Probability 1 – Exercises

Tutorial no. 7 — 5th November 2026

7.1 Choose a point uniformly from the three-dimensional unit ball, and let D denote its distance from the center. Determine the CDF and the expected value of D.

7.2 A stick of length ℓ is broken at two independently and uniformly chosen points. What is the expected length of the shortest of the three resulting pieces?

7.3 Choose a point uniformly from a regular hexagon with side length 1. Let ξ be its distance from the closest side. Determine the CDF and the probability density function (PDF) of ξ.

HW 7.4 A rectangle has side lengths 1 and $a > 0$. Choose one point on each of its two sides of length 1, independently and uniformly. Let X be the distance between the two points. Find the PDF of X.

7.5 The waiting time between successive metro trains in Budapest is modeled by an exponential distribution with mean 2 minutes. I have just missed a train.

(a) What is the probability that I must wait at least 5 minutes for the next train?

(b) After waiting 3 minutes, what is the probability that I must wait at least another 5 minutes?

7.6 The expected lifetime of a radioactive isotope called *probabilitium* is 1 year. Assuming its lifetime is exponentially distributed, find its half-life.

7.7 The lifetime of a "Mehr Licht" light bulb is exponentially distributed. According to the manufacturer, 90% of the bulbs last at least one year. How long can the warranty period be if the company wants at most 1% of bulbs to fail during the warranty period?

HW 7.8 Selena watches shooting stars on a summer night. Model the stars she sees as a Poisson process: the mean number of sightings is 3 per half-hour. She starts watching on August 19 at 22:00.

(a) What is the probability that she sees no shooting stars between 22:00 and 22:25?

(b) Let T be the waiting time, in minutes, until she sees her first shooting star. Find the CDF and PDF of T.

(c) What does this example illustrate about the relationship between Poisson counts and exponential waiting times?

7.9 Suppose IQ scores have a normal distribution with mean 100 and standard deviation 15. Assume that these scores accurately reflect the population.

(a) What percentage of people have an IQ score between 95 and 110?

(b) How wide must an interval centered at 100 be to contain 50% of the population?

(c) In a town with 2500 inhabitants, what is the expected number of people with an IQ score of at least 125?

7.10 Approximate the probability that, among 100 000 independently dealt five-card poker hands, the number of full houses is between 128 and 158 (inclusive). A full house consists of three cards of one rank and two cards of another rank, from a standard 52-card deck.

7.11 An unknown proportion p of Budapest residents smoke. We ask n independently sampled residents, and k of them report that they smoke. We estimate p by $\widehat{p} = k/n$. How large should n be if we want

$$\mathbb{P}\big(|\widehat{p} - p| \le 0.02\big) \ge 0.93$$

regardless of the true value of $p \in [0, 1]$?

7.12 The lifetime of a randomly chosen glass in the university cafeteria is exponentially distributed with mean 48 months. Assume the lifetimes of different glasses are independent. Find the probability that within two years

(a) at most 3 out of 6 glasses break;

(b) at most 210 out of 500 glasses break.

Instruction for (b): Use the de Moivre–Laplace theorem.

HW 7.13 In Budapest, 43% of residents support banning smoking in public places. Suppose independently chosen respondents give their true opinions. Let A_n be the event that at least 40% of a sample of n residents support the ban.

(a) Calculate $\mathbb{P}(A_{11})$; **(b)** approximate $\mathbb{P}(A_{101})$; **(c)** approximate $\mathbb{P}(A_{1001})$.

(d) How large should n be to ensure that $\mathbb{P}(A_n) \geq 0.95$?

7.14 Determine the PDFs of the following transformed random variables.

(a) If ξ is standard normal, find the PDF of $X = \xi^2$.

HW (b) If $\xi \sim \mathrm{EXP}(\lambda)$, with rate $\lambda > 0$, find the PDFs of $X = 3\xi - 2$ and $Z = \sqrt[4]{\xi}$.

(c) If $\xi \sim \mathrm{UNI}[-1, 1]$, find the PDFs of $X = \xi^4$ and $U = \sin(\pi\xi)$.

(d) If $\xi \sim \mathrm{EXP}(\lambda)$, with rate $\lambda > 0$, find the PDF of $Y = e^{\xi}$.

HW (e) If $\xi \sim \mathrm{UNI}[-3, 4]$, find the PDF of $Z = \xi^2$.

7.15 Let X be a random variable with a continuous CDF F. What is the distribution of $Y = F(X)$?

7.16 Let $P = (1, 1)$ and $Q = (U, 0)$, where U is uniformly distributed on an interval of the x-axis. Find the PDF of the distance $\xi = \|P - Q\|$ in each of the following cases:

(a) $U \sim \mathrm{UNI}[0, 1]$;

(b) $U \sim \mathrm{UNI}[0, 2]$.

Compare the two distributions.

7.17 Choose two points independently and uniformly on the unit circle. Determine the cumulative distribution function (CDF) of their Euclidean distance.