

Exercise 2.2

There are 7 horses in the race, and we assume that every ordering of the horses is equally likely. Let

$$A = \{\text{Horse A finishes first, second, or third}\},$$

and

$$B = \{\text{Horse B finishes second}\}.$$

Clearly,

$$\mathbb{P}(A) = \frac{3}{7}, \quad \mathbb{P}(B) = \frac{1}{7}.$$

To calculate $\mathbb{P}(A \cap B)$, condition on Horse B finishing second. Then Horse A is equally likely to occupy any of the remaining 6 positions, and it belongs to the top three precisely when it finishes first or third. Hence

$$\mathbb{P}(A \mid B) = \frac{2}{6} = \frac{1}{3},$$

and therefore

$$\mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(A \mid B) = \frac{1}{7} \cdot \frac{1}{3} = \frac{1}{21}.$$

By the inclusion-exclusion formula,

$$\mathbb{P}(A \cup B) = \frac{3}{7} + \frac{1}{7} - \frac{1}{21} = \frac{11}{21}.$$

Exercise 2.4

Think of the distribution as randomly assigning the 60 chocolate bars to 60 positions, grouped into 10 groups of 6 positions, one group for each child. The 20 positions occupied by white chocolate bars form a uniformly chosen 20-element subset of the 60 positions. Thus there are

$$\binom{60}{20}$$

equally likely possibilities.

For $i = 1, \dots, 10$, let

$$A_i = \{\text{child } i \text{ receives no white chocolate}\}.$$

If k specified children receive no white chocolate, then all 20 white chocolate bars have to occupy positions among the other $60 - 6k$ positions. Therefore

$$\mathbb{P}(A_{i_1} \cap \dots \cap A_{i_k}) = \frac{\binom{60-6k}{20}}{\binom{60}{20}}.$$

By inclusion-exclusion,

$$\mathbb{P}(\text{every child receives at least one white chocolate}) = \sum_{k=0}^{10} (-1)^k \binom{10}{k} \frac{\binom{60-6k}{20}}{\binom{60}{20}}.$$

For $k \geq 7$ the corresponding terms are zero, since fewer than 20 positions remain. Hence

$$\mathbb{P}(\text{every child receives at least one white chocolate}) = \frac{1}{\binom{60}{20}} \sum_{k=0}^6 (-1)^k \binom{10}{k} \binom{60-6k}{20}.$$

Numerically, this is approximately

$$0.398619.$$

Exercise 2.9

We deal all 52 cards of a well-shuffled French deck.

(a) Every card is equally likely to occur in each of the 52 positions. Therefore

$$\mathbb{P}(\text{the ace of hearts is the 11th card}) = \frac{1}{52}.$$

(b) The four positions occupied by the aces form a uniformly chosen 4-element subset of $\{1, \dots, 52\}$. For the first ace to occur in position 11, position 11 must contain an ace and the other three aces must occur among positions $12, \dots, 52$.

There are $\binom{52}{4}$ possible sets of positions of the four aces, and $\binom{41}{3}$ favorable ones. Hence

$$\mathbb{P}(\text{the first ace is the 11th card}) = \frac{\binom{41}{3}}{\binom{52}{4}}.$$

(c) The suit of the first card can be arbitrary. Once it has been dealt, 39 of the remaining 51 cards have a different suit. After two different suits have appeared, 26 of the remaining 50 cards have one of the two unused suits. Finally, 13 of the remaining 49 cards have the fourth suit. Thus

$$\mathbb{P}(\text{the first four cards have different suits}) = \frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49}.$$

(d) For each suit s , let

$$A_s = \{\text{the king of suit } s \text{ is immediately followed by the ace of suit } s\}.$$

If we require this to happen for one specified suit, regard the pair $K_s A_s$ as a single block. There are then 51! possible arrangements, so

$$\mathbb{P}(A_s) = \frac{51!}{52!} = \frac{1}{52}.$$

More generally, if this condition is required for k specified suits, we regard the corresponding k king-ace pairs as k blocks. This leaves $52 - k$ objects to arrange, and consequently

$$\mathbb{P}\left(\bigcap_{s \in S} A_s\right) = \frac{(52 - k)!}{52!}, \quad |S| = k.$$

Applying inclusion-exclusion to the four suits gives

$$\begin{aligned} & \mathbb{P}(\text{no ace immediately follows the king of the same suit}) \\ &= \sum_{k=0}^4 (-1)^k \binom{4}{k} \frac{(52 - k)!}{52!} \\ &= 1 - \frac{4}{52} + \frac{6}{52 \cdot 51} - \frac{4}{52 \cdot 51 \cdot 50} + \frac{1}{52 \cdot 51 \cdot 50 \cdot 49}. \end{aligned}$$

Thus

$$\mathbb{P}(\text{no such king-ace pair occurs}) = 1 - \frac{4}{52} + \frac{6}{52 \cdot 51} - \frac{4}{52 \cdot 51 \cdot 50} + \frac{1}{52 \cdot 51 \cdot 50 \cdot 49}.$$

Exercise 2.12

The players are equally strong, so in every round either of the two players has probability $1/2$ of winning.

The game must last for at least two rounds. Before the final round, the winner has to change after every round; otherwise the game would already have ended. Once the winner of the first round is known, the sequence of winners is therefore determined until the final round.

For example, the possible outcomes begin as

$$AA, \quad BB, \quad ACC, \quad BCC, \quad ACBB, \quad BCBA, \quad ACBAA, \quad BCABB, \dots$$

Here the last letter is always equal to the preceding letter, since the game ends when someone wins twice in a row.

More formally, if the first winner is A , then before the final repetition the winners follow the periodic sequence

$$A, C, B, A, C, B, \dots,$$

whereas if the first winner is B , they follow

$$B, C, A, B, C, A, \dots$$

The sample space consists of all finite sequences obtained by stopping one of these two sequences by repeating its final entry.

(a) A particular outcome consisting of n rounds specifies the winner of each of the n rounds. Since in every round the specified player wins with probability $1/2$, we have

$$\boxed{\mathbb{P}(\text{a particular outcome of length } n) = 2^{-n}.}$$

(b) Suppose first that A wins the first round. The possible winners of the whole game, according to its length, are

n	2	3	4	5	6	7
winner	A	C	B	A	C	B

and this pattern repeats with period 3.

If B wins the first round, the corresponding pattern is

n	2	3	4	5	6	7
winner	B	C	A	B	C	A

Consequently, A wins either on an outcome beginning with A whose length is

$$2, 5, 8, \dots,$$

or on an outcome beginning with B whose length is

$$4, 7, 10, \dots$$

Thus

$$\begin{aligned} \mathbb{P}(A \text{ wins}) &= \sum_{k=0}^{\infty} 2^{-(3k+2)} + \sum_{k=0}^{\infty} 2^{-(3k+4)} \\ &= \frac{1/4}{1 - 1/8} + \frac{1/16}{1 - 1/8} \\ &= \frac{2}{7} + \frac{1}{14} = \boxed{\frac{5}{14}}. \end{aligned}$$

By symmetry,

$$\boxed{\mathbb{P}(B \text{ wins}) = \frac{5}{14}}.$$

Player C wins precisely when the game has length

$$3, 6, 9, \dots,$$

and there are two possible outcomes of each such length, according as A or B won the first round. Hence

$$\begin{aligned}\mathbb{P}(C \text{ wins}) &= 2 \sum_{k=0}^{\infty} 2^{-(3k+3)} \\ &= 2 \frac{1/8}{1 - 1/8} = \boxed{\frac{2}{7}}.\end{aligned}$$

As a check,

$$\frac{5}{14} + \frac{5}{14} + \frac{2}{7} = 1.$$

Exercise 2.14

Let R denote the event that a cake is ruined. We have

$$\mathbb{P}(A) = 0.5, \quad \mathbb{P}(B) = 0.3, \quad \mathbb{P}(C) = 0.2,$$

and

$$\mathbb{P}(R \mid A) = 0.02, \quad \mathbb{P}(R \mid B) = 0.03, \quad \mathbb{P}(R \mid C) = 0.05.$$

By the law of total probability,

$$\begin{aligned}\mathbb{P}(R) &= 0.5 \cdot 0.02 + 0.3 \cdot 0.03 + 0.2 \cdot 0.05 \\ &= 0.01 + 0.009 + 0.01 \\ &= 0.029.\end{aligned}$$

By Bayes' formula,

$$\begin{aligned}\mathbb{P}(A \mid R) &= \frac{\mathbb{P}(R \mid A)\mathbb{P}(A)}{\mathbb{P}(R)} \\ &= \frac{0.02 \cdot 0.5}{0.029} = \frac{10}{29} \approx 0.34483.\end{aligned}$$

Thus approximately

$$\boxed{34.48\%}$$

of the ruined cakes were baked by A .

Exercise 2.15

Let

$$D = \{\text{the fetus has Down syndrome}\},$$

and let $+$ denote a positive test result. We are given

$$\mathbb{P}(D) = 0.001, \quad \mathbb{P}(D^c) = 0.999.$$

The false negative probability is 0.01, so

$$\mathbb{P}(+ \mid D) = 0.99.$$

The false positive probability is 0.05, so

$$\mathbb{P}(+ \mid D^c) = 0.05.$$

By Bayes' formula,

$$\begin{aligned}\mathbb{P}(D \mid +) &= \frac{\mathbb{P}(+ \mid D)\mathbb{P}(D)}{\mathbb{P}(+ \mid D)\mathbb{P}(D) + \mathbb{P}(+ \mid D^c)\mathbb{P}(D^c)} \\ &= \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 + 0.05 \cdot 0.999} \\ &= \frac{0.00099}{0.05094} = \frac{11}{566} \approx 0.01943.\end{aligned}$$

Therefore

$$\boxed{\mathbb{P}(D \mid +) \approx 1.94\%}.$$