Probability 1 — Tutorial 4
Solutions to Exercises 4.4, 4.9, and 4.12–4.16

Exercise 4.4

Let K be the number of times our chosen number appears in the three throws. Then $K \sim \mathrm{Bin}(3, 1/6)$. Our net gain G is K when $K \geq 1$, and -1 when $K = 0$. Thus

$$G = K - \mathbf{1}_{\{K=0\}}.$$

By linearity of expectation,

$$\mathbb{E}G = \mathbb{E}K - \mathbb{P}(K = 0) = 3 \cdot \frac{1}{6} - \left(\frac{5}{6}\right)^3 = \frac{1}{2} - \frac{125}{216} = \boxed{-\frac{17}{216}}.$$

The expected gain is negative, so the game is *not fair* to the player (it favors the casino).

Exercise 4.9

Call one round the sequence consisting of Anna's throw, followed (if necessary) by Bianca's throw. In any round, the probability that Anna wins is

$$a = \mathbb{P}(\text{Anna rolls a double}) = \frac{6}{36} = \frac{1}{6} = \frac{36}{216}.$$

Bianca gets to throw only if Anna does not roll a double. Conditional on Bianca throwing, the probability that she rolls at least one six is $1 - (5/6)^2 = 11/36$. Thus the probability that Bianca wins a round is

$$b = \frac{5}{6} \cdot \frac{11}{36} = \frac{55}{216}.$$

The probability that neither player wins in a round is

$$r = \frac{5}{6}\left(\frac{5}{6}\right)^2 = \frac{125}{216}.$$

Consequently, each new round ends the game with probability

$$q = a + b = 1 - r = \frac{91}{216}.$$

(a) The rounds are independent, and therefore the number X of rounds has a geometric distribution with parameter q:

$$\boxed{\mathbb{P}(X = k) = \left(\frac{125}{216}\right)^{k-1} \frac{91}{216}, \qquad k = 1, 2, \ldots.}$$

In particular,

$$\boxed{\mathbb{E}X = \frac{1}{q} = \frac{216}{91}.}$$

(b) For Anna to win in round k, the first $k - 1$ rounds must have no winner, and she must win round k. Hence

$$\mathbb{P}(\text{Anna wins the game}) = \sum_{k=1}^{\infty} r^{k-1} a = \frac{a}{1 - r} = \frac{1/6}{91/216} = \boxed{\frac{36}{91}}.$$

Exercise 4.12

Assume the handedness of the selected people is independent. If L is the number of left-handed people among the 200 selected people, then $L \sim \text{Bin}(200, 0.01)$. Therefore

$$\boxed{\mathbb{P}(L \geq 4) = 1 - \sum_{k=0}^{3} \binom{200}{k} (0.01)^k (0.99)^{200-k} \approx 0.141966.}$$

Since 200 is large and 0.01 is small, we can also use the Poisson approximation $L \approx \text{Poi}(200 \cdot 0.01) = \text{Poi}(2)$:

$$\mathbb{P}(L \geq 4) \approx 1 - e^{-2} \sum_{k=0}^{3} \frac{2^k}{k!} \approx 0.142877.$$

Exercise 4.13

Under the usual Poisson model for the number R of raisins in a randomly chosen muffin, write $R \sim \text{Poi}(\lambda)$, where $\lambda = \mathbb{E}R$ is the average number of raisins per muffin. We require

$$\mathbb{P}(R \geq 1) = 1 - \mathbb{P}(R = 0) = 1 - e^{-\lambda} \geq 0.99.$$

Equivalently, $e^{-\lambda} \leq 0.01$, or

$$\boxed{\lambda \geq \log 100 \approx 4.60517.}$$

Thus the mean should be at least about 4.61 raisins per muffin. The Poisson assumption is essential: the mean alone does not determine $\mathbb{P}(R \geq 1)$ for an arbitrary distribution.

Exercise 4.14

We use the Poisson approximation appropriate for a large number of independent obstacles, each with a small probability of being knocked down. Let $K \sim \text{Poi}(\lambda)$ be the number of knocked-down obstacles in one lap. Since a perfect lap means $K = 0$,

$$e^{-\lambda} = \mathbb{P}(K = 0) = 0.1, \qquad \text{so} \qquad \lambda = \log 10.$$

(a) The probability of at least four mistakes is

$$\boxed{\mathbb{P}(K \geq 4) = 1 - e^{-\lambda} \sum_{k=0}^{3} \frac{\lambda^k}{k!} = 1 - \frac{1}{10} \sum_{k=0}^{3} \frac{(\log 10)^k}{k!} \approx 0.201179.}$$

(b) Each mistake independently causes an injury with probability 1/2. Conditional on $K = k$, the probability of no injury is 2^{-k}. Hence

$$\begin{aligned} \mathbb{P}(\text{no injuries}) &= \mathbb{E}\left[2^{-K}\right] = \sum_{k=0}^{\infty} 2^{-k} e^{-\lambda} \frac{\lambda^k}{k!} \\ &= e^{-\lambda} e^{\lambda/2} = e^{-\lambda/2} = \boxed{\frac{1}{\sqrt{10}} \approx 0.316228}. \end{aligned}$$

These answers use the Poisson approximation. For a fixed but unspecified number n of obstacles, each knocked down with probability p, the exact assumptions only give $(1-p)^n = 0.1$. In that model the exact answers depend on n and p, so cannot be determined uniquely from the stated information.

Exercise 4.15

Suppose that the number of ticks found by each runner is approximately Poisson with mean λ, and let N be the total number of runners. Then the expected numbers of runners finding exactly one and exactly two ticks are, respectively,

$$N\,\mathbb{P}(K=1) = Ne^{-\lambda}\lambda, \qquad N\,\mathbb{P}(K=2) = Ne^{-\lambda}\frac{\lambda^2}{2}.$$

Using the observed counts 300 and 75 to estimate these quantities,

$$Ne^{-\lambda}\lambda \approx 300, \qquad Ne^{-\lambda}\frac{\lambda^2}{2} \approx 75.$$

Dividing the second relation by the first gives

$$\frac{\lambda}{2} \approx \frac{75}{300} = \frac{1}{4}, \qquad \text{hence} \qquad \lambda \approx \frac{1}{2}.$$

Substituting into the first relation yields

$$N \cdot \frac{1}{2}e^{-1/2} \approx 300, \qquad \text{so} \qquad \boxed{N \approx 600e^{1/2} \approx 989.}$$

Thus we estimate that roughly 990 people ran in the race.

Exercise 4.16

Let R denote a rainy day, S a sunny day, and A the event that exactly 17 accidents occur. We are given

$$\mathbb{P}(R) = 0.6, \quad \mathbb{P}(S) = 0.4, \quad \mathbb{P}(A \mid R) = e^{-20}\frac{20^{17}}{17!}, \quad \mathbb{P}(A \mid S) = e^{-10}\frac{10^{17}}{17!}.$$

By Bayes' formula,

$$\begin{aligned}
\mathbb{P}(R \mid A) &= \frac{\mathbb{P}(R)\mathbb{P}(A \mid R)}{\mathbb{P}(R)\mathbb{P}(A \mid R) + \mathbb{P}(S)\mathbb{P}(A \mid S)} \\
&= \frac{0.6e^{-20}20^{17}}{0.6e^{-20}20^{17} + 0.4e^{-10}10^{17}} \\
&= \boxed{\frac{3 \cdot 2^{17}}{3 \cdot 2^{17} + 2e^{10}} \approx 0.899254}.
\end{aligned}$$

Therefore, given that there were 17 accidents, the probability that the day was rainy is approximately 89.9%.