Exercise 1.7

There are 20 families in the community. Their distribution according to the number of children is

i	1	2	3	4	5
number of families with i children	5	7	4	3	1.

(a) If a family is chosen uniformly at random from the 20 families, then the probability that it has exactly i children is simply the proportion of families with i children. Let A_i denote the event that we pick a family with i children.

Hence

$$\begin{aligned}
\mathbb{P}(A_1) &= \frac{5}{20} = \frac{1}{4}, \\
\mathbb{P}(A_2) &= \frac{7}{20}, \\
\mathbb{P}(A_3) &= \frac{4}{20} = \frac{1}{5}, \\
\mathbb{P}(A_4) &= \frac{3}{20}, \\
\mathbb{P}(A_5) &= \frac{1}{20}.
\end{aligned}$$

(b) Now a child is chosen uniformly at random rather than a family. First, let us compute the total number of children:

$$5 \cdot 1 + 7 \cdot 2 + 4 \cdot 3 + 3 \cdot 4 + 1 \cdot 5 = 5 + 14 + 12 + 12 + 5 = 48.$$

The families with i children contain altogether i times as many children as there are such families. Let B_i denote the event that we pick a child that comes from a family with i children. We have

$$\begin{aligned}
\mathbb{P}^*(B_1) &= \frac{5}{48}, \\
\mathbb{P}^*(B_2) &= \frac{14}{48} = \frac{7}{24}, \\
\mathbb{P}^*(B_3) &= \frac{12}{48} = \frac{1}{4}, \\
\mathbb{P}^*(B_4) &= \frac{12}{48} = \frac{1}{4}, \\
\mathbb{P}^*(B_5) &= \frac{5}{48}.
\end{aligned}$$

We used $\mathbb{P}^*$ instead of $\mathbb{P}$ because now we use a different sampling procedure, thus we use a different probability space.

Exercise 1.8

We assume that each ball is thrown independently and uniformly at random into one of the urns.

(a) We throw five balls into two urns. There are

$$2^5$$

possible allocations of the balls.

Exactly one urn remains empty if and only if all five balls are thrown into the same urn. There are two such allocations: either all balls go into the first urn or all balls go into the second urn. Thus

$$\boxed{\mathbb{P}(\text{exactly one urn is empty}) = \frac{2}{2^5} = \frac{1}{16}.}$$

(b) Now five balls are thrown into five urns. The total number of possible allocations is

$$5^5.$$

If exactly one urn remains empty, then each of the other four urns must contain at least one ball. Since there are five balls in total, the numbers of balls in the urns must, in some order, be

$$(2, 1, 1, 1, 0).$$

The empty urn can be chosen in 5 ways. Among the remaining four urns, the urn containing two balls can be chosen in 4 ways. The two balls that go into this urn can be chosen in

$$\binom{5}{2}$$

ways, and the remaining three balls can be assigned to the remaining three urns in 3! ways. Therefore the number of favorable allocations is

$$5 \cdot 4 \cdot \binom{5}{2} \cdot 3! = 5 \cdot 4 \cdot 10 \cdot 6 = 1200.$$

Hence

$$\boxed{\mathbb{P}(\text{exactly one urn is empty}) = \frac{1200}{5^5} = \frac{48}{125}.}$$

Exercise 1.10

Among the 18 elks, 5 are marked and 13 are unmarked. We choose 4 elks uniformly at random, so the total number of possible groups is

$$\binom{18}{4}.$$

We can choose exactly two marked elks in

$$\binom{5}{2}$$

ways and exactly two unmarked elks in

$$\binom{13}{2}$$

ways. Therefore

$$\begin{aligned}\mathbb{P}(\text{exactly 2 marked elks}) &= \frac{\binom{5}{2}\binom{13}{2}}{\binom{18}{4}} \\ &= \frac{10 \cdot 78}{3060} \\ &= \boxed{\frac{13}{51}}.\end{aligned}$$

Exercise 1.12

The committee consists of 4 women and 3 men. The total number of possible committees is

$$\binom{8}{4}\binom{7}{3}.$$

Let the two men who are unwilling to serve together be denoted by A and B. The committee is incapable of making decisions exactly when both A and B are selected.

If both of them are selected, then the third man must be chosen from the remaining five men. Hence the number of such committees is

$$\binom{8}{4}\binom{5}{1}.$$

Therefore

$$\mathbb{P}(\text{committee cannot make decisions}) = \frac{\binom{8}{4}\binom{5}{1}}{\binom{8}{4}\binom{7}{3}} = \frac{5}{35} = \frac{1}{7}.$$

Thus

$$\boxed{\mathbb{P}(\text{committee can make decisions}) = 1 - \frac{1}{7} = \frac{6}{7}.}$$

Exercise 1.13

A directed path from $(0,0)$ to $(8,6)$ consists of 8 steps to the right and 6 steps upward. Therefore the total number of directed paths is

$$N = \binom{14}{6} = 3003.$$

Let

$$A = (2,3), \qquad B = (6,4)$$

denote the locations of the two land mines.

Paths passing through A.

To get from $(0,0)$ to $(2,3)$, a path must make 2 right steps and 3 upward steps, so there are

$$\binom{5}{2} = 10$$

such paths.

From $(2,3)$ to $(8,6)$, the path must make 6 right steps and 3 upward steps, so there are

$$\binom{9}{3} = 84$$

such paths. Hence the number of paths passing through A is

$$N_A = \binom{5}{2}\binom{9}{3} = 10 \cdot 84 = 840.$$

Paths passing through B.

To get from $(0,0)$ to $(6,4)$, a path must make 6 right steps and 4 upward steps, so there are

$$\binom{10}{4} = 210$$

such paths.

From $(6,4)$ to $(8,6)$, the path must make 2 right steps and 2 upward steps, so there are

$$\binom{4}{2} = 6$$

such paths. Hence

$$N_B = \binom{10}{4}\binom{4}{2} = 210 \cdot 6 = 1260.$$

Paths passing through both land mines.

Any path passing through both land mines must first pass through A and then through B. Between $A = (2, 3)$ and $B = (6, 4)$, the path must make 4 right steps and 1 upward step, so there are

$$\binom{5}{1} = 5$$

possibilities. Therefore

$$N_{A,B} = \binom{5}{2}\binom{5}{1}\binom{4}{2} = 10 \cdot 5 \cdot 6 = 300.$$

By the inclusion–exclusion principle, the number of paths passing through at least one land mine is

$$N_A + N_B - N_{A,B} = 840 + 1260 - 300 = 1800.$$

Thus

$$\boxed{\mathbb{P}(\text{the path hits a land mine}) = \frac{1800}{3003} = \frac{600}{1001}.}$$

Exercise 1.14

The urn contains 3 red, 4 white, and 5 blue balls, for a total of 12 balls.

(a) Drawing without replacement.

The total number of possible samples of five balls is

$$\binom{12}{5}.$$

It is convenient to count the complementary event. All three colors fail to appear exactly when at least one color is missing.

If no red ball is drawn, then all five balls must be chosen from the 9 non-red balls, giving

$$\binom{9}{5}$$

possibilities.

If no white ball is drawn, there are

$$\binom{8}{5}$$

possibilities, and if no blue ball is drawn, there are

$$\binom{7}{5}$$

possibilities.

We must also take intersections into account. A sample containing balls of only one color can have size five only if all five blue balls are chosen, which can happen in

$$\binom{5}{5} = 1$$

way. There are not enough red or white balls to form a sample of five using only one of those colors.

Thus, by inclusion–exclusion, the number of samples in which at least one color is missing is

$$\binom{9}{5} + \binom{8}{5} + \binom{7}{5} - \binom{5}{5} = 126 + 56 + 21 - 1 = 202.$$

Hence the number of favorable samples is

$$\binom{12}{5} - 202 = 792 - 202 = 590.$$

Therefore

$$\boxed{\mathbb{P}(\text{all three colors appear}) = \frac{590}{792} = \frac{295}{396}.}$$

(b) Drawing with replacement.

For a single draw, the probabilities of the three colors are

$$p_R = \frac{3}{12} = \frac{1}{4}, \qquad p_W = \frac{4}{12} = \frac{1}{3}, \qquad p_B = \frac{5}{12}.$$

Again, we use the complementary event and the inclusion–exclusion principle. The probability that no red ball is drawn is

$$\left(\frac{9}{12}\right)^5 = \left(\frac{3}{4}\right)^5.$$

Similarly,

$$\mathbb{P}(\text{no white ball}) = \left(\frac{8}{12}\right)^5 = \left(\frac{2}{3}\right)^5,$$

and

$$\mathbb{P}(\text{no blue ball}) = \left(\frac{7}{12}\right)^5.$$

The pairwise intersections correspond to all five draws having the same color:

$$\left(\frac{1}{4}\right)^5, \qquad \left(\frac{1}{3}\right)^5, \qquad \left(\frac{5}{12}\right)^5.$$

Therefore

$$\begin{aligned}\mathbb{P}(\text{all three colors appear}) &= 1 - \left(\frac{3}{4}\right)^5 - \left(\frac{2}{3}\right)^5 - \left(\frac{7}{12}\right)^5 \\ &\quad + \left(\frac{1}{4}\right)^5 + \left(\frac{1}{3}\right)^5 + \left(\frac{5}{12}\right)^5 \\ &= \boxed{\frac{6025}{10368}}.\end{aligned}$$