Problem Sheet 12

1. Determine the cardinality of the factor ring $K[x]/(x^2+x+1)$ if $K = \mathbb{Z}_2$ or $\mathbb{Z}_3$. Which of the two factor rings is a field?

2. Let $K = \mathbb{Z}_2$ and $p(x) = x^3 + x + 1$. Show that $R = K[x]/(p(x))$ is a field of 8 elements. Find all the roots of $x^3 + x^2 + 1$ in R as polynomials of $\alpha = x + (p(x))$.

3. Prove that $\mathbb{Q}[x]/(x^2-2) \cong \mathbb{Q}[x]/(x^2-2x-1)$.

4. What is the minimal polynomial of $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$ and over $\mathbb{Q}(\sqrt{6})$?

5. Suppose that for some $\alpha, \beta \in \mathbb{C}$, the numbers $\alpha+\beta$ and $\alpha\beta$ are algebraic over $\mathbb{Q}$. Prove that α and β are also algebraic.

6. Let $\alpha \in \mathbb{C}$ be a root of the polynomial $x^3 - 2x^2 + x + 1 \in \mathbb{Q}[x]$. Express the reciprocal of $\alpha^2 + 2$ as an at most second degree polynomial of α.

7. Determine the degrees of the following extensions over $\mathbb{Q}$.

a) $\mathbb{Q}(i+\sqrt{3})$ b) $\mathbb{Q}(\sqrt[3]{2})$ c) $\mathbb{Q}(\sqrt[3]{2}+\sqrt[3]{4})$ d) $\mathbb{Q}(\sqrt[3]{2}+\sqrt{2})$

8. Let α be a root of the polynomial x^3+x+1 over $\mathbb{Z}_2$, and let $K = \mathbb{Z}_2(\alpha)$. Is the polynomial $x^2+x+\alpha$ irreducible over K?

HW. Let $K = \mathbb{Z}_2(\alpha)$ be the extension of $\mathbb{Z}_2$ by a root α of the polynomial $x^4+x+1 \in \mathbb{Z}_2[x]$. Write the element $\dfrac{\alpha^2}{\alpha+1} \in K$ as an at most third degree polynomial of α.