A ONE PARAMETER EXTENSION OF THE BURES AND HELLINGER DISTANCES, AND TRACE CHARACTERIZATIONS

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MASTER THESIS

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Contents

1. Introduction and overview of the results

All results that follow are new and original, they constitute a part of the paper [11] joint with the supervisor Lajos Molnár, which is published in a special issue of J. Math. Anal. Appl. Below only those proofs are presented which were given by the author of the thesis. Hence, despite Proposition 1 and Theorem 3 formulates results for all $p \geq 0$, the proofs are presented only for the case where $p = 0$ since the case $p > 0$ was resolved in [11] by the supervisor.

The quantum versions of the classical Hellinger distance (of probability distributions) play important roles in quantum information theory. The two of the most studied versions defined on the positive semidefinite cone $\mathbb{M}_n(\mathbb{C})^+$ in the algebra $\mathbb{M}_n(\mathbb{C})$ of all n by n complex matrices are the following (see, e.g., [2]):

$$d_H(A,B) = \sqrt{2}\left(\mathrm{Tr}\left((A+B)/2 - A^{1/4}B^{1/2}A^{1/4}\right)\right)^{1/2}, \quad A,B \in \mathbb{M}_n(\mathbb{C})^+ \tag{1}$$

and

$$d_B(A,B) = \sqrt{2}\left(\mathrm{Tr}\left((A+B)/2 - (A^{1/2}BA^{1/2})^{1/2}\right)\right)^{1/2}, \quad A,B \in \mathbb{M}_n(\mathbb{C})^+. \tag{2}$$

These two functions are true metrics on $\mathbb{M}_n(\mathbb{C})^+$. The latter one is called Bures or Bures-Wasserstein metric and the former one, being the most trivial extension of the classical Hellinger distance to the quantum case, is also called Hellinger distance, see, e.g., [18]. In the formulas (1), (2) above, under the trace denoted by Tr we see that the difference of the arithmetic mean and two variants of the geometric mean show up. It is easy to see that those latter two operations are in fact members of a natural parametric family of geometric mean type operations which we can define also in the general setting of C^*-algebras as follows.

Let $\mathscr{A}$ be a C^*-algebra. In what follows we always assume that $\mathscr{A}$ is unital. Let us denote by $\mathscr{A}^+$ the positive semidefinite cone of $\mathscr{A}$ (the collection of all self-adjoint elements of $\mathscr{A}$ with non-negative spectrum) and by $\mathscr{A}^{++}$ the positive definite cone of $\mathscr{A}$ (the collection of all self-adjoint elements of $\mathscr{A}$ with strictly positive spectrum). We can introduce the parametric family of operations in question as follows. For any positive real number $p > 0$, we define

$$A\kappa_p B = (A^{p/4}B^{p/2}A^{p/4})^{1/p}, \quad A,B \in \mathscr{A}^+.$$

The so-called symmetrized Lie-Trotter formula asserts that for any $X, Y \in \mathscr{A}$, the convergence

$$\left(e^{(p/2)X}e^{pY}e^{(p/2)X}\right)^{1/p} \xrightarrow{p\to 0} e^{X+Y}$$

holds in norm. This implies that for any $A,B \in \mathscr{A}^{++}$, we have

$$A\kappa_p B \xrightarrow{p\to 0} \exp((\log A + \log B)/2)$$

which limit is usually called the log-Euclidean mean of A and B. Therefore, for any $A,B \in \mathscr{A}^{++}$ we also set

$$A\kappa_0 B = \exp((\log A + \log B)/2).$$

Apparently, all those κ_p-s are variants of the classical Kubo-Ando (or originally Pusz-Woronowicz) geometric mean defined by

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}, \quad A,B \in \mathscr{A}^{++}.$$

Indeed, for commuting A, B all those operations coincide. It should be mentioned that the κ_p-s also appear in relation with some variants of the quantum Rényi relative entropy. See, for example, [14] and its references.

In what follows we will also use the quite standard notation for the arithmetic mean

$$A\nabla B = \frac{A+B}{2}, \quad A, B \in \mathscr{A}^+.$$

Clearly, for any $A, B \in \mathbb{M}_n(\mathbb{C})^+$, we have

$$d_H(A,B) = \sqrt{2}\,(\mathrm{Tr}\,(A\nabla B - A\kappa_1 B))^{1/2}$$

and

$$d_B(A,B) = \sqrt{2}(\mathrm{Tr}(A\nabla B - A\kappa_2 B))^{1/2}.$$

Those quantities can easily be defined in the setting of C^*-algebras. Assume that τ is a positive linear functional on $\mathscr{A}$ which is faithful and tracial. The former property means that for any $A \in \mathscr{A}^+$, $\tau(A) = 0$ implies $A = 0$, the latter one means that $\tau(XY) = \tau(YX)$ holds for all $X, Y \in \mathscr{A}$.

In this work, for any non-negative real number p, we would like to set

$$d_p^\tau(A,B) = \left(\tau(A\nabla B - A\kappa_p B)\right)^{1/2}, \quad A, B \in \mathscr{A}^+$$

and study these quantities as possible sorts of distance measures.

If we want to study d_p^τ, we have to observe whether it is well-defined, that is, if we have a non-negative expression under the square root. First, we examine a stronger question: when do we have that the elements $A\nabla B - A\kappa_p B$ are positive for any $A, B \in \mathscr{A}^{++}$? As an apparent consequence of Proposition 1, we obtain that this happens only in commutative algebras. After this, we show that $d_p = d_p^{\mathrm{Tr}}$ is well-defined on $\mathbb{M}_n(\mathbb{C})^+$ for all $p \ge 0$ and that d_p^τ is also well-defined on the positive cone of a general C^*-algebras if $p \le 2$.

After this, it is natural to ask whether d_p^τ is a true metric. In the $p = 1$ and $p = 2$ cases, it is well known that d_1^τ, d_2^τ are indeed true metrics, for which [11] provides a new, more elementary and transparent argument. As for the general case d_p^τ, $p \ge 0$, the natural problem that for which values of p we have that d_p^τ is a true metric seems to be very difficult, we have only partial results. Namely, in Proposition 2 we show that $d_p = d_p^{\mathrm{Tr}}$ is a true metric on the set of rank-one projections (the set of pure states) of a Hilbert space exactly when $p \le 2$.

Approaching the positivity of $\tau(A\nabla B - A\kappa_p B)$ from another perspective, Theorem 3 shows that if a positive linear functional takes non-negative values on the elements $A\nabla B - A\kappa_p B$, $A, B \in \mathscr{A}^{++}$, then it is necessarily tracial. This means that if there is any positive linear functional φ on $\mathscr{A}$ for which d_p^φ is well-defined for all $A, B \in \mathscr{A}^{++}$, then φ must be tracial. This provides a new characterization of the trace.

In the last part of the thesis we consider a result by Sra [19] which says that on the positive definite cone of a matrix algebra the square root of the symmetric Stein divergence is a true metric. We observe that this quantity equals the square root of the trace of the difference of the logarithm of the arithmetic mean and the logarithm of any member κ_p of our parametric family of operations. Indeed, on matrix algebras the trace of the logarithm of any κ_p actually does not depend on the parameter p which is a simple consequence of the properties of the determinant. We show that this independence in fact distinguishes the trace among all bounded linear functionals. Namely, in Theorem 4 we show that for von Neumann algebras, if a bounded linear functional φ has the property that for one given pair p, q of different non-negative real numbers we have $\varphi(\log A\kappa_p B) = \varphi(\log A\kappa_q B)$, $A, B \in \mathscr{A}^{++}$, then φ is necessarily tracial. The result can also be interpreted as a certain characterization of the determinant (on positive definite matrices, the trace of the logarithm equals the logarithm of the determinant).

Now we turn to the precise presentation of our results.

2. A CENTRALITY CHARACTERIZATION UTILIZING d_τ^p

If we want to study d_p^τ on positive cones, we have to start by discussing the question of the possible validity of the inequalities

$$\tau(A\nabla B - A\kappa_p B) \geq 0,$$

which would prove that d_p^τ is well-defined.

In the next result we consider the stronger question: when do we have the positivity of the elements $A\nabla B - A\kappa_p B$? The result gives a characterization of the centrality of positive definite elements.

Proposition 1. *Let $\mathscr{A}$ be a C^*-algebra and p be a non-negative real number. If $B \in \mathscr{A}^{++}$ is fixed and*

$$A\kappa_p B \leq A\nabla B$$

holds for every $A \in \mathscr{A}^{++}$, then B is necessarily a central element of $\mathscr{A}$ (it commutes with every element of $\mathscr{A}$). The same conclusion holds if A is fixed and B runs through the set $\mathscr{A}^{++}$.

Proof: As in Proposition 4 in [16], due to Kadison's celebrated transitivity theorem, we have the somewhat surprising fact that the above general statement follows if we can verify statement for the particular algebra $\mathbb{M}_2(\mathbb{C})$. In what follows we check the statement for that algebra.

For the case $p > 0$, see [11]. Assume that $p = 0$. To treat that case we need two tools. First, in the proof of Proposition 8 in [16] it was proved that for any self-adjoint $T \in \mathbb{M}_2(\mathbb{C})$ and rank-one projection $P \in \mathbb{M}_2(\mathbb{C})$, we have

$$e^{(T-n(I-P))} \to e^{\operatorname{Tr} TP} P \tag{3}$$

as $n \to \infty$. The second observation that we will use is the following. Assume that P is a rank-one projection, t is a positive number, A is a positive definite matrix and we have

$$tP \leq A.$$

Multiplying by $A^{-1/2}$ from both sides, we have $tA^{-1/2}PA^{-1/2} \leq I$. The only positive eigenvalue of $tA^{-1/2}PA^{-1/2}$ can easily be shown to equal $t \operatorname{Tr} A^{-1}P$ which implies that

$$t \leq \frac{1}{\operatorname{Tr} A^{-1}P}. \tag{4}$$

Now we turn to the proof of the case where $p = 0$. Assume that

$$\exp\left(\frac{\log A + \log B}{2}\right) \leq \frac{A+B}{2}$$

holds for some fixed $B \in \mathbb{M}_2(\mathbb{C})^+$ and for all $A \in \mathbb{M}_2(\mathbb{C})^+$. Apparently, A and B can be written in the form e^C and e^D for some self-adjoint elements C and D of $\mathbb{M}_2(\mathbb{C})$ (D is fixed, C is varying). Let $P \in \mathbb{M}_2(\mathbb{C})$ be a rank-one projection, and set $C_n = D - n(I-P)$. We have

$$\exp\left(\frac{2D - n(I-P)}{2}\right) \leq \frac{e^{D-n(I-P)} + e^D}{2}$$

Taking limits, by (3) we can see that

$$e^{\operatorname{Tr} DP} P \leq \frac{e^{\operatorname{Tr} DP} P + e^D}{2}. \tag{5}$$

If we multiply both sides by 2 and rearrange, we get

$$e^{\operatorname{Tr} DP} P \leq e^{D}. \tag{6}$$

Referring back to the discussion leading to (4), this inequality implies

$$e^{\operatorname{Tr} DP} \leq \frac{1}{\operatorname{Tr} e^{-D} P}. \tag{7}$$

Let e^D have spectral decomposition

$$e^{D} = d_1 R + d_2 (I - R) \tag{8}$$

for some $d_1, d_2 > 0$ and rank-one projection R. The inequality (7) tells us that

$$e^{\log(d_1) \operatorname{Tr} RP + \log(d_2)(1 - \operatorname{Tr} RP)} \leq \frac{1}{d_1^{-1} \operatorname{Tr} RP + d_2^{-1}(1 - \operatorname{Tr} RP)}.$$

If we denote $t = \operatorname{Tr} RP$, we get

$$d_1^t d_2^{1-t} \leq \frac{1}{t d_1^{-1} + (1-t) d_2^{-1}}.$$

But t can be any number between 0 and 1 (recall that P is an arbitrary rank-one projection). So If we choose $t = 1/2$, we get

$$\sqrt{d_1 d_2} \leq \frac{2}{d_1^{-1} + d_2^{-1}}.$$

This means that the harmonic mean of d_1 and d_2 is not less than their geometric mean, which implies $d_1 = d_2$, thus $B = e^D = d_1 I$. It completes the proof of our statement. ■

It might be interesting to note that both of the statements in Proposition 1 and in Theorem 3 are proved if they are verified only for the matrix algebra $\mathbb{M}_2(\mathbb{C})$. However, the reasons for this, as we will see, are different.

We turn back to the original question whether the d_p-s or d_p^τ-s are well-defined. The famous Araki-Lieb-Thirring inequality says that for any $r \geq 1$, $p > 0$ we have

$$\operatorname{Tr}(A^{1/2} B A^{1/2})^{rp}) \leq \operatorname{Tr}(A^{r/2} B^r A^{r/2})^p \tag{9}$$

for any $A, B \in \mathbb{M}_n(\mathbb{C})^+$ (see (6.36) in [10] or the original source [1]). Considering arbitrary two positive real numbers $0 < p \leq q$ and then writing $1/q$ in the place of p, setting $r = q/p$, and then replacing A by A^p and B by B^p, the inequality (9) becomes

$$\operatorname{Tr}(A^{p/2} B^p A^{p/2})^{1/p} \leq \operatorname{Tr}(A^{q/2} B^q A^{q/2})^{1/q} \tag{10}$$

for any $A, B \in \mathbb{M}_n(\mathbb{C})^+$. It follows that

$$0 \leq \operatorname{Tr}(A \nabla B - A \kappa_2 B) \leq \operatorname{Tr}(A \nabla B - A \kappa_p B), \quad A, B \in \mathbb{M}_n(\mathbb{C})^+$$

holds for all $0 \leq p \leq 2$. (We admit that in the cases where $1 \leq p \leq 2$, it was proved in [8] that d_p is actually a so-called quantum divergence, see Definition 2.1 and Theorem 3 in that paper.) But what about the non-negativity of $\operatorname{Tr}(A \nabla B - A \kappa_p B)$ in the cases where $p > 2$? It turns out that this is true, for the details see [11]. As a consequence, d_p is well-defined on $\mathbb{M}_n(\mathbb{C})^+$ for all $p \geq 0$.

What can survive from the above observations in the far more general setting of C^*-algebras? The $p \leq 2$ case is treated in [11], we have the non-negativity, the $p > 2$ case however still unresolved, hence we form the following problem.

Problem 1. Is it true for an arbitrary C^*-algebra $\mathscr{A}$ endowed with a faithful positive tracial linear functional τ, that $\tau(A \nabla B - A \kappa_p B) \geq 0$ holds for all $A, B \in \mathscr{A}$ and $p > 2$?

3. d_p ON THE SET OF RANK ONE PROJECTIONS

We know that $d_1 = d_H, d_2 = d_B$ are true metrics on $\mathbb{M}_n(\mathbb{C})^+$. We investigate the natural question of what happens to the remaining p-s? In our first proposition we prove that for any $p > 2$, the triangle inequality does not hold for d_p even in the case of 2 by 2 matrices.

Before the formulation of the result we remark the following. Let H be a Hilbert space (of any dimension) and denote by $P_1(H)$ the set of all rank-one projections on H. If p is a positive real number, then for any $P, Q \in P_1(H)$, the quantity $d_p(P,Q) = (1 - \mathrm{Tr}(PQP)^{1/p})^{1/2}$ is well-defined and equals $\sqrt{1-\cos^{2/p}\alpha}$, where $\alpha \in [0,\pi/2]$ is the angle between the ranges of P and Q.

Now, our statement reads as follows.

Proposition 2. *Let H be a Hilbert space with* $\dim H > 1$. *Then d_p is a metric on $P_1(H)$ if and only if $p \le 2$.*

Proof: Suppose that $p > 2$. We prove that d_p does not satisfy the triangle inequality even when $H = \mathbb{C}^2$. Let P, Q two mutually orthogonal rank-one projections and assume that R is a rank-one projection such that the angle between ranges of P and R is $\alpha \in [0,\pi/2]$. We claim that for some α we have

$$1 = d_p(P,Q) > d_p(P,R) + d_p(R,Q) = \sqrt{1-\cos^{2/p}\alpha} + \sqrt{1-\sin^{2/p}\alpha}.$$

Taking squares and reordering, this inequality is equivalent to

$$\cos^{2/p}\alpha + \sin^{2/p}\alpha - 1 > 2\sqrt{1-\cos^{2/p}\alpha}\sqrt{1-\sin^{2/p}\alpha}.$$

Taking square again,

$$\begin{aligned}\cos^{4/p}\alpha + \sin^{4/p}\alpha + 1 - 2\cos^{2/p}\alpha - 2\sin^{2/p}\alpha + 2\cos^{2/p}\alpha\sin^{2/p}\alpha \\ > 4 - 4\cos^{2/p}\alpha - 4\sin^{2/p}\alpha + 4\cos^{2/p}\alpha\sin^{2/p}\alpha.\end{aligned}$$

Reordering we have

$$0 > 3 - 2(\cos^{2/p}\alpha + \sin^{2/p}\alpha) - (\cos^{2/p}\alpha - \sin^{2/p}\alpha)^2.$$

Denoting $t = \cos^2\alpha$, we need to verify that for some $t \in [0,1]$ we have

$$0 > 3 - 2(t^{1/p} + (1-t)^{1/p}) - (t^{1/p} - (1-t)^{1/p})^2.$$

Let

$$f(t) = 3 - 2(t^{1/p} + (1-t)^{1/p}) - (t^{1/p} - (1-t)^{1/p})^2, \quad t \in [0,1].$$

We have $f(0) = 0$. The derivative of f at any point $0 < t < 1$ is

$$f'(t) = -(2/p)\left((t^{1/p-1} - (1-t)^{1/p-1}) + (t^{1/p} - (1-t)^{1/p})(t^{1/p-1} + (1-t)^{1/p-1}\right).$$

We clearly have

$$\begin{aligned}-(p/2)f'(t) &= (t^{1/p-1} - (1-t)^{1/p-1}) + (t^{1/p} - (1-t)^{1/p})(t^{1/p-1} + (1-t)^{1/p-1}) \\ &= t^{2/p-1} + t^{1/p-1}(1-(1-t)^{1/p}) - (1-t)^{1/p-1}(1 - t^{1/p} + (1-t)^{1/p}) \\ &\ge t^{2/p-1} - (1-t)^{1/p-1}(1 - t^{1/p} + (1-t)^{1/p}).\end{aligned}$$

As $t \to 0$, this latter expression tends to ∞. Hence, for small enough positive t, the derivative $f'(t)$ is negative. Since $f(0) = 0$, we have that $f(t) < 0$ for small enough positive t. This completes the proof in the case where $p > 2$.

Assume next that $0 < p < 2$. For any $P,Q \in P_1(H)$, let α be the angle between their ranges. We have mentioned above that $d_p(P,Q) = \sqrt{1-\cos^{2/p}\alpha}$. Now denote by f this function of α what we consider on $[0,\pi/2]$.

Select arbitrary $P,Q,R \in P_1(H)$. Let the angle between the ranges of P,R be α and the angle between the ranges R,Q be β.

Let us assume for a moment that we know that the inequality

$$f(\alpha+\beta) \le f(\alpha)+f(\beta) \tag{11}$$

holds for any $\alpha,\beta \in [0,\pi/2]$ with $\alpha+\beta \in [0,\pi/2]$.

Let the angle between the ranges of P,Q be γ. Clearly, $\gamma \le \alpha+\beta$, so if $\alpha+\beta \le \pi/2$, then by (11) and the monotonicity of f we have the triangle inequality

$$d_p(P,Q) = f(\gamma) \le f(\alpha+\beta) \le f(\alpha)+f(\beta) = d_p(P,R)+d_p(R,Q).$$

The case where $\alpha+\beta > \pi/2$ is easy. Again by (11) and the monotonicity of f we then have

$$\begin{aligned} d_p(P,Q) =& f(\gamma) \le f(\pi/2) \le f(\pi/2-\beta)+f(\beta) \\ \le& f(\alpha)+f(\beta) = d_p(P,R)+d_p(R,Q). \end{aligned}$$

So, it remains to prove that

$$f(s)+f(t-s)-f(t) \ge 0, \quad 0 \le s \le t \le \pi/2.$$

Fix a t_0, and consider the right hand side of the above inequality as a function of s. For $s=0$, $s=t_0$ we have the required inequality. We prove that this function is concave on $[0,t_0]$ which will verify our claim. The second derivative of the function in question is just the double of the second derivative of f. So, let us compute the second derivative of f. The first derivative is as follows

$$\begin{aligned} f'(t) &= (1/2)(1-\cos^{2/p}t)^{-1/2}(-2/p)\cos^{2/p-1}t(-\sin t)) \\ &= (1/p)(1-\cos^{2/p}t)^{-1/2}\cos^{2/p-1}t\sin t) \\ &= (1/p)\frac{\cos^{2/p-1}t\sin t}{f(t)}. \end{aligned}$$

We compute the second derivative

$$\begin{aligned} f''(t) &= (1/p)\frac{\big((2/p-1)\cos^{2/p-2}t(-\sin t)\sin t+\cos^{2/p-1}t\cos t\big)f(t)-\cos^{2/p-1}t\sin t(1/p)\frac{\cos^{2/p-1}t\sin t}{f(t)}}{f(t)^2} \\ &= \frac{\cos^{2/p}t}{pf(t)^2}\left(-((2/p-1)\cos^{-2}t\sin^2 tf(t)+f(t)-(1/p)\frac{\cos^{2/p}t\cos^{-2}t\sin^2 t}{f(t)}\right) \\ &= \frac{\cos^{2/p}t}{pf(t)^3}\left(-(2/p)\tan^2 tf(t)^2+\tan^2 tf(t)^2+f(t)^2-(1/p)\cos^{2/p}t\tan^2 t\right) \\ &= \frac{\cos^{2/p}t}{pf(t)^3}\left(-(2/p)\tan^2 tf(t)^2+\tan^2 tf(t)^2+f(t)^2-(1/p)(1-f(t)^2)\tan^2 t\right) \\ &= \frac{\cos^{2/p}t}{pf(t)^3}\left((1+\tan^2 t)f(t)^2-(1/p)(1+f(t)^2)\tan^2 t\right). \end{aligned}$$

We prove that

$$(1+\tan^2 t)f(t)^2-(1/p)(1+f(t)^2)\tan^2 t \le 0,$$

or equivalently that

$$\frac{f(t)^2}{1+f(t)^2} \le \frac{1}{p}\frac{\tan^2 t}{1+\tan^2 t} = \frac{1}{p}\sin^2 t.$$

We can rewrite this as

$$\frac{1-\cos^{2/p} t}{2-\cos^{2/p} t} \le \frac{1}{p}(1-\cos^2 t).$$

Changing the variables, the desired equality becomes

$$\frac{1-t}{2-t} \le \frac{1}{p}(1-t^p).$$

We assert that for any $t \in [0,1]$ we have

$$\frac{1-t}{2-t} \le \frac{1}{2}(1-t^2) \le \frac{1}{p}(1-t^p) \tag{12}$$

The first inequality is easy to prove. We need to verify the second one. Examine the derivative of the function

$$\frac{t^p}{p} - \frac{t^2}{2}.$$

We see that this derivative is 0 at 0, positive on]0,1[and again 0 at 1. It follows that the function displayed above takes its maximal value at 1, i.e., we have

$$\frac{t^p}{p} - \frac{t^2}{2} \le \frac{1}{p} - \frac{1}{2}, \quad t \in [0,1].$$

After reordering we obtain the second inequality in (12) and this finishes the proof for $0 < p < 2$. The case $p = 0$ is obtained by taking the limit $p \to 0$. ■

As for the metric behaviour of d_p or d_p^τ on full cones, we formulate the following most probably quite difficult questions.

Problem 2. Given a positive integer $n \ge 2$, is it true that d_p on $\mathbb{M}_n(\mathbb{C})^+$ is a metric for all $1 < p < 2$ and it is not a metric for all $0 < p < 1$? What about the case of general C^*-algebras? Is it true that d_p^τ is a metric if $1 < p < 2$?

Concerning matrix algebras we remark that using a mathematical software, computer tests convince us to believe that the answers to the first two questions are affirmative.

We also remark that if for $0 < p < 1$, d_p is really not a metric on $\mathbb{M}_n(\mathbb{C})^+$ $(n \ge 2)$, then using a method presented in [11], one could prove that if d_p^τ is a true metric on the positive cone $\mathscr{A}^+$ of a C^*-algebra $\mathscr{A}$ for some $0 < p < 1$, then $\mathscr{A}$ is necessarily commutative.

4. A trace characterization utilizing d_p^τ

According to Proposition 1 the well-definedness of d_p^τ does not hold in the most trivial way, namely, the positivity of an element $A\nabla B - A\kappa_p B$ is guaranteed only in special cases. We showed that for certain values of p all elements of the preceding form has a non-negative image under a tracial positive linear functional. Our following result shows that there can not exist another type of positive linear functional with this property.

Theorem 3. *Let $\mathscr{A}$ be a C^*-algebra and p be any non-negative real number. If φ is a positive linear functional on $\mathscr{A}$ such that*

$$\varphi(A\kappa_p B) \le \varphi(A\nabla B), \quad A, B \in \mathscr{A}^{++}, \tag{13}$$

then φ is necessarily tracial.

Before presenting the proof we make some short remarks.

First, referring back to Proposition 1, the result above can also be interpreted saying that the inequality $A\kappa_p B \leq A\nabla B$ is so much untrue that only a very few positive linear functionals φ can satisfy the inequality $\varphi(A\kappa_p B) \leq \varphi(A\nabla B)$, $A, B \in \mathscr{A}^{++}$. Indeed, in the particular case of a finite factor von Neumann algebra, φ is necessarily a scalar multiple of the unique trace.

We must point out that a number of results on the traciality of positive linear functionals similar to the one in Theorem 3 were obtained by Bikchentaev in a series of papers. His investigations were motivated by the work [9] in which Gardner proved that if the positive linear functional φ on the C^*-algebra $\mathscr{A}$ satisfies $|\varphi(A)| \leq \varphi(|A|)$ for all $A \in \mathscr{A}$, then φ is tracial.

The one among Bikchentaev's results which is the closest to ours is Theorem 2 in [3]. It is shown there that the Araki-Lieb-Thirring inequality (9) for any given pair $r > 1, p > 0$ distinguishes the tracial ones among all positive linear functionals on a C^*-algebra. Although we present a similar result here but observe that our assumption is much weaker in some sense: we require φ to respect only a sort of geometric-arithmetic mean inequality while in [3] an inequality involving two variants of the same geometric mean is assumed.

Proof: As Bikchentaev did, we also follow the scheme of Gardner's proof and reduce the problem to the particular case when the domain is the algebra of 2 by 2 matrices.

First, we can obviously assume that φ has norm 1, i.e., it is a state. Consider the image of $\mathscr{A}$ under its universal representation ω. The second commutant $\omega(\mathscr{A})'' = \mathscr{M}$ is a von Neumann algebra. The sate φ is a vector state for $\omega(A)$ hence extends to a normal state $\tilde{\varphi}$ on $\mathscr{M}$, see III.5.2.6 Proposition in [5]. Since the normal states are weakly/strongly continuous on the unit ball of $\mathscr{M}$ (see, e.g., III.2.1.4 Theorem in [5]), by the Kaplansky density theorem (and using the strong continuity of bounded continuous functions, see, e.g., 4.3.2. Theorem in [17]) we can conclude that $\tilde{\varphi}$ satisfies (13) on the positive definite cone of $\mathscr{M}$.

Therefore, it is sufficient to prove our result in the case where $\mathscr{A}$ is a von Neumann algebra and our positive linear functional φ is normal.

To do that we refer to Lemma 2 in [20] which says that a normal positive linear functional on a von Neumann algebra is tracial if (and only if) its values on mutually orthogonal equivalent projections are equal. This observation was actually made (but not explicitly stated) already in [9].

Select two mutually orthogonal equivalent projections $P, Q \in \mathscr{A}$. As in the proof of Theorem 7 in [15], choose a partial isometry $V \in \mathscr{A}$ such that $VV^* = P, V^*V = Q$ and consider the map

$$\Phi : \begin{bmatrix} u & v & 0 \\ z & w & 0 \\ 0 & 0 & w' \end{bmatrix} \mapsto uP + vV + zV^* + wQ + w'(I - (P+Q))$$

from $M_2(\mathbb{C}) \oplus \mathbb{C} \subset M_3(\mathbb{C})$ into $\mathscr{A}$. It is easy to see that Φ is an injective unital *-algebra homomorphism and we can deduce that the functional

$$X \mapsto \varphi(\Phi(X \oplus 0))$$

is a positive linear functional on $\mathbb{M}_2(\mathbb{C})$ which satisfies (13) on $M_2(\mathbb{C})^{++}$. We need to prove that this takes equal values on

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and it will be done as soon as we verify the statement of our proposition in the particular case of the matrix algebra $\mathbb{M}_2(\mathbb{C})$.

The proof for the case $p > 0$ can be found in [11]. We consider here the case $p = 0$ (which is definitely more complicated than the one where $p > 0$). Assume that the positive liner functional φ on $\mathbb{M}_2(\mathbb{C})$ satisfies

$$\varphi(\exp((\log A + \log B)/2)) \leq \varphi((A+B)/2)$$

for all positive definite A and B. Obviously, A and B are of the form $A = e^C$ and $B = e^D$ for some self-adjoint $C, D \in \mathbb{M}_2(\mathbb{C})$. If we take $r \in \mathbb{R}$, and instead of e^C, we write $e^{(C+2rI)}$, and rearrange, we get

$$0 \le e^{2r} \frac{\varphi\left(e^C\right)}{2} - e^r \varphi\left(e^{\frac{C+D}{2}}\right) + \frac{\varphi\left(e^D\right)}{2}.$$

Just as a few times earlier, this is a second order polynomial of e^r, thus the discriminant is non-positive, which means that

$$\left(\varphi\left(e^{\frac{C+D}{2}}\right)\right)^2 \le \varphi\left(e^C\right)\varphi\left(e^D\right). \tag{14}$$

Let $t \in [0,1]$, and choose the unit vectors

$$x = \begin{bmatrix} \sqrt{1-t} \\ \sqrt{t} \end{bmatrix}$$

and

$$y = \begin{bmatrix} \sqrt{t} \\ \sqrt{1-t} \end{bmatrix}.$$

Set the projections $P = x \otimes x$ and $Q = y \otimes y$, and let $C = c_1 Q + c_2(I - Q)$ for some $c_1, c_2 \in \mathbb{R}$, and $D_n = -n(I - P)$. We will use the following. Using the limit formula (3), we deduce

$$\lim_{n\to\infty} e^{\frac{C+D_n}{2}} = e^{\frac{\operatorname{Tr} PC}{2}} P.$$

Thus we can take limits on both sides of the inequality (14) to get

$$e^{\operatorname{Tr} PC} \varphi(P)^2 \le \varphi\left(e^C\right)\varphi(P).$$

As φ is a positive linear functional on $\mathbb{M}_2(\mathbb{C})$, we have $T \in \mathbb{M}_2(\mathbb{C})^+$ such that $\varphi(X) = \operatorname{Tr}(XT)$, $X \in \mathbb{M}_2(\mathbb{C})$. Clearly, we can assume that T is diagonal, its largest eigenvalue is 1, the other eigenvalue is α that we assume to be strictly smaller than 1

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}. \tag{15}$$

Our aim is to prove that $\alpha = 1$. Since

$$\operatorname{Tr} PC = c_1 \operatorname{Tr} PQ + c_2(1 - \operatorname{Tr} PQ) = (c_1 - c_2)4t(1-t) + c_2,$$

we have

$$e^{(c_1-c_2)4t(1-t)+c_2} \varphi(P) \le (e^{c_1} - e^{c_2})\varphi(Q) + e^{c_2}\varphi(I).$$

If we divide by e^{c_2}, and set $c = c_1 - c_2 \in \mathbb{R}$, we get

$$e^{c4t(1-t)} \varphi(P) \le (e^c - 1)\varphi(Q) + \varphi(I)$$

which implies

$$e^{c4t(1-t)}(1 - t + \alpha t) \le (e^c - 1)(t + \alpha(1-t)) + (1 + \alpha).$$

If we rearrange this latter inequality, it becomes

$$e^{c4t(1-t)}(1-t) + (1 - e^c)t - 1 \le \alpha(1 - e^{c4t(1-t)} t - (1 - e^c)(1-t)).$$

Introduce the following auxiliary functions

$$f(t,c) = (1 - e^c - e^{c4t(1-t)})t + e^{c4t(1-t)} - 1$$
$$g(t,c) = (1 - e^c - e^{c4t(1-t)})t + e^c$$

defined for all $t \in [0,1]$ and $c \in \mathbb{R}$. What we have to prove is that the inequality $f \le \alpha g$ for some $\alpha \in [0,1]$ implies $\alpha = 1$.

To verify this, first observe that since f has positive values (e.g., $f(1/4,2) > 0$), hence $\alpha > 0$. Thus it is enough to show that the function f/g can approach 1 arbitrarily while its denominator is positive.

We will look at the function $q : [0,1] \to \mathbb{R}$ defined by

$$q(t) = \frac{f\left(t, -\frac{\log(1-4t(1-t))}{4t(1-t)}\right)}{g\left(t, -\frac{\log(1-4t(1-t))}{4t(1-t)}\right)} = \frac{\left(1-(1-4t(1-t))^{\frac{-1}{4t(1-t)}} - (1-4t(1-t))^{-1}\right)t + (1-4t(1-t))^{-1} - 1}{\left(1-(1-4t(1-t))^{\frac{-1}{4t(1-t)}} - (1-4t(1-t))^{-1}\right)t + (1-4t(1-t))^{\frac{-1}{4t(1-t)}}}.$$

We will show that $q(t) \to 1$ as $t \nearrow 1/2$. Let us apply a change of variable. For any $0 \le t < 1/2$ we write $s = 1-4t(1-t)$ which runs through the interval $]0,1]$ and we also have $t = (1-\sqrt{s}/2)$. With this, we introduce a new function $\tilde{q} : [0,1] \to \mathbb{R}$:

$$\tilde{q}(s) = q\left(\frac{1-\sqrt{s}}{2}\right) = \frac{-1 - s^{\frac{-1}{1-s}} + s^{-1} + \sqrt{s}(s^{\frac{-1}{1-s}} + s^{-1} - 1)}{1 + s^{\frac{-1}{1-s}} - s^{-1} + \sqrt{s}(s^{\frac{-1}{1-s}} + s^{-1} - 1)}$$

and here we can see that as $s \searrow 0$, the denominator, which is the double of the new parametrization of g, is positive. We are going to prove that $\tilde{q}(s) - 1 \to 0$ as $s \searrow 0$. We have

$$\tilde{q}(s) - 1 = \frac{-2(1 + s^{\frac{-1}{1-s}} - s^{-1})}{1 + s^{\frac{-1}{1-s}} - s^{-1} + \sqrt{s}(s^{\frac{-1}{1-s}} + s^{-1} - 1)}, \quad s \in [0,1].$$

Since the first term in the denominator is the same as the numerator up to a constant multiplier, thus we only have to show that

$$\frac{1 + s^{\frac{-1}{1-s}} - s^{-1}}{\sqrt{s}(s^{\frac{-1}{1-s}} + s^{-1} - 1)} \to 0 \quad \text{as} \quad s \searrow 0.$$

For $s \in]0,1]$ we have

$$0 \le \frac{1 + s^{\frac{-1}{1-s}} - s^{-1}}{\sqrt{s}(s^{\frac{-1}{1-s}} + s^{-1} - 1)} \le \frac{1 + s^{\frac{-1}{1-s}} - s^{-1}}{\sqrt{s}(2s^{-1} - 1)} = \frac{1 + s^{\frac{-1}{1-s}} - s^{-1}}{2s^{-\frac{1}{2}} - s^{\frac{1}{2}}}.$$

Since the denominator goes to infinity, we can neglect the 1 in the numerator when taking limits. We will also expand by s, to get

$$\frac{s^{\frac{-1}{1-s}} - s^{-1}}{2s^{-\frac{1}{2}} - s^{\frac{1}{2}}} = \frac{s^{\frac{-s}{1-s}} - 1}{2s^{\frac{1}{2}} - s^{\frac{3}{2}}}.$$

The denominator clearly goes to zero, and using $s\log(s) \to 0$, one can see that the numerator also goes to zero. Hence we can apply L'Hospital's rule:

$$\lim_{s \searrow 0} \frac{s^{\frac{-s}{1-s}} - 1}{2s^{\frac{1}{2}} - s^{\frac{3}{2}}} = \lim_{s \searrow 0} \frac{-s^{\frac{-s}{1-s}}\left(\frac{1}{1-s} + \frac{1}{(1-s)^2}\log(s)\right)}{\frac{1}{2}(2s^{-\frac{1}{2}} - 3s^{\frac{1}{2}})}$$

$$= \lim_{s \searrow 0} -2\frac{s^{\frac{-s}{1-s}}\left(\frac{1}{1-s} + \frac{1}{(1-s)^2}\log(s)\right)}{2s^{-\frac{1}{2}} - 3s^{\frac{1}{2}}} = \lim_{s \searrow 0} -2\frac{s^{\frac{-s}{1-s}}\left(\frac{1}{1-s} + \frac{1}{(1-s)^2}\log(s)\right)}{s^{-\frac{1}{2}}(2-3s)}$$

$$= \lim_{s \searrow 0} -2s^{\frac{-s}{1-s}+\frac{1}{2}} \frac{1 - s + \log(s)}{2(1-s)^2 - 3s(1-s)^2} = \lim_{s \searrow 0} -2\frac{s^{\frac{1-3s}{2(1-s)}} - s^{\frac{3-5s}{2(1-s)}} + s^{\frac{1-3s}{2(1-s)}}\log(s)}{2(1-s)^2 - 3s(1-s)^2}.$$

The first two terms in the numerator go to 0, and the denominator goes to 2, so we only have to look at the last term of the numerator. For small enough $s>0$ we have

$$0 \geq s^{\frac{1-3s}{2(1-s)}} \log(s) \geq s^{\frac{1}{3}} \log(s) \to 0$$

With this, we have proved the desired equality

$$\lim_{s\searrow 0} \tilde{q}(s) - 1 = 0.$$

This implies that $\alpha = 1$ completing the proof of our statement in the remaining case $p = 0$. ■

Observe that in the case $p = 0$, the statement above gives us that if a positive linear functional φ "makes" the exponential function convex meaning that $\varphi \circ \exp$ is convex on $\mathscr{A}^{++}$, then φ is necessarily tracial. In other words, the exponential function is so much non operator convex that there is only one type of positive linear functional which "makes" it convex.

5. A TRACE CHARACTERIZATION WITH κ_p AND THE STEIN DIVERGENCE

Above we have studied certain distance measures, each of which is defined in the case of matrix algebras as the square root of the trace of the difference of the arithmetic mean and a certain variant of the geometric mean. In a few particular cases we know that those are actually metrics even in the general setting of C^*-algebras, in some cases we know that they are not, and in certain cases we do not know only conjecture that they are true metrics or that they are not.

Let us recall now the following. For any pair $A, B \in \mathbb{M}_n(\mathbb{C})^{++}$ of positive definite matrices, the symmetric Stein divergence is defined by

$$S(A,B) = \log\det\left(\frac{A+B}{2}\right) - \frac{1}{2}\log\det(AB).$$

Actually, it is just the Jensen-Shannon symmetrization of the divergence called Stein's loss (see the first two sections in [19]). In fact, the measure S was originally introduced in [7]. The authors of that work claimed that

$$\delta_S(A,B) = \sqrt{S(A,B)}, \quad A, B \in \mathbb{M}_n(\mathbb{C})^{++}$$

is not a metric while the authors of [6] conjectured that it is. The problem was solved by Sra in [19]. Namely, in Theorem 5 in [19] it was proved that δ_S is a true metric on $\mathbb{M}_n(\mathbb{C})^{++}$, furthermore, several interesting results were presented there concerning the properties of δ_S.

Observe that since $\mathrm{Tr} \circ \log$ equals $\log \circ \det$ on $\mathbb{M}_n(\mathbb{C})^{++}$, we can rewrite $S(A,B)$ as

$$S(A,B) = \mathrm{Tr}\left(\log(A\nabla B) - \log(A\kappa_p B)\right), \quad A, B \in \mathbb{M}_n(\mathbb{C})^{++}$$

for any real number $p \geq 0$. Therefore, in contrast with the case of the trace of the difference of the arithmetic mean and the operation κ_p which really depends on the value of the parameter p, the square root of the trace of the difference of the logarithm of the arithmetic mean and the logarithm of the operation κ_p does not depend on p and is a true metric.

The natural problem arises whether a similar assertion is true in the general case of C^*-algebras. Let $\mathscr{A}$ be a C^*-algebra with a faithful tracial positive linear functional τ. Then we indeed have that for any non-negative real numbers p, q

$$\tau(\log(A\kappa_p B)) = \tau(\log(A\kappa_q B)), \quad A, B \in \mathscr{A}^{++}. \tag{16}$$

To see this, we remark the following. Employing an idea given in the proof of Theorem 2 in [12], in Lemma 17 in [13] we proved that for any bounded linear functional φ on a von Neumann algebra, φ is tracial if and only if it satisfies

$$\varphi(\log(ABA)) = 2\varphi(\log A) + \varphi(\log B) \tag{17}$$

for all positive invertible A, B. (To tell the truth, in [13] that result was proved for self-adjoint linear functionals, but knowing the characterization for such functionals, one can deduce the same for general linear functionals.)

It was proved in [11] that $\mathscr{A}$ carrying a faithful tracial positive linear functional τ can be assumed to be a weakly/strongly dense subalgebra of a von Neumann algebra onto which τ extends as a normal faithful tracial positive linear functional, and one can deduce that τ satisfies (17). It then easily implies that

$$\tau(\log(A\kappa_p B)) = (1/2)\tau(\log A + \log B) = \tau(\log(A\kappa_q B)), \quad A, B \in \mathscr{A}^{++}.$$

Therefore, we have the independence of the quantity

$$\delta_S^\tau(A, B) = \left(\tau\left(\log(A\nabla B) - \log(A\kappa_p B)\right)\right)^{1/2}, \quad A, B \in \mathscr{A}^{++} \tag{18}$$

of p. But we do not know the answer to the following probably difficult question.

Problem 3. Is the function δ_S^τ in (18) a true metric in the general case of a C^*-algebra equipped with a faithful tracial positive linear functional τ?

To this we note that since

$$\tau(\log(A\kappa_p B)) = \tau(\log(A\# B)), \quad A, B \in \mathscr{A}^{++}.$$

is also a consequence of the validity of (17) for τ, using the operator monotonicity of the logarithm function, one case see that $\delta_S^\tau(A, B)$ is always non-negative, takes the value 0 only if $A = B$ and it is symmetric in its variables. Hence the real question is whether the triangle inequality holds true for δ_S^τ. Again we believe that this is a hard question which opinion might be justified by the complexity of the proof for matrix algebras, see [19].

We close our work with the following result concerning the equality (16). It shows that for any given pair p, q of different non-negative numbers, the validity of that equality actually characterizes the traciality of bounded linear functionals on von Neumann algebras. So, in contrast to Theorem 3, here we do not need to assume that the linear functional in question is positive. Of course there is a price of this generality of functionals that we have to pay: the result concerns von Neumann algebras and not general C^*-algebras.

Theorem 4. *Let $\mathscr{A}$ be a von Neumann algebra. Assume the φ is a bounded linear functional on $\mathscr{A}$. Let p, q be different non-negative real numbers. Then*

$$\varphi(\log A\kappa_p B) = \varphi(\log A\kappa_q B), \quad A, B \in \mathscr{A}^{++} \tag{19}$$

if and only if φ is tracial.

Proof: Because of the discussion above we only need to verify the necessity part of the statement.

Assume first that $p = 0$ and $q > 0$. If we replace A by $A^{4/q}$ and B by $B^{2/q}$ in the equality

$$\varphi(\log A\kappa_0 B) = \varphi(\log A\kappa_q B),$$

we easily obtain the equality

$$\varphi(2\log A + \log B) = \varphi(\log ABA), \quad A, B \in \mathscr{A}^{++}. \tag{20}$$

Referring to Lemma 17 in [13], we have the traciality of φ. Let us recall the idea of the proof of that lemma. For a given pair P, Q of projections, we wrote $I + tP$ in the place of A and $I + tQ$ in the place of B, $t > -1$, and used the power series representation

$$\log(I + X) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{X^n}{n} \tag{21}$$

for any $X \in \mathscr{A}$ with $\|X\| < 1$. This way, from the equality in (20), we obtained the equality of two power series of the variable t in a neighborhood of zero. Comparing the coefficients of t^3, we got the equality

$$\varphi(PQP) = \varphi(QPQ).$$

Finally, applying an idea from the proof of Lemma 1 in [4], we deduced that this latter equality for any pair P, Q of projections implies that φ is tracial.

Assume now that $0 < p < q$. We employ a similar argument but from the technical point of view the case is definitely more difficult. Let us substitute A and B with $A^{p/4}$ and $B^{q/2}$, and bring the roots out of the logarithms. This way, (19) becomes

$$\frac{1}{q}\varphi\left(\log(ABA)\right) = \frac{1}{p}\varphi\left(\log\left(A^{\frac{q}{p}} B^{\frac{q}{p}} A^{\frac{q}{q}}\right)\right).$$

From this, we deduce that, writing q in the place of q/p, the equality (19) is equivalent to

$$\varphi\left(\log(ABA)\right) = \frac{1}{q}\varphi\left(\log\left(A^q B^q A^q\right)\right), \quad A, B \in \mathscr{A}^{++}. \tag{22}$$

Let P and Q now be two projections and set $A = I + tP$ and $B = I + tQ$ for any real number $t > -1$. We will use power series representations to obtain the following equalities:

$$\frac{1}{q}\varphi\left(\log\left(A^q B^q A^q\right)\right) = \frac{1}{q}\varphi\left(\log\left(\sum_{n=0}^{\infty} \upsilon_n t^n\right)\right) = \frac{1}{q}\varphi\left(\sum_{n=0}^{\infty} \Upsilon_n t^n\right)$$

and

$$\varphi\left(\log(ABA)\right) = \varphi\left(\log\left(\sum_{n=0}^{\infty} \omega_n t^n\right)\right) = \varphi\left(\sum_{n=0}^{\infty} \Omega_n t^n\right).$$

The coefficients $\upsilon_n, \Upsilon_n, \omega_n, \Omega_n$ are, of course, elements of $\mathscr{A}$. Since the two power series define the same function in a neighbourhood of $t = 0$, thus the coefficients must coincide. In particular, we have

$$\varphi\left(\Omega_3 - \frac{\Upsilon_3}{q}\right) = 0. \tag{23}$$

We will show that this implies $\varphi(PQP) = \varphi(QPQ)$. As we have already mentioned, the validity of this latter equality for any pair P, Q of projections in $\mathscr{A}$ gives the traciality of φ.

First, we use the power series representation of $(1+t)^q$ to get

$$\begin{aligned} A^q B^q A^q &= (I + ((1+t)^q - 1)P)(I + ((1+t)^q - 1)Q)(I + ((1+t)^q - 1)P) \\ &= \left(I + \sum_{i=1}^{\infty} \binom{q}{i} t^i P\right)\left(I + \sum_{k=1}^{\infty} \binom{q}{k} t^k Q\right)\left(I + \sum_{j=1}^{\infty} \binom{q}{j} t^j P\right) \\ &= \log(I + \sum_{n=1}^{\infty} \upsilon_n t^n) \end{aligned}$$

and then we use (21) to obtain

$$\log(I + \sum_{n=1}^{\infty} \upsilon_n t^n) = \sum_{m=0}^{\infty} (-1)^{m+1} \frac{\left(\sum_{n=1}^{\infty} \upsilon_n t^n\right)^m}{m}.$$

It is not difficult to see that

$$\Upsilon_3 = \upsilon_3 - \frac{\upsilon_1\upsilon_2 + \upsilon_2\upsilon_1}{2} + \frac{\upsilon_1^3}{3}, \tag{24}$$

thus we only have to examine v_1, v_2, v_3. We compute

$$v_1 = q(2P+Q),$$

and

$$\begin{aligned} v_2 =& \frac{q(q-1)}{2} 2P + q^2 P + q^2(PQ+QP) + \frac{q(q-1)}{2} Q \\ =& (q^2-q)P + q^2 P + q^2(PQ+QP) + \frac{q^2-q}{2} Q \\ =& q^2\left(PQ+QP+2P+\frac{1}{2}Q\right) - q\left(P+\frac{1}{2}Q\right) \\ =& \frac{1}{2}q^2(2P+Q)^2 - \frac{1}{2}q\,(2P+Q)\,. \end{aligned}$$

As for v_3, we have

$$\begin{aligned} v_3 =& \frac{q(q-1)(q-2)}{6} 2P + \frac{q^2(q-1)}{2} 2P + \frac{q^2(q-1)}{2}(PQ+QP) \\ &+ q^3 PQP + \frac{q^2(q-1)}{2}(PQ+QP) + \frac{q(q-1)(q-2)}{6} Q \\ =& \frac{q^3-3q^2+2q}{3} P + (q^3-q^2)P + \frac{q^3-q^2}{2}(PQ+QP) + q^3 PQP \\ &+ \frac{q^3-q^2}{2}(PQ+QP) + \frac{q^3-3q^2+2q}{6} Q \\ =& q^3\left(PQP+PQ+QP+\frac{4}{3}P+\frac{1}{6}Q\right) \\ &- q^2\left(PQ+QP+2P+\frac{1}{2}Q\right) + q\left(\frac{2}{3}P+\frac{1}{3}Q\right). \end{aligned}$$

Hence

$$\begin{aligned} v_1v_2+v_2v_1 =& \frac{1}{2}q^3\left((2P+Q)(2P+Q)^2+(2P+Q)^2(2P+Q)\right) \\ &- \frac{1}{2}q^2\left((2P+Q)\,(2P+Q)+(2P+Q)\,(2P+Q)\right) \\ =& q^3\left(4PQP+2QPQ+6(PQ+QP)+8P+Q\right) \\ &- q^2\left(2(PQ+QP)+4P+Q\right) \end{aligned}$$

and

$$v_1^3 = q^3(2P+Q)^3 = q^3(4PQP+2QPQ+6(PQ+QP)+8P+Q).$$

Therefore, by (24) we can compute

$$\begin{aligned} \Upsilon_3 =& v_3 - \frac{v_1v_2+v_2v_1}{2} + \frac{v_1^3}{3} = q^3\left(PQP+PQ+QP+\frac{4}{3}P+\frac{1}{6}Q-2PQP-QPQ\right. \\ &\left.-3(PQ+QP)-4P-\frac{1}{2}Q+\frac{4}{3}PQP+\frac{2}{3}QPQ+2(PQ+QP)+\frac{8}{3}P+\frac{1}{3}Q\right) \\ &+ q^2\left(PQ+QP+2P+\frac{1}{2}Q-(PQ+QP)-2P-\frac{1}{2}Q\right) + q\left(\frac{2}{3}P+\frac{1}{3}Q\right) \\ =& q^3\left(\frac{1}{3}PQP-\frac{1}{3}QPQ\right) + q\left(\frac{2}{3}P+\frac{1}{3}Q\right). \end{aligned}$$

We have an easier job with Ω_3, since it is the $q = 1$ case of Υ_3. Thus we get

$$\Omega_3 = \frac{1}{3}PQP - \frac{1}{3}QPQ + \frac{2}{3}P + \frac{1}{3}Q.$$

With all this done, we can see that

$$\begin{aligned}\Omega_3 - \frac{\Upsilon_3}{q} =& \left(\frac{1}{3}PQP - \frac{1}{3}QPQ + \frac{2}{3}P + \frac{1}{3}Q\right) - \left(\frac{2}{3} + P\frac{1}{3}Q\right) - q^2\left(\frac{1}{3}PQP - \frac{1}{3}QPQ + \frac{2}{3}P + \frac{1}{3}Q\right)\\ =&(1-q^2)\left(\frac{1}{3}PQP - \frac{1}{3}QPQ\right).\end{aligned}$$

Hence, by (23) we obtain

$$\varphi\left(\frac{1}{3}PQP - \frac{1}{3}QPQ\right) = 0,$$

which implies $\varphi(PQP) = \varphi(QPQ)$ for any pair P, Q of projections in $\mathscr{A}$. As mentioned above, this implies that φ is tracial. This completes the proof of our result. ■

We finish our present work with summarizing the results. First, we studied whether there is an inequality between the arithmetic mean and our κ_p operation on C^*-algebras (whether the difference of the arithmetic mean and κ_p is non-negative) which would trivially imply that d_p^τ is well-defined. In Proposition 1 we showed however, that this happens only in special cases, precisely on commutative algebras. Following this, we referred to [11] where the author proves that d_p is well-defined on $\mathbb{M}_n(\mathbb{C})^+$ for all $p \geq 0$, and that d_p^τ is also well-defined on the positive cone of a general C^*-algebra in the case of $p \leq 2$. (The $p > 2$ case seems to hold quite a challenge.) After this, we studied the metric properties of d_p. In Proposition 2 we showed that d_p is a true metric on the set $P_1(H)$ of the rank one projections of a Hilbert space H if and only if $p \leq 2$. This implies that for any $p > 2$, d_p fails to be a metric even on the positive cone of the algebra of 2 by 2 matrices. The question, whether d_p^τ is a true metric in these cases, is certainly a hard problem, the further research of which is required. In Theorem 2 we proved that if there exists a φ positive linear functional on a C^*-algebra such that d_p^φ is well defined, then it is necessarily tracial. We note, that the $p = 0$ case of both this result and Proposition 1 tell us about how much the exponential function fails to be operator convex. Motivated by the symmetric Stein divergence, we studied the differences of the logarithm of the arithmetic mean and the logarithm of the κ_p-s under a faithful tracial positive linear functional on the positive definite cone of an arbitrary C^*-algebra. It is natural to ask whether these functions coincide mimicking the behavior on $\mathbb{M}_n(\mathbb{C})^{++}$. Our last result shows that this happens to a bounded linear functional of a von Neumann algebra precisely when it is tracial. We note that one can prove a similar result in the case of a C^*-algebra and a positive linear functional. Whether the square root of δ_S^τ is a true metric in the latter context, is a deep problem as well, which we would like to examine in the future.

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