1-\frac{9}{10}} = 290$

2. Határozza meg a gyökkritériumot haasználva, hogy az alábbi konvergens sorok konvergensek vagy divergensek (lehet, hogy nem használható a gyökkritérium :():

a. $\sum_{n=1}^{\infty} \frac{5^n}{7^n}$: $\lim_{n\to\infty} \sqrt[n]{\frac{5^n}{7^n}} = \lim_{n\to\infty} \frac{5}{7} = \frac{5}{7} < 1 \to$ konvergens

b. $\sum_{n=1}^{\infty} \frac{6^{n+2}}{4^n}$: $\lim_{n\to\infty} \sqrt[n]{\frac{6^{n+2}}{4^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{36 \cdot 6^n}}{\sqrt[n]{4^n}} = \lim_{n\to\infty} \frac{6}{4}\sqrt[n]{36} = \frac{6}{4} > 1 \to$ divergens

c. $\sum_{n=1}^{\infty} \frac{4^{3n}}{10^{2n}}$: $\lim_{n\to\infty} \sqrt[n]{\frac{4^{3n}}{10^{2n}}} = \lim_{n\to\infty} \frac{4^3}{10^2} = \frac{64}{100} < 1 \to$ konvergens

d. $\sum_{n=1}^{\infty} \frac{n3^n}{4^n}$: $\lim_{n\to\infty} \sqrt[n]{\frac{n3^n}{4^n}} = \lim_{n\to\infty} \frac{3}{4}\sqrt[n]{n} = \frac{3}{4} < 1 \to$ konvergens

e. $\sum_{n=1}^{\infty} \frac{n^2 11^n}{10^{n+1}}$: $\lim_{n\to\infty} \sqrt[n]{\frac{n^2 11^n}{10^{n+1}}} = \lim_{n\to\infty} \frac{11\sqrt[n]{n^2}}{\sqrt[n]{10 \cdot 10^n}} = \lim_{n\to\infty} \frac{11}{10}\frac{\sqrt[n]{n^2}}{\sqrt[n]{10}} = \frac{11}{10} > 1 \to$ divergens

f. $\sum_{n=1}^{\infty} \frac{2^n + n^2}{3^n}$: $\lim_{n\to\infty} \sqrt[n]{\frac{2^n + n^2}{3^n}} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n(1 + \frac{n^2}{2^n})}{3^n}} = \lim_{n\to\infty} \frac{2}{3}\sqrt[n]{1 + \frac{n^2}{2^n}} = \frac{2}{3} < 1 \to$ konvergens

g. $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$: $\lim_{n\to\infty} \sqrt[n]{\frac{2^n + 3^n}{4^n}} = \lim_{n\to\infty} \sqrt[n]{\frac{3^n(1 + \frac{2^n}{3^n})}{4^n}} = \lim_{n\to\infty} \frac{3}{4}\sqrt[n]{1 + (\frac{2}{3})^n} = \frac{3}{4} < 1 \to$ konvergens

h. $\sum_{n=1}^{\infty} \frac{6^n}{4^n + 5^{n+1}}$: $\lim_{n\to\infty} \sqrt[n]{\frac{6^n}{5^n(5 + \frac{4^n}{5^n})}} = \lim_{n\to\infty} \frac{6}{5}\sqrt[n]{5 + (\frac{4}{5})^n} = \frac{6}{5} > 1 \to$ divergens

i. $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$: $\lim_{n\to\infty} \sqrt[n]{\frac{n}{n^3+1}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n]{n^3+1}} = \frac{1}{1} = 1 \to$ nem használható a gyökkritérium

3. Határozza meg a hányadoskritériumot haasználva, hogy az alábbi konvergens sorok konvergensek vagy divergensek:

a. $\sum_{n=1}^{\infty} \frac{5^n}{7^n}$: $a_n = \frac{5^n}{7^n}$, $a_{n+1} = \frac{5^{n+1}}{7^{n+1}}$, $\lim_{n\to\infty} \frac{\frac{5^{n+1}}{7^{n+1}}}{\frac{5^n}{7^n}} = \lim_{n\to\infty} \frac{5^{n+1}7^n}{7^{n+1}5^n} = \lim_{n\to\infty} \frac{5}{7} = \frac{5}{7} < 1 \to$ konvergens

b. $\sum_{n=1}^{\infty} \frac{n}{2^n}$: $a_n = \frac{n}{2^n}$, $a_{n+1} = \frac{n+1}{2^{n+1}}$, $\lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{(n+1)2^n}{n2^{n+1}} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1 \to$ konvergens

c. $\sum_{n=1}^{\infty} \frac{(n^2+1)3^n}{2^{n+1}}$: $a_n = \frac{(n^2+1)3^n}{2^{n+1}}$, $a_{n+1} = \frac{((n+1)^2+1)3^{n+1}}{2^{(n+1)+1}} = \frac{(n^2+2n+2)3^{n+1}}{2^{n+2}}$,

$\lim_{n\to\infty} \frac{\frac{(n^2+2n+2)3^{n+1}}{2^{n+2}}}{\frac{(n^2+1)3^n}{2^{n+1}}} = \lim_{n\to\infty} \frac{(n^2+2n+2)2^{n+1}3^{n+1}}{(n^2+1)2^{n+2}3^n} = \lim_{n\to\infty} \frac{3}{2}\frac{n^2+2n+2}{n^2+1} = \frac{3}{2} > 1 \to$ divergens

d. $\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n}$: $a_n = \frac{2^n+3^n}{4^n}$, $a_{n+1} = \frac{2^{n+1}+3^{n+1}}{4^{n+1}}$,

$\lim_{n\to\infty} \frac{\frac{2^{n+1}+3^{n+1}}{4^{n+1}}}{\frac{2^n+3^n}{4^n}} = \lim_{n\to\infty} \frac{4^n(2^{n+1}+3^{n+1})}{4^{n+1}(2^n+3^n)} = \lim_{n\to\infty} \frac{1}{4}\frac{2^{n+1}+3^{n+1}}{2^n+3^n} = \lim_{n\to\infty} \frac{1}{4}\frac{3^n(3+\frac{2^{n+1}}{3^n})}{3^n(1+\frac{2^n}{3^n})} = \frac{3}{4} < 1 \to$ konvergens

e. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$: $a_n = \frac{2^n}{n!}$, $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$, $\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2^{n+1}n!}{2^n(n+1)!} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1 \to$ konvergens

f. $\sum_{n=1}^{\infty} \frac{(n+1)!}{(2n)!}$: $a_n = \frac{(n+1)!}{(2n)!}$, $a_{n+1} = \frac{((n+1)+1)!}{(2(n+1))!} = \frac{(n+2)!}{(2n+2)!}$, $\lim_{n\to\infty} \frac{\frac{(n+2)!}{(2n+2)!}}{\frac{(n+1)!}{(2n)!}} = \lim_{n\to\infty} \frac{(n+2)!(2n)!}{(n+1)!(2n+2)!} =$ $\lim_{n\to\infty} \frac{n+2}{(2n+1)(2n+2)} = \lim_{n\to\infty} \frac{n+2}{4n^2+6n+1} = 0 < 1 \to$ konvergens

g. $\sum_{n=1}^{\infty} \frac{n+1}{n!}$: $a_n = \frac{n+1}{n!}$, $a_{n+1} = \frac{(n+1)+1}{(n+1)!}$, $\lim_{n\to\infty} \frac{\frac{(n+1)+1}{(n+1)!}}{\frac{n+1}{n!}} = \lim_{n\to\infty} \frac{(n+2)n!}{(n+1)(n+1)!} = \lim_{n\to\infty} \frac{n+2}{(n+1)^2} =$ $\lim_{n\to\infty} \frac{n+2}{n^2+2n+1} = 0 \to$ konvergens

h.* $\sum_{n=1}^{\infty} \frac{n!}{n^n}$: $a_n = \frac{n!}{n^n}$, $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$, $\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} =$ $\lim_{n\to\infty} \frac{1}{\frac{(n+1)^n}{n^n}} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1 \to$ konvergens

4. Határozza meg az integrálkritériummal, hogy az alábbi végtelen sorok konvergensek vagy divergensek:

a. $\sum_{n=1}^{\infty} \frac{1}{n^2}$: $f(x) = \frac{1}{x^2}$, $\int_1^{\infty} \frac{1}{x^2}dx = \lim_{B\to\infty} \int_1^B \frac{1}{x^2}dx = \lim_{B\to\infty} [\frac{-1}{x}]_1^B = \lim_{B\to\infty} \left(\frac{-1}{B} - \frac{-1}{1}\right) = 1 \to$ konvergens

b. $\sum_{n=1}^{\infty} \frac{3}{2n}$: $f(x) = \frac{3}{2x}$, $\int_1^{\infty} \frac{3}{2x}dx = \lim_{B\to\infty} \int_1^B \frac{3}{2x}dx = \lim_{B\to\infty} [\frac{3}{2}\ln x]_1^B = \lim_{B\to\infty} \left(\frac{3}{2}\ln B - \frac{3}{2}\ln 1\right) =$ $+\infty \to$ divergens

c. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$: $f(x) = \frac{1}{\sqrt[3]{x}}$, $\int_1^{\infty} \frac{1}{\sqrt[3]{x}} dx = \lim_{B\to\infty} \int_1^B x^{-1/3} dx = \lim_{B\to\infty} [\frac{x^{2/3}}{2/3}]_1^B = \lim_{B\to\infty} \left(\frac{3}{2}B^{2/3} - \frac{2}{3}\right) = \infty \to$ divergens

d. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$: $f(x) = \frac{1}{x^2+1}$, $\int_1^{\infty} \frac{1}{x^2+1} dx = \lim_{B\to\infty} \int_1^B \frac{1}{x^2+1} dx = \lim_{B\to\infty} [\operatorname{arctg} x]_1^B = \lim_{B\to\infty} (arctgB - arctg1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \to$ konvergens

e. $\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$: $f(x) = \frac{1}{x \ln^2 x}$, $\int_2^{\infty} \frac{1}{x \ln^2 x} dx = \lim_{B\to\infty} \int_2^B \frac{1}{x} \ln^{-2} x dx = \lim_{B\to\infty} [\frac{-1}{\ln x}]_2^B = \lim_{B\to\infty} \left(\frac{-1}{\ln B} - \frac{-1}{\ln 2}\right) = \frac{1}{\ln 2} \to$ konvergens

f. $\sum_{n=2}^{\infty} \frac{\ln^3 n}{n}$: $f(x) = \frac{\ln^3 x}{x}$, $\int_2^{\infty} \frac{\ln^3 x}{x} dx = \lim_{B\to\infty} \int_2^B \frac{1}{x} \ln^3 x dx = \lim_{B\to\infty} [\frac{\ln^4 x}{4}]_2^B = \lim_{B\to\infty} \left(\frac{\ln^4 B}{4} - \frac{\ln^4 2}{4}\right) = \infty \to$ divergens

g. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$: $f(x) = \frac{1}{x \ln x}$, $\int_2^{\infty} \frac{1}{\ln x} dx = \lim_{B\to\infty} \int_2^B \frac{\frac{1}{x}}{\ln x} dx = \lim_{B\to\infty} [\ln |lnx|]_2^B = \lim_{B\to\infty} (\ln |\ln B| - \ln |\ln 2|) = \infty \to$ divergens

5. Döntse el az alábbi alternáló sorokról, hogy konvergensek vagy divergensek:

a. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \cdots + \frac{(-1)^n}{2n-1} +$: $a_n = \frac{1}{2n-1}$ monoton csökkenve tart a 0-hoz, tehát konvergens.

b. $1 - 2 + 3 - 4 + 5 - +$: $a_n = n$, tehát $\lim_{n\to\infty} a_n = 0$ nem teljesül, ezért divergens

c. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$: $a_n = |\frac{(-1)^{n-1}}{\sqrt{n}}| = \frac{1}{\sqrt{n}}$, monoton csökkenve tart a 0-hoz, tehát konvergens.

d. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$: $a_n = |\frac{(-1)^n}{2^n}| = \frac{1}{2^n}$, ami monoton csökkenve tart a 0-hoz, így konvergens.

e. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}$: $a_n = |\frac{(-1)^n n}{2^n}| = \frac{n}{2^n}$, ami 0-hoz tart; még a monoton csökkenést kell ellenőrizni: $a_{n+1} = \frac{n+1}{2^{n+1}}$,

$$a_n \geq a_{n+1} \Leftrightarrow \frac{n}{2^n} \geq \frac{n+1}{2^{n+1}} \Leftrightarrow n2^{n+1} \geq (n+1)2^n \Leftrightarrow 2n \geq n+1 \Leftrightarrow n \geq 1,$$

ami teljesül, tehát monoton csökkenve tart a_n a 0-hoz, így konvergens.

f. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$: $a_n = |\frac{(-1)^n n}{n^2+1}| = \frac{n}{n^2+1}$, ami 0-hoz tart; még a monoton csökkenést kell ellenőrizni: $a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$,

$$a_n \geq a_{n+1} \Leftrightarrow \frac{n}{n^2+1} \geq \frac{n+1}{n^2+2n+2} \Leftrightarrow n(n^2+2n+2) \geq (n+1)(n^2+1) \Leftrightarrow n^3+2n^2+2n \geq n^3+n^2+n+1 \Leftrightarrow n^2+n \geq 1,$$

ami teljesül, tehát monoton csökkenve tart a_n a 0-hoz, így konvergens.

g. $\sum_{n=1}^{\infty} \frac{(-4)^{n-1}}{n2^{2n}}$: $\sum_{n=1}^{\infty} \frac{(-4)^{n-1}}{n2^{2n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}4^{n-1}}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n}$, ami egy alternáló sor, $a_n = |\frac{(-1)^{n-1}}{4n}| = \frac{1}{4n}$, ami monoton csökkenve tart a 0-hoz, tehát konvergens

h. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{n!}$: $a_n = |\frac{(-1)^{n-1}2^n}{n!}| = \frac{2^n}{n!}$, ami 0-hoz tart; még a monoton csökkenést kell ellenőrizni: $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$,

$$a_n \geq a_{n+1} \Leftrightarrow \frac{2^n}{n!} \geq \frac{2^{n+1}}{(n+1)!} \Leftrightarrow 2^n(n+1)! \geq 2^{n+1}n! \Leftrightarrow n+1 \geq 2 \Leftrightarrow n \geq 1,$$

ami teljesül, tehát monoton csökkenve tart a_n a 0-hoz, így konvergens.