Problem Sheet 3

1. Prove that in a group of even order there always exists an element of order 2.

2. Show that 6 is a divisor of $|S_4|$ but S_4 has no element of order 6.

3. Prove that every group of prime order is cyclic.

4. Prove that every cyclic group is commutative. Give an example for a commutative subgroup in S_4 which is not cyclic.

5. Let $A, B \le G$ and $|G| < \infty$.
Prove that the cardinality of the subset $AB = \{\, ab \,|\, a \in A,\ b \in B \,\}$ is

$$|AB| = \frac{|A| \cdot |B|}{|A \cap B|}.$$

6. a) Let $A, B \le G$. Show that AB is also a subgroup if and only if $AB = BA$.
b) Check that for $A = \langle (12) \rangle$, $B = \langle (123) \rangle$ and $C = \langle (13) \rangle$, the subset AB is a subgroup of S_3 but AC is not a subgroup.
c) Show that for $A = \langle (12) \rangle \le S_4$ and $B = \langle (234) \rangle \le S_4$ the cardinality $|AB|$ of the subset AB is a divisor of $|S_4|$ but AB not a subgroup of S_4.

7. Show that $Hg \leftrightarrow g^{-1}H$ is a bijection between the right and left cosets of $H \le G$, and for a right transversal R the set R^{-1} is a left transversal.

8. Prove that the cyclic group C_n has exactly $\varphi(d)$ elements of order d for every divisor d of n, where $\varphi(d) = \{\, m \,|\, 1 \le m \le d,\ \gcd(m, d) = 1 \,\}$.

9. What are the possible orders of the elements of D_n, and what is the number of elements for each order?

10. Show that every nontrivial (that is, $\neq 1$) subgroup of C_∞ has a finite index, i.e. it has finitely many cosets.

HW1. Let $A, B \le G$, where G is a finite group. Show that the subgroups A and B are contained in the set AB, and both $|A|$ and $|B|$ are divisors of $|AB|$ (though AB is not necessarily a group).

HW2. Prove that the (multiplicative) group of invertible 3×3 upper triangular matrices over $\mathbb{Z}_2$ is not a cyclic group. (For example, you may show that it is not commutative.)