3. MATEMATIKA A2 FELADATSOR

1. Határozza meg az alábbi 2π-szerint periodikus függvények Fourier-sorának első öt nemnulla tagját:

a. $f(x) = \begin{cases} 0 & \text{ha } 0 \le x \le \pi, \\ 1 & \text{ha } \pi < x < 2\pi, \end{cases}$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{\pi} 0dx + \frac{1}{2\pi}\int_{\pi}^{2\pi} 1dx = 0 + \frac{1}{2\pi}[x]_{\pi}^{2\pi} = \frac{1}{2\pi}(2\pi - \pi) = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_0^{\pi} 0\cdot\cos nxdx + \frac{1}{\pi}\int_{\pi}^{2\pi} 1\cdot\cos nxdx =$$

$$0 + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi} = \frac{1}{\pi}\left(\frac{\sin 2n\pi}{n} - \frac{\sin n\pi}{n}\right) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_0^{\pi} 0\cdot\sin nxdx + \frac{1}{\pi}\int_{\pi}^{2\pi} 1\cdot\sin nxdx =$$

$$0 + \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{\pi}^{2\pi} = \frac{1}{\pi}\left(-\frac{\cos 2n\pi}{n} - \left(-\frac{\cos n\pi}{n}\right)\right) = \frac{1}{\pi}\frac{(-1)^n - 1}{n} = \begin{cases} 0 & \text{ha } n \text{ páros} \\ -\frac{2}{n\pi} & \text{ha } n \text{ páratlan} \end{cases}$$

A Fourier-sor:

$$a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx) = \frac{1}{2} - \frac{2}{\pi}\sin x - \frac{2}{3\pi}\sin 3x - \frac{2}{5\pi}\sin 5x - \frac{2}{7\pi}\sin 7x -$$

b. $f(x) = \begin{cases} 1 & \text{ha } 0 \le x \le \frac{2\pi}{3}, \\ 0 & \text{ha } \frac{2\pi}{3} < x < 2\pi, \end{cases}$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{\frac{2\pi}{3}} 1dx + \frac{1}{2\pi}\int_{\frac{2\pi}{3}}^{2\pi} 0dx = \frac{1}{2\pi}[x]_0^{\frac{2\pi}{3}} + 0 = \frac{1}{2\pi}\frac{2\pi}{3} = \frac{1}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_0^{\frac{2\pi}{3}} 1\cdot\cos nxdx + \frac{1}{\pi}\int_{\frac{2\pi}{3}}^{2\pi} 0\cdot\cos nxdx =$$

$$\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\frac{2\pi}{3}} + 0 = \frac{1}{\pi}\left(\frac{\sin\frac{2n\pi}{3}}{n} - \frac{\sin 0}{n}\right) = \frac{\sin\frac{2n\pi}{3}}{n\pi} =$$

$$= \begin{cases} \frac{\sin\frac{2\cdot 3k\pi}{3}}{\cdot 3k\pi} = \frac{\sin 2k\pi}{3k\pi} = 0 & \text{ha } n = 3k \\ \frac{1\cdot\sin\frac{(3k+1)\pi}{3}}{(3k+1)\pi} = \frac{\sin(2k\pi + \frac{2\pi}{3})}{(3k+1)\pi} = \frac{\frac{\sqrt{3}}{2}}{(3k+1)\pi} = \frac{\sqrt{3}}{(6k+2)\pi} & \text{ha } n = 3k+1 \\ \frac{\sin\frac{2\cdot(3k+2)\pi}{3}}{(3k+2)\pi} = \frac{\sin(2k\pi + \frac{4\pi}{3})}{(3k+2)\pi} = \frac{-\frac{\sqrt{3}}{2}}{(3k+2)\pi} = -\frac{\sqrt{3}}{(6k+4)\pi} & \text{ha } n = 3k+2 \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_0^{\frac{2\pi}{3}} 1\cdot\sin nxdx + \frac{1}{\pi}\int_{\frac{2\pi}{3}}^{2\pi} 0\cdot\sin nxdx =$$

$$\frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\frac{2\pi}{3}} + 0 = \frac{1}{\pi}\left(-\frac{\cos\frac{2n\pi}{3}}{n} - \left(-\frac{\cos 0}{n}\right)\right) = \frac{1 - \cos\frac{2n\pi}{3}}{n\pi} =$$

$$= \begin{cases} \frac{1-\cos\frac{2\cdot 3k\pi}{3}}{3k\pi} = \frac{1-\cos 2k\pi}{3k\pi} = 0 & \text{ha } n = 3k \\ \frac{1-\cos\frac{2\cdot(3k+1)\pi}{3}}{(3k+1)\pi} = \frac{1-\cos(2k\pi + \frac{2\pi}{3})}{(3k+1)\pi} = \frac{1-\left(-\frac{1}{2}\right)}{(3k+1)\pi} = \frac{3}{(6k+2)\pi} & \text{ha } n = 3k+1 \\ \frac{1-\cos\frac{2\cdot(3k+2)\pi}{3}}{(3k+2)\pi} = \frac{1-\cos(2k\pi + \frac{4\pi}{3})}{(3k+2)\pi} = \frac{1-\left(-\frac{1}{2}\right)}{(3k+2)\pi} = \frac{3}{(6k+4)\pi} & \text{ha } n = 3k+2 \end{cases}$$

A Fourier-sor:

$$a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{1}{3}+\left(\frac{\sqrt{3}}{2\pi}\cos x+\frac{3}{2\pi}\sin x\right)+\left(-\frac{\sqrt{3}}{4\pi}\cos 2x+\frac{3}{4\pi}\sin 2x\right)+\dots$$

c. $f(x)=\begin{cases} x & \text{ha } 0\le x\le\pi, \\ 0 & \text{ha } \pi<x<2\pi, \end{cases}$

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)dx=\frac{1}{2\pi}\int_0^{\pi}xdx+\frac{1}{2\pi}\int_{\pi}^{2\pi}0dx=\frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{\pi}+0=\frac{1}{2\pi}(\frac{\pi^2}{2}-0)=\frac{\pi}{4}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_0^{\pi}x\cdot\cos nxdx+\frac{1}{\pi}\int_{\pi}^{2\pi}0\cdot\cos nxdx=$$

Parciálisan integrálunk: $u=x$, $v'=\cos nx$, $u'=1$, $v=\frac{\sin nx}{n}$

$$\frac{1}{\pi}\left(\left[x\frac{\sin nx}{n}\right]_0^{\pi}-\int_0^{\pi}1\cdot\frac{\sin nx}{n}dx\right)+0=\frac{1}{\pi}\left(\pi\frac{\sin n\pi}{n}-\frac{\sin 0}{n}-\left[-\frac{\cos nx}{n^2}\right]_0^{\pi}\right)=$$

$$\frac{1}{\pi}\left(0+\frac{\cos n\pi}{n^2}-\frac{\cos 0}{n^2}\right)=\frac{(-1)^n-1}{n^2\pi}=\begin{cases} 0 & \text{ha } n \text{ páros} \\ \frac{-2}{n^2\pi} & \text{ha } n \text{ páratlan} \end{cases}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_0^{\pi}x\cdot\sin nxdx+\frac{1}{\pi}\int_{\pi}^{2\pi}0\cdot\sin nxdx=$$

Parciálisan integrálunk: $u=x$, $v'=\sin nx$, $u'=1$, $v=-\frac{\cos nx}{n}$

$$\frac{1}{\pi}\left(\left[-x\frac{\cos nx}{n}\right]_0^{\pi}-\int_0^{\pi}1\cdot\left(-\frac{\cos nx}{n}\right)dx\right)+0=\frac{1}{\pi}\left(-\pi\frac{\cos n\pi}{n}-\left(-0\frac{\cos 0}{n}\right)+\left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right)=$$

$$\frac{1}{\pi}\left(\frac{-\pi(-1)^n}{n}+\left(\frac{\sin n\pi}{n^2}-\left(\frac{\sin 0}{n^2}\right)\right)\right)=\frac{-\pi(-1)^n}{n\pi}=\frac{(-1)^{n+1}}{n}$$

A Fourier-sor:

$$a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{\pi}{4}+(-\frac{2}{\pi}\cos x+\sin x)-\frac{1}{2}\sin 2x+(-\frac{2}{9\pi}\cos 3x+\frac{1}{3}\sin 3x)-+$$

d. $f(x)=2\pi-x$, ha $0\le x<2\pi$

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)dx=\frac{1}{2\pi}\int_0^{2\pi}2\pi-xdx=\frac{1}{2\pi}\left[2\pi x-\frac{x^2}{2}\right]_0^{2\pi}=\frac{1}{2\pi}(4\pi^2-\frac{4\pi^2}{2})=\pi$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_0^{\pi}(2\pi-x)\cdot\cos nxdx=$$

Parciálisan integrálunk: $u=2\pi-x$, $v'=\cos nx$, $u'=-1$, $v=\frac{\sin nx}{n}$

$$\frac{1}{\pi}\left(\left[(2\pi-x)\frac{\sin nx}{n}\right]_0^{2\pi}-\int_0^{\pi}-1\cdot\frac{\sin nx}{n}dx\right)+0=\frac{1}{\pi}\left((2\pi-2\pi)\frac{\sin 2n\pi}{n}-(2\pi-0)\frac{\sin 0}{n}+\left[-\frac{\cos nx}{n^2}\right]_0^{2\pi}\right)=$$

$$\frac{1}{\pi}\left(-\frac{\cos 2n\pi}{n^2}-\left(-\frac{\cos 0}{n^2}\right)\right)=\frac{-1+1}{n^2\pi}=0$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx=\frac{1}{\pi}\int_0^{2\pi}(2\pi-x)\cdot\sin nx dx=$$

Parciálisan integrálunk: $u=2\pi-x$, $v'=\sin nx$, $u'=-1$, $v=-\frac{\cos nx}{n}$

$$\frac{1}{\pi}\left(\left[-(2\pi-x)\frac{\cos nx}{n}\right]_0^{2\pi}-\int_0^{2\pi}(-1)\left(-\frac{\cos nx}{n}\right)dx\right)=$$

$$\frac{1}{\pi}\left(-(2\pi-2\pi)\frac{\cos 2n\pi}{n}-\left(-(2\pi-0)\frac{\cos 0}{n}\right)-\left[\frac{\sin nx}{n^2}\right]_0^{2\pi}\right)=$$

$$\frac{1}{\pi}\left(\frac{2\pi}{n}-\left(\frac{\sin 2n\pi}{n^2}-\frac{\sin 0}{n^2}\right)\right)=\frac{2\pi}{n\pi}=\frac{2}{n}$$

A Fourier-sor:

$$a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\pi+2\sin x+\sin 2x+\frac{2}{3}\sin 3x+\frac{2}{4}\sin 4x+\dots$$