Problem Sheet 1

1. Determine the number of isometries (in other word, symmetries) for a
a) square;
b) regular tetrahedron;
c) cube.
How many of these are reflections or rotations?

2. a) Show that exactly as many of the isometries in problem 1 are orientation preserving as orientation reversing.
b) Let $1, 2, 3, 4$ denote the corners of the tetrahedron in problem 1.b), and $1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 1$ be the permutation of the corners determining an isometry f. Show that f is neither a rotation nor a reflection. Try to obtain it as the composition (product) of two of those.

3. Consider the following permutations (bijections) of the set $\{1, 2, 3, 4, 5\}$ (where each i is mapped to the element below).

$$f: \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix} \qquad g: \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$$

Show the action of each permutation on an oriented graph with vertices $1, 2, 3, 4, 5$. How many times do we need to apply the permutation to get the identity map ($i \mapsto i\ \forall i$)?

4. Let $GL_n(K)$ (general linear group) denote the set of invertible $n \times n$ matrices over the field K. The *order* of a matrix A in this group is the smallest positive integer k such that $A^k = I$, where I denotes the identity matrix (the order is ∞ if such a k does not exist).
a) What is the order of the following matrices?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \in GL_5(\mathbb{R}), \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{R})$$

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \in GL_2(\mathbb{Z}_3) \qquad D = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \in GL_3(\mathbb{Z}_2)$$

b) What is the number of elements in the group $G = GL_2(\mathbb{Z}_3)$ and in $G = GL_3(\mathbb{Z}_2)$?
c) What are the possible orders of the elements of $GL_2(\mathbb{R})$?

5*. What are the possible orders of the elements of $GL_3(\mathbb{Z}_2)$? (Hint: The order of a matrix is k if and only if k is the smallest positive integer such that the minimal polinomial of the matrix divides $x^k - 1$.)

HW1. Let C be a square based rectangular cuboid with vertices $(\pm 1, \pm 1, \pm 2)$. Determine the number of isometries of C. How many of these are reflections across planes or points and how many are rotations about a straight line?

HW2. Draw the action of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 6 & 5 & 4 & 8 & 7 & 2 \end{pmatrix}$ as an oriented graph as in problem 3. Determine the order of the permutation, that is, the smallest positive integer k such that if we apply the permutation k times we get the identity map.