Probability 1 – Exercises

Tutorial no. 4 — 8nd October 2026

4.1 We toss a coin 3 times. Let A be the event that there is both at least one heads and at least one tails tossed. Let B be the event, that there is at most one tails in the outcomes. Are the two events independent?

4.2 Let's pick a random number from the set $\{1, 2, 3, \ldots, n\}$ with an equal probability. Let A_p be the event that the chosen number is divisible by p, which is a prime.

(a) Let's show that if $p_1, p_2, \ldots, p_k$ are prime numbers and n is divisible by $p_1, p_2, \ldots, p_k$, then $A_{p_1}, A_{p_2}, \ldots, A_{p_k}$ events are (totally) independent.

(b) Let C_n be the event that the randomly chosen number is a relative prime to n. Prove that

$$\mathbb{P}(C_n) = \prod_{p \text{ prime}, p|n} \left(1 - \frac{1}{p}\right)$$

4.3 We put a knight on a random position on a chess board. What is the expected value of the possible moves from that position?

4.4 In a casino, we play the following game: we bet a number from one to six, then a die is thrown three times. If our number appeared at least once then we get a peso for each appearance of our number, but we have to pay one peso if our number did not appear at all. Is the game fair?

HW **4.5** Andrew and Bob are playing a game. There are 5 red and 5 green balls in an urn. They draw 2 balls from the urn. If they're both the same color, Andrew pays Bob 1000 forints. If they're different colors, Bob pays Andrew x forint. How much should x be to make the game fair?

4.6 (St. Petersburg paradox) We toss a coin until it lands on heads. If the n^{th} toss is heads, the player wins 2^n forint. Show that the expected value of the reward is infinite.

(a) Is it worth paying 1 million Ft to play the game?

(b) Is it worth paying 1 million Ft per game if we can play as much as we want, and only have to pay after we decide to stop?

4.7 Show that for a non-negative, integer valued random variable N, where $\mathbb{E}(N) < \infty$

$$\mathbb{E}(N) = \sum_{i=1}^{\infty} P(N \geq i)$$

4.8 On 4 buses 148 students are travelling all together. The number of students on each bus is 40, 33, 25 and 50, respectively. Choose a student uniformly at random, and let X denote the number of students on their bus. Also choose a bus driver uniformly at random, and let Y denote the number of students on their bus.

(a) Which is larger, $\mathbb{E}(X)$ or $\mathbb{E}(Y)$?

(b) Calculate $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

(c) Calculate $\mathbb{D}^2(X)$ and $\mathbb{D}^2(Y)$.

4.9 Anna and Bianca have two regular dice, one white and one yellow, and they play according to the following rules. First Anna rolls the two dice, and if the two dice have the same number on them, she wins. If Anna rolls different numbers then Bianca gets the dice and now she rolls: if she manages to roll at least one six, she wins. If not then there is no winner in the first round and they repeat the same process, so they roll alternately until one of them wins.

(a) Denote by X the total number of rounds that they play. What is the distribution and the expected value of X?

(b) What is the probability that the entire game ends with Anna winning?

HW 4.10 (3 points) We have two dice, and we're only focusing on the sum. We keep throwing until the sum of the two dice is either 6 or 9.

(a) What is the probability that we stop on 6?

(b) Prove that the number of throws and the sum when we stop are independent. In other words, show that if

$$A_k = \{\text{We throw k times}\} \qquad \text{and} \qquad B = \{\text{We stop on 6}\}$$

then for every $k \geq 1$, A_k and B are independent events.

4.11 There are $m+1$ urns with m balls in each. In the i-th urn, there are $i-1$ red and $m-i+1$ blue balls ($i = 1, \ldots, m+1$). First, we choose an urn uniformly randomly, then we draw n balls from the chosen urn with replacement.

(a) What is the probability that we only draw red balls?

(b) Assuming that the first n draws were red, what is the probability that the $n+1$-th draw will be red as well?

(c) Calculate the limit of the probability obtained in part (b) for fixed n as $m \to \infty$.

4.12 Assuming that the ratio of left-handed people is 1% on average, what is the probability that out of 200 randomly selected people, at least 4 are left-handed?

4.13 I operate the giant dough mixing machine at the muffin factory. On average, how many raisins need to be in a muffin, if we want to make sure that for a randomly selected muffin, the probability of it containing at least 1 raisin is at least 99%?

4.14 At a horse race, the horses run laps and there are many obstacles that they have to jump over. My horse, "Lucky Star", has an equally small probability to knock each obstacle down, independently of the other obstacles, and the probability that she does a lap perfectly is 0.1.

(a) What is the probability that she knocks down at least 4 obstacles in one lap?

(b) At each mistake she gets an injury with probability 1/2, independently of what happens at any other obstacle. What is the probability that she finishes a lap without any injuries?

4.15 A Marathon is held next to the Danube Bend. Sadly, the route was led through a section of tall grass, which was full of ticks. We later found out, that 300 people found one tick on their skin, and 75 found two. Based on this, approximate how many people ran in the race, overall.

4.16 In the center of London, the number of traffic accidents follows a Poisson distribution with parameter 20 on rainy days and parameter 10 on sunny days. In early Novermber in London, the probability that it rains (the whole day) is 0.6; the probability that the weather is clear is thus 0.4. According to The Times, 17 car accidents happened in London's center last Thursday. What is the probability that it was a rainy day?

HW 4.17 In Myocardistan, the rate of heart attacks is 1 per $10,000$ people every month. Let's examine the city of New Cholesterol in Myocardistan, which has $30,000$ inhabitants.

(a) What is the probability that at least 4 people will have a heart attack in a given month?

(b) What is the probability that in a given year, there will be at least 1 month with at least 4 heart attacks?

(c) Starting now, what is the probability that the first month with at least 4 heart attacks will be i months from now? ($i \geq 1$)