VSZ1 VIZSGA, 2022 JAN 14.

$\boxed{\text{ECM 1}}$ a) $X \sim POI(\lambda)$, HA $\mathbb{P}(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$, $k \in \mathbb{N}_0$

$$M(t) = \mathbb{E}(e^{t \cdot X}) = \sum_{k=0}^{\infty} e^{t \cdot k} \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{(e^t \cdot \lambda)^k}{k!} =$$

$$= e^{-\lambda} \cdot e^{e^t \cdot \lambda} = \exp(\lambda \cdot (e^t - 1))$$

$M'(t) = M(t) \cdot \lambda \cdot e^t \qquad M''(t) = M(t) \cdot \lambda^2 \cdot e^{2t} + M(t) \cdot \lambda \cdot e^t$

$\mathbb{E}(X) = M'(0) = \lambda \qquad \mathbb{E}(X^2) = M''(0) = \lambda^2 + \lambda$

$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda$

(b) TÉTEL: HA $X_n \sim BIN(n, p_n)$ ÉS $\boxed{\lim_{n \to \infty} n \cdot p_n = \lambda}$

AKKOR $\forall k \in \mathbb{N}_0$: $\lim_{n \to \infty} \mathbb{P}(X_n = k) = \mathbb{P}(X = k)$,

AHOL $X \sim POI(\lambda)$.

BIZ: $\lim_{n \to \infty} \mathbb{P}(X_n = k) = \lim_{n \to \infty} \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k!} \cdot p_n^k \cdot (1-p_n)^{n-k} =$

$$= \frac{1}{k!} \cdot \lim_{n \to \infty} \underbrace{(n \cdot p_n)}_{\xrightarrow{n \to \infty} \lambda} \cdot \underbrace{((n-1) \cdot p_n)}_{\xrightarrow{n \to \infty} \lambda} \cdot \ldots \cdot \underbrace{((n-k+1) \cdot p_n)}_{\xrightarrow{n \to \infty} \lambda} \cdot \underbrace{(1-p_n)^{n-k}}_{\xrightarrow{n \to \infty} e^{-\lambda}} =$$

$$= \frac{1}{k!} \cdot \lambda^k \cdot e^{-\lambda} = \mathbb{P}(X = k)$$

1. OLDAL

$\boxed{\text{ELM 2}}$ a) $f_X(x) := \int_{-\infty}^{\infty} f(x,y)\,dy$, $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

X ÉS Y FÜGGETLENEK $\iff f(x,y) = f_X(x)\cdot f_Y(y)$

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b) $Z_1 := X + Y \quad F_{Z_1}(z) := \mathbb{P}(Z_1 < z) = \mathbb{P}(X + Y < z) =$

$= \mathbb{P}((X,Y) \in S_z)$, AHOL $S_z = \{(x,y) : x+y < z\}$

$= \iint_{S_z} f(x,y)\,dx\,dy = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f(x,y)\,dx\,dy =$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} g(x)\cdot h(y)\,dx\,dy$

$X + Y$ SŰRŰSÉGFÜGGVÉNYE: $\frac{d}{dz} F_{Z_1}(z) = F'_{Z_1}(z) =$

$= \frac{d}{dz} \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} g(x)\cdot h(y)\,dx\,dy = \int_{-\infty}^{\infty} \frac{d}{dz}\left(\int_{-\infty}^{z-y} g(x)\cdot h(y)\,dx\right)dy$

$= \int_{-\infty}^{\infty} g(z-y)\cdot h(y)\,dy = (g*h)(z)$

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c) $f_X(x) = C\cdot e^{-x}\cdot \mathbb{1}[x>0]\cdot \int_0^{\infty} e^{-y/2}\,dy = 2\cdot C\cdot e^{-x}\cdot \mathbb{1}[x>0]$

$\int_{-\infty}^{\infty} f_X(x)\,dx = 2\cdot C\cdot \int_0^{\infty} e^{-x}\,dx = 2\cdot C$, TEHÁT $\boxed{C = \frac{1}{2}}$

$f(x,y) = \frac{1}{2}\cdot e^{-(x+y/2)}\cdot \mathbb{1}[x>0, y>0] = f_X(x)\cdot f_Y(y)$, AHOL

$f_X(x) = e^{-x}\cdot \mathbb{1}[x>0]$ ÉS $f_Y(y) = \frac{1}{2}\cdot e^{-y/2}\cdot \mathbb{1}[y>0]$

TEHÁT...

2. OLDAL

$\boxed{\text{ELM 2}}$ (c) FOLYTATÁS:

TEHÁT $X \sim EXP(1)$, $Y \sim EXP\left(\frac{1}{2}\right)$, TOVÁBBÁ X ÉS Y FÜGGETLENEK. ÍGY $2X$ ÉS Y F.A.E.

ÍGY $\mathbb{P}(2X < Y) = \mathbb{P}(2X > Y) = \frac{1}{2}$

$\boxed{\text{ELM 3}}$ a) $\operatorname{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X) \cdot \mathbb{E}(Y)$

HA X ÉS Y FÜGGETLENEK, AKKOR

$\mathbb{E}(XY) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$, ÉS ÍGY $\operatorname{Cov}(X, Y) = 0 - 0$

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b) $\underline{\underline{C}} = (c_{i,j})_{i,j=1}^{n}$ AHOL $c_{i,j} = \operatorname{Cov}(X_i, X_j)$

$\underline{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ TETSZŐLEGES

$$\underline{v}^T \cdot \underline{\underline{C}} \cdot \underline{v} = \sum_{i,j=1}^{n} v_i \cdot c_{i,j} \cdot v_j \underset{\text{BILINEARITÁS}}{=} \sum_{i,j=1}^{n} \operatorname{Cov}(v_i \cdot X_i, v_j \cdot X_j) =$$

$$= \operatorname{Cov}\left(\sum_{i=1}^{n} v_i \cdot X_i, \sum_{j=1}^{n} v_j \cdot X_j\right) = \operatorname{Cov}(Z_1, Z_1) = \operatorname{Var}(Z_1) \geq 0$$

(AHOL $Z_1 = \sum_{i=1}^{n} v_i \cdot X_i$) TEHÁT $\underline{v}^T \cdot \underline{\underline{C}} \cdot \underline{v} \geq 0 \;\; \forall \underline{v}$

AZAZ $\underline{\underline{C}}$ POZITÍV SZEMIDEFINIT.

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c) $\operatorname{corr}(X, Y) = \dfrac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var}(X)} \cdot \sqrt{\operatorname{Var}(Y)}} \in [-1, 1]$ KELL, TEHÁT

KELL: $\operatorname{Var}(X) \cdot \operatorname{Var}(Y) \geq \operatorname{Cov}(X, Y)^2$,

FOLYT. KÖV.

$\boxed{\text{ELM 3}}$ (C) FOLYT:

$$\underbrace{\det\begin{pmatrix} \mathrm{Cov}(X,X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(X,Y) & \mathrm{Cov}(Y,Y) \end{pmatrix}}_{\geq 0 \text{ HISZ A POZITÍV SZEMIDEFINIT MÁTRIXOK DETERMINÁNSA NEMNEGATÍV}} = \underbrace{\mathrm{Cov}(X,X)}_{\mathrm{Var}(X)} \cdot \underbrace{\mathrm{Cov}(Y,Y)}_{\mathrm{Var}(Y)} - \mathrm{Cov}(X,Y) \quad \checkmark$$

$\boxed{\text{GYAK 1}}$ $p_{i,j} := \mathbb{P}(X = i, Y = j)$ $\quad i = 1, 2, \dots, 6$, $\; j = 2, 3, \dots, 12$

LEGYEN X_1 = ELSŐ KOCKA ÉRTÉKE

X_2 = MÁSODIK –"– –"–

$$p_{i,j} = \mathbb{P}(X = i, Y = j, X_1 > X_2) + \mathbb{P}(X = i, Y = j, X_1 < X_2) + \mathbb{P}(X = i, Y = j, X_1 = X_2) \overset{\text{SZIMMETRIA}}{=}$$

$$= 2 \cdot \mathbb{P}(X = i, Y = j, X_1 > X_2) + \mathbb{P}(X = i, Y = j, X_1 = X_2)$$

$$= 2 \cdot \mathbb{P}(X_1 = i, X_2 = j - i, i > j - i) + \mathbb{P}(X_1 = X_2 = i, j = 2i)$$

$$= 2 \cdot \frac{1}{36} \cdot \mathbb{1}[1 \leq j - i \leq 6, i > j - i] + \frac{1}{36} \cdot \mathbb{1}[j = 2i]$$

$i = 1, 2, \dots, 6$, $\; j = 2, 3, \dots, 12$

$\boxed{\text{GYAK 2}}$ $\mathfrak{X}_T :=$ KETYEGÉSEK SZÁMA A $[0,T]$ INTERVALLUMON

$\mathfrak{X}_T \sim POI(\mu \cdot T)$ (HISZ SOK ATOM EGYMÁSTÓL FÜGGETLENÜL BOMLIK KIS VAL.SÉGGEL; BINOMIÁLIS POI KÖZELÍTÉSE)

$$\frac{\mathfrak{X}_T - \mu \cdot T}{\sqrt{\mu \cdot T}} \approx \mathcal{N}(0,1) \quad \text{(C.H.T. MIATT)}$$

$$\mathbb{P}(|\bar{\mu} - \mu| \leq 0.02) = \mathbb{P}\left(\left|\frac{\mathfrak{X}_T}{T} - \mu\right| \leq 0.02\right) =$$

$$\mathbb{P}\left(\left|\frac{\mathfrak{X}_T - \mu \cdot T}{T}\right| \leq 0.02\right) = \mathbb{P}(|\mathfrak{X}_T - \mu \cdot T| \leq 0.02 \cdot T) =$$

$$\mathbb{P}\left(\left|\frac{\mathfrak{X}_T - \mu \cdot T}{\sqrt{\mu \cdot T}}\right| \leq 0.02 \cdot \frac{\sqrt{T}}{\sqrt{\mu}}\right) \overset{\text{C.H.T.}}{\approx} \Phi\left(0.02 \cdot \frac{\sqrt{T}}{\sqrt{\mu}}\right) - \Phi\left(-0.02 \cdot \frac{\sqrt{T}}{\sqrt{\mu}}\right)$$

$$= 2 \cdot \Phi\left(0.02 \cdot \frac{\sqrt{T}}{\sqrt{\mu}}\right) - 1 \overset{\text{KELL}}{\geq} 0.95$$

KELL: $\Phi\left(0.02 \cdot \frac{\sqrt{T}}{\sqrt{\mu}}\right) \geq 0.975$

TÁBLÁZATBÓL: $0.02 \cdot \frac{\sqrt{T}}{\sqrt{\mu}} \overset{\text{KELL}}{\geq} 1.96$

KELL: $\frac{\sqrt{T}}{\sqrt{\mu}} \geq 98$ KELL: $T \geq 9604 \cdot \mu$

TUDJUK, HOGY $2 \leq \mu \leq 3$, ÍGY A LEGROSSZABB ESET A $\mu = 3$, TEHÁT $T = 28812$

GYAK 3

(a) $f(x,y) = f_Y(y)\cdot \frac{1}{y}\cdot \mathbb{1}[0<x<y] = a^2\cdot e^{-a\cdot y}\cdot \mathbb{1}[0<x<y]$

(b) $f_X(x) = \int\limits_{-\infty}^{\infty} f(x,y)\,dy = a^2\cdot \int\limits_{x}^{\infty} e^{-a\cdot y}\,dy = a\cdot e^{-a\cdot x}\cdot \mathbb{1}[0<x]$

TEHÁT $X \sim EXP(a)$, ÍGY $\mathbb{E}(X) = \frac{1}{a}$

(c) $\mathbb{E}(X \mid Y=y) = y/2$ (HISZ $Y=y \Rightarrow X \sim UNI[0,y]$)

$$\underbrace{f_{Y|X}(y|x)}_{} = \frac{f(x,y)}{f_X(x)} \underset{\boxed{x>0}}{\overset{}{=}} \frac{a^2\cdot e^{-a\cdot y}\cdot \mathbb{1}[x<y]}{a\cdot e^{-a\cdot x}} = a\cdot e^{-a\cdot(y-x)}\cdot \mathbb{1}[y-x>0]$$

TEHÁT EZ UGYANAZ, MINT $X+Z_1$ SŰ. FV.-E, $Z_1 \sim EXP(a)$

ÍGY $\mathbb{E}(Y \mid X=x) = \mathbb{E}(x+Z_1) = x + \frac{1}{a}$

(d) $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y \mid X)) = \mathbb{E}\left(X + \frac{1}{a}\right) = \mathbb{E}(X) + \frac{1}{a} = \frac{2}{a}$

<u>BÓNUSZ</u>: $\mathrm{Var}(Y) = \mathbb{E}(\mathrm{Var}(Y \mid X)) + \mathrm{Var}(\mathbb{E}(Y \mid X)) =$ ☺

$\mathrm{Var}(Y \mid X=x) = \mathrm{Var}(x+Z_1) = \mathrm{Var}(Z_1) = \frac{1}{a^2}$

$\mathrm{Var}(\mathbb{E}(Y \mid X)) = \mathrm{Var}\left(X + \frac{1}{a}\right) = \mathrm{Var}(X) = \frac{1}{a^2}$

☺ $= \mathbb{E}\left(\frac{1}{a^2}\right) + \frac{1}{a^2} = \frac{2}{a^2}$